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MULTITURN INJECTION OF HEAVY IONS INTO THE BOOSTER

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Collider Accelerator Department

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1 Introduction

Positvely charged heavy ions, which are transfered from the Tandem to the Booster via the Heavy Ion Transfer Line (HITL) and the HTB (HITL To Booster) line, are injected into the booster by means of an electrostatic inflector and four fast ferrite dipole kickers. The inflector, which consists of a cathode and a thin septum, bends the incoming beam along a circular arc of approximately 15 degrees bringing the beam to the outside edge of the booster acceptance region. The septum separates and shields the region of circulating beam in the booster from the electrostatic fields of the inflector. The four dipole kickers produce a local distortion—i.e. a bump—of the closed orbit which places the distorted orbit near the septum. This allows beam emerging from the inflector to enter the acceptance region of the booster. Since the beam arrives from the Tandem in pulses of 200–400 microsecond duration, and the period for one revolution around the booster is 5–17 microseconds (depending on the ion species), the injection process takes place over several revolutions—or turns—of beam around the machine, and as such is refered to as multiturn injection.

As the beam enters the booster it occupies a region of phase space which is beyond the outer side of the septum. On subsequent turns around the machine the beam will eventually return to this region and hit the septum if the distorted orbit is not moved away. For a given position and angle of the distorted orbit, the number of turns that can be injected before having to move the orbit, and the amount by which the orbit must be moved, depend on the incoming beam ellipse parameters and the horizontal tune. In this report we obtain conditions for beam survival during injection in terms of these parameters. The results are consistent with those of Ref. 1.

2 General Considerations

2.1 The Injection Bump

The injection bump is produced by four dipole magnets located in the injection region with positions

$$s_1 < s_2 < s_I < s_3 < s_4 \tag{1}$$

where s_I is the point of injection. Let ϕ_j be the angular kick produced by the dipole at s_j and let

$$\mathbf{\Phi}_j = \begin{pmatrix} 0 \\ \phi_j \end{pmatrix}, \quad \mathbf{X}_c = \begin{pmatrix} x_c \\ x'_c \end{pmatrix}, \quad \mathbf{M}_{ji} = \mathbf{M}(s_j, s_i) = \begin{pmatrix} a_{ji} & b_{ji} \\ c_{ji} & d_{ji} \end{pmatrix} \tag{2}$$

where x_c, x'_c are the horizontal position and angle of the closed orbit at s_I, and $\mathbf{M}(s_j, s_i)$ is the unperturbed—i.e. without the four dipole kicks—linear transfer matrix from s_i to s_j. Then if we require that the four dipoles produce no closed orbit distortion outside the injection region, we must have

$$\mathbf{\Phi}_4 + \mathbf{M}_{43}\mathbf{\Phi}_3 + \mathbf{M}_{42}\mathbf{\Phi}_2 + \mathbf{M}_{41}\mathbf{\Phi}_1 = \mathbf{0}, \tag{3}$$

and the position and angle of the closed orbit at the injection point s_I are given by

$$\mathbf{X}_c = \mathbf{M}_{I2}\mathbf{\Phi}_2 + \mathbf{M}_{I1}\mathbf{\Phi}_1. \tag{4}$$

Using Eqs. (2) in (3) and (4) we find

$$b_{41}\phi_1 + b_{42}\phi_2 + b_{43}\phi_3 = 0, \tag{5}$$

$$d_{41}\phi_1 + d_{42}\phi_2 + d_{43}\phi_3 + \phi_4 = 0, \tag{6}$$

$$b_{I1}\phi_1 + b_{I2}\phi_2 = x_c, \tag{7}$$

$$d_{I1}\phi_1 + d_{I2}\phi_2 = x'_c, \tag{8}$$

where

$$b_{ji} = \sqrt{\beta_j\beta_i}\sin(\psi_j - \psi_i), \tag{9}$$

$$d_{ji} = \sqrt{\frac{\beta_i}{\beta_j}}\cos(\psi_j - \psi_i) - \frac{\alpha_j}{\beta_j}b_{ji}, \tag{10}$$

and α_j, β_j, and ψ_j are the horizontal lattice parameters at the point s_j.

Thus, if we specify certain values for the position and angle of the closed orbit at the injection point, then equations (5–8) amount to four equations which can be solved for ϕ_1, ϕ_2, ϕ_3, and ϕ_4. Using Eqs. (9–10) in (5–8) we find

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{b_{21}} \begin{pmatrix} d_{I2} & -b_{I2} \\ -d_{I1} & b_{I1} \end{pmatrix} \begin{pmatrix} x_c \\ x_c' \end{pmatrix}, \tag{11}$$

and

$$\begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{b_{43}} \begin{pmatrix} -b_{41} & -b_{42} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{12}$$

These formulae are consistent with those given in Ref. 2.

2.2 Injection onto the Bumped Orbit

Now let

$$\mathbf{X} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad \mathbf{X}_4 = \begin{pmatrix} x_4 \\ x_4' \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} y \\ y' \end{pmatrix} \tag{13}$$

where x, x' are the position and angle of a beam particle at the injection point, x_4, x_4' are the subsequent position and angle at s_4, and y, y' are the position and angle on the next pass by the injection point. Then

$$\mathbf{X}_4 = \mathbf{M}_{4I}\mathbf{X} + \mathbf{M}_{43}\mathbf{\Phi}_3 + \mathbf{\Phi}_4 = \mathbf{M}_{4I}\mathbf{X} - \mathbf{M}_{42}\mathbf{\Phi}_2 - \mathbf{M}_{41}\mathbf{\Phi}_1, \tag{14}$$

in which we have used (3), and using (4) we find

$$\mathbf{X}_4 = \mathbf{M}_{4I}\mathbf{X} - \mathbf{M}_{4I}\mathbf{X}_c = \mathbf{M}_{4I}(\mathbf{X} - \mathbf{X}_c). \tag{15}$$

If $\mathbf{M}_I$ is the unperturbed transfer matrix for one turn around the machine begining at the injection point, then we have

$$\mathbf{Y} = \mathbf{M}_I\mathbf{M}_{4I}^{-1}\mathbf{X}_4 + \mathbf{M}_{I1}\mathbf{\Phi}_1 + \mathbf{M}_{I2}\mathbf{\Phi}_2, \tag{16}$$

and using (4) and (15) we find

$$\mathbf{Y} - \mathbf{X}_c = \mathbf{M}_I(\mathbf{X} - \mathbf{X}_c). \tag{17}$$

This equation gives the turn-by-turn evolution of the beam position and angle with respect to the closed orbit. The unperturbed transfer matrix, $\mathbf{M}_I$, is given by

$$\mathbf{M}_I = \begin{pmatrix} \cos\mu + \alpha_I \sin\mu & \beta_I \sin\mu \\ -\gamma_I \sin\mu & \cos\mu - \alpha_I \sin\mu \end{pmatrix}, \quad \mu = 2\pi Q \tag{18}$$

where α_I, β_I, $\gamma_I = (1+\alpha_I^2)/\beta_I$ are the machine lattice parameters at s_I and Q is the horizontal tune.

Now suppose the beam entering the machine is contained within the beam ellipse

$$(\mathbf{X}^\dagger - \mathbf{X}_0^\dagger)\mathbf{E}^{-1}(\mathbf{X} - \mathbf{X}_0) = \epsilon, \tag{19}$$

where

$$\mathbf{X}_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \quad \beta\gamma - \alpha^2 = 1, \tag{20}$$

x_0 and x_0' are the position and angle of the center of the ellipse, and $\pi\epsilon$ is the incoming beam emittance. Then it follows from (17) that on the next pass by the injection point this injected beam will be contained within the ellipse

$$(\mathbf{Y}^\dagger - \mathbf{Y}_0^\dagger)\mathbf{F}^{-1}(\mathbf{Y} - \mathbf{Y}_0) = \epsilon \tag{21}$$

where

$$\mathbf{F} = \mathbf{M}_I \mathbf{E} \mathbf{M}_I^\dagger, \tag{22}$$

and

$$(\mathbf{Y}_0 - \mathbf{X}_c) = \mathbf{M}_I(\mathbf{X}_0 - \mathbf{X}_c). \tag{23}$$

These equations give the turn-by-turn evolution of the injected beam ellipse.

2.3 Conditions for Survival of the Injected Beam

2.3.1 Phase Space Layers and Optimum Beam Ellipse Parameters

As the beam ellipse (19) enters the machine, it occupies a region of phase space for which the horizontal positions of the beam particles are greater than the horizontal position, x_s, of the outer side of the septum. On subsequent turns around the machine the beam ellipse remains in the region between two ellipses,

$$(\mathbf{X}^\dagger - \mathbf{X}_c^\dagger)\mathbf{E}_I^{-1}(\mathbf{X} - \mathbf{X}_c) = \epsilon_I, \quad \mathbf{E}_I = \begin{pmatrix} \beta_I & -\alpha_I \\ -\alpha_I & \gamma_I \end{pmatrix}, \tag{24}$$

and

$$(\mathbf{X}^\dagger - \mathbf{X}_c^\dagger)\mathbf{E}_I^{-1}(\mathbf{X} - \mathbf{X}_c) = \epsilon_s, \tag{25}$$

which are centered on the closed orbit and are matched to the machine lattice at the injection point. Here ϵ_I is chosen so that the first ellipse is the smallest such ellipse which contains the beam ellipse, and ϵ_s is chosen so that the second ellipse just touches the outer side of the septum—i.e. so that

$$x_c + \sqrt{\epsilon_s \beta_I} = x_s. \tag{26}$$

Figure 1 shows the relative positions of these ellipses. The labels A, B, and C refer to the ellipses defined by (19), (24), and (25) respectively. Although the beam ellipse may occupy various positions between the two ellipses on subsequent turns around the machine, it will eventually return to the region of phase space it occupied as it entered the machine. In order to avoid loosing beam on the septum one must therefore, at an appropriate time, move the closed orbit away from the septum to a new position, x_{c2}, for which

$$x_{c2} + \sqrt{\epsilon_I \beta_I} \leq x_s - t, \tag{27}$$

where t is the septum thickness. Beam entering the machine with the closed orbit in this new position will remain inside the region between two new ellipses similar to (24) and (25) but centered on the new closed orbit position. As before, the beam in this region will eventually hit the septum unless the closed orbit is moved still further away from the septum. During the multiturn injection process, the beam is therefore injected into a series of phase space regions—or layers—between ellipses similar to (24) and (25). The process continues until the injection bump has collapsed and beam emerging from the inflector is no longer placed inside the machine acceptance region. Clearly, the less one has to move the closed orbit away from the septum for a given layer, the more layers, and hence, the more beam, one can get into the machine. For a given position and angle (x_c, x'_c) of the closed orbit, if one chooses the incoming beam ellipse parameters α, β, x_0, and x'_0 such that ϵ_I is minimized, then it follows from (27) that the amount the closed orbit has to be moved away from the septum is minimized. In Appendix I it is shown that ϵ_I can be made smallest when

$$\alpha = \alpha_I \beta / \beta_I \tag{28}$$

and

$$x_0 = x_s + \sqrt{\epsilon \beta}, \quad x'_0 - x'_c = -\alpha_I (x_0 - x_c)/\beta_I. \tag{29}$$

One then finds that for a fixed β, the minimum ϵ_I is given by

$$\epsilon_I = \epsilon \left\{ \frac{1 + b\eta^2 + 2b^{3/2}\eta}{b(1 - b^2)} \right\}, \quad B < 1 \tag{30}$$

and

$$\epsilon_I = \epsilon(\eta + 2\sqrt{b})^2, \quad B \geq 1, \tag{31}$$

where

$$b = \beta/\beta_I, \quad \eta = (x_s - x_c)/\sqrt{\epsilon\beta_I}, \quad B = b^{3/2}(\eta + 2\sqrt{b}). \tag{32}$$

If β is allowed to vary, then for a given η, the value of the parameter b for which ϵ_I is smallest, is given by

$$1 - 3b^2 = 2\eta b^{3/2} \tag{33}$$

and in this case we have

$$\epsilon_I = \epsilon \left(\frac{1 + 3b^2}{4b^3} \right). \tag{34}$$

Figure 2 shows a plot of b versus η obtained from Eq. (33). Here we see that as the closed orbit is moved away from the septum the optimum β for the incoming beam ellipse decreases. For the case in which $\eta = 0$ we have $b = 1/\sqrt{3}$ and $\epsilon_I/\epsilon = 3\sqrt{3}/2$.

Now let x_{cn} be the position of the closed orbit (at the injection point) for the nth phase space layer. Then it follows from (27) that the position, x_{cn+1}, of the closed orbit for the $(n+1)$th layer should be given by

$$\eta_{n+1} = \rho_n + \tau, \tag{35}$$

where

$$\eta_{n+1} = (x_s - x_{cn+1})/\sqrt{\epsilon\beta_I}, \quad \rho_n = \sqrt{\epsilon_{In}/\epsilon}, \quad \tau = t/\sqrt{\epsilon\beta_I}, \tag{36}$$

and ϵ_{In} is the value of ϵ_I obtained from Eqs. (30–31) or (33–34) with $\eta = \eta_n$. Using the recursion relation (35) with $\rho_0 = 0$ one obtains η_n for each phase space layer. Table I lists the values of η_n, ρ_n, and b_n for the case in which the septum thickness is zero ($\tau = 0$). Here b_n is obtained from (33) with $\eta = \eta_n$, and ρ_n is obtained from (34) with $b = b_n$. The corresponding positions of the closed orbit and beam ellipse parameters can be obtained from the table using Eqs. (28–29) and (32). The number of phase space layers possible during the multiturn injection process can be determined from the table if the limiting aperture, A_L, in the machine is known. If β_L is the horizontal beta function at the limiting aperture, then we must have $\sqrt{\epsilon_I \beta_L} \leq A_L$ and ρ_n must therefore satisfy

$$\rho_n \leq A_L / \sqrt{\epsilon \beta_L}. \tag{37}$$

Table I: Phase Space Layers with $\tau = 0$						
Layer (n)	η_n	b_n	ρ_n	Q_n	$1 - Q_n$	Turns
1	0.00000	0.57735	1.61185	0.50000	0.50000	2
2	1.61185	0.34289	2.89628	0.25685	0.74315	3
3	2.89628	0.26485	4.03580	0.20074	0.79926	4
4	4.03580	0.22312	5.08627	0.17027	0.82973	5
5	5.08627	0.19626	6.07396	0.15042	0.84958	6
6	6.07396	0.17716	7.01390	0.13617	0.86383	7
7	7.01390	0.16270	7.91568	0.12531	0.87469	7
8	7.91568	0.15126	8.78592	0.11668	0.88332	8
9	8.78592	0.14193	9.62943	0.10962	0.89038	9
10	9.62943	0.13413	10.44985	0.10369	0.89631	9
11	10.44985	0.12748	11.25001	0.09863	0.90137	10
12	11.25001	0.12173	12.03217	0.09424	0.90576	10
13	12.03217	0.11668	12.79818	0.09038	0.90962	11
14	12.79818	0.11222	13.54958	0.08697	0.91303	11
15	13.54958	0.10823	14.28766	0.08391	0.91609	11

The sequence of layers and corresponding parameters given in Table I are those one would obtain for the ideal case in which the septum thickness is zero and the beam ellipse parameters are programed so that equations (28–29) and (33–34) are satisfied. In practice it can be difficult to program the incoming beam ellipse parameters as the injection bump collapses, and a real septum must, of course, have a finite thickness. For the case of injection into the AGS Booster, the beam ellipse parameters α, β, x_0, and x'_0 are fixed, and we will find that $b = 0.3$ and $\tau = 0.3$. Table II lists the values of η_n and ρ_n obtained from (30–31) and (35) in this case. (The columns labeled 'Q' and 'Turns' in tables I and II will be discussed in the following section.)

Table II: Phase Space Layers with $\tau = 0.3$, $b = 0.3$						
Layer (n)	η_n	b_n	ρ_n	Q	$1-Q$	Turns
1	0.30000	0.3	2.03053	0.2	0.8	1
2	2.33053	0.3	3.52661	0.2	0.8	1
3	3.82661	0.3	4.93564	0.2	0.8	5
4	5.23564	0.3	6.33109	0.2	0.8	5
5	6.63109	0.3	7.72653	0.2	0.8	5
6	8.02653	0.3	9.12198	0.2	0.8	5
7	9.42198	0.3	10.51743	0.2	0.8	5
8	10.81743	0.3	11.91287	0.2	0.8	5
9	12.21287	0.3	13.30832	0.2	0.8	5
10	13.60832	0.3	14.70376	0.2	0.8	5

2.3.2 The Horizontal Tune

The number of turns that can be injected into a layer before loosing beam on the septum depends on the horizontal tune. In Appendix II it is shown that for a given tune, Q, the number of turns that can be injected into a layer with no loss on the septum is the smallest integer, $m \geq 1$, for which

$$C(\eta + \sqrt{b}) + \sqrt{C^2 b + S^2/b} > \eta - \tau, \tag{38}$$

where

$$C = \cos(2\pi m Q), \quad S = \sin(2\pi m Q). \tag{39}$$

The number of turns obtained in this way for the case in which $Q = 0.2$, $b = 0.3$, and $\tau = 0.3$ is listed in Table II for each phase space layer. (We note that if (38) is satisfied for tune, Q, then it is also satisfied for tune $1 - Q$. The number of turns listed in Table II is therefore the same for $Q = 0.2$ and $1 - Q = 0.8$.)

The tunes, $Q_n \leq 0.5$ and $1 - Q_n$, for which the beam ellipse just touches the inner side of the septum after one turn around the machine, satisfy the equation

$$C(\eta_n + \sqrt{b_n}) + \sqrt{C^2 b_n + S^2/b_n} = \eta_n - \tau, \tag{40}$$

where

$$C = \cos 2\pi Q_n, \quad S = \sin 2\pi Q_n. \tag{41}$$

These are the tunes for which the largest number of turns can be injected into the nth layer without loss on the septum. The number of turns injected in this case is the largest integer in $1/Q_n$. The tunes obtained from (40–41), and the corresponding number of turns are listed in Table I for each phase space layer. Here we see that Q_n and $1 - Q_n$ move away from 0.5 as the closed orbit is moved away from the septum. In practice it is not always possible to program the horizontal tune to achieve the desired Q_n as the injection bump collapses. In this case one generally keeps the tune fixed during the multiturn injection.

So far we have considered only conditions for which no beam is lost during injection. If we allow some loss on the septum as beam is injected into a given layer, then provided one has an ample supply of beam, it is possible, with an appropriate choice of the tune, Q, to inject more beam into the layer than one could with no loss on the septum. Beam will be lost on the septum on the mth turn after entering the machine if m is the smallest integer (greater than or equal to one) for which equation (38) is satisfied. The amount of beam lost on the septum is conveniently expressed in terms of the parameter

$$d = \frac{\eta - \tau - C(\eta + \sqrt{b})}{\sqrt{C^2 b + S^2/b}}, \tag{42}$$

where C and S are given by (39). If $d \leq -1$ all beam is lost on the mth turn; if $d \geq 1$ no beam is lost. In Appendix III it is shown that if

$$-1 < d < 1, \tag{43}$$

the fraction, f, of beam lost on the septum on the mth turn is

$$f = (\arccos|d| - |d|\sqrt{1 - d^2})/\pi, \tag{44}$$

for $0 \leq d < 1$, and

$$f = 1 - (\arccos|d| - |d|\sqrt{1 - d^2})/\pi, \tag{45}$$

for $-1 < d \leq 0$. The fraction of the beam remaining in the beam ellipse after the mth turn is then $1 - f$.

Let us suppose that $-1 < d < 1$ and that after loosing beam on the mth turn the beam ellipse goes another n turns around the machine before it again looses beam on the septum. If the closed orbit is then moved away from the septum, so that no more beam loss can occur, a total of $m + n$ turns will be injected into the layer, and $m + (1 - f)n$ of these will survive. The injection efficiency for the layer is then

$$e = \frac{m + (1 - f)n}{m + n}. \tag{46}$$

Table III lists the values of m, n, and $m + n - fn$ obtained for the case in which $\tau = 0.3$, $b = 0.3$, and $Q = 0.1$.

Table III: Phase Space Layers with $\tau = 0.3$, $b = 0.3$							
Layer (k)	η_k	b_k	ρ_k	Q	m	n	$m + n - fn$
1	0.30000	0.3	2.03053	0.1	1	5	1.736
2	2.33053	0.3	3.52661	0.1	1	7	3.369
3	3.82661	0.3	4.93564	0.1	1	8	4.946
4	5.23564	0.3	6.33109	0.1	1	8	6.117
5	6.63109	0.3	7.72653	0.1	1	8	7.215
6	8.02653	0.3	9.12198	0.1	1	8	8.175
7	9.42198	0.3	10.51743	0.1	1	8	8.872
8	10.81743	0.3	11.91287	0.1	10	0	10.000
9	12.21287	0.3	13.30832	0.1	10	0	10.000
10	13.60832	0.3	14.70376	0.1	10	0	10.000

3 Injection into the AGS Booster

3.1 The Electrostatic Inflector

The electrostatic inflector, described in Ref. 3, consists of a cathode and thin septum which are separated by 17 mm. The septum is approximately 1 mm thick and its inner side is 47.5 mm from the unperturbed booster orbit. (This is 2.5 mm outside the limiting aperture at this point.) The parameter, x_s, introduced in the section 2 is therefore 48.5 mm. Beam entering the inflector from the HTB line follows a trjectory which is nominally a circular arc of approximately 15 degrees centered halfway between the cathode and septum. This brings the beam near the outside edge of the booster acceptance region at the inflector exit where the position and angle of the beam with respect to the unperturbed booster orbit are 57.0 mm and 9.654 milliradians. The electrostatic field required between the cathode and septum is approximately 35 kV/cm.

Although the inflector was designed so that beam would follow the nominal trajectory halfway between the cathode and septum, we have found in the previous sections that the beam should be as close to the septum as possible and should have position and angle, x_0 and x'_0, given by (29). In order to obtain the desired position and angle at the inflector exit, two trim dipoles, 29TDH2 and 29TDH3, have been provided in the

HTB line. They are located respectively 13.44 and 1.60 meters upstream of the inflector entrance and have integrated strengths of 6.10×10^{-4} and 1.10×10^{-3} T-m/A. The currents in these dipoles are not programable and so x_0 and x_0' must remain constant during injection.

3.2 The Injection Bump

The four ferrite dipoles used to produce the injection bump in the AGS Booster are located at C1, C3, C7, and D1. Each dipole has an integrated strength of 1.33×10^{-5} T-m/A (as reported in Ref. 4) and is powered by a programable monopolar power supply. Table IV lists the machine lattice parameters (Ref. 5) at the locations of the dipoles and at the exit of the inflector for the case in which the horizontal tune is 4.8.

Table IV: Lattice Parameters			
Location	ψ	β(m)	α
s_1 (C1)	0.054	4.889	0.878
s_2 (C3)	1.723	4.169	−0.726
s_I (Inflector Exit)	2.143	10.96	−1.736
s_3 (C7)	3.816	4.819	0.845
s_4 (D1)	5.087	4.889	0.878

Using these values in equations (9–12) one can find the kick angles, ϕ_j, required for various values of x_c and x_c'. Rewriting the second of Eqs. (29) as

$$x_c' + \alpha_I x_c/\beta_I = x_0' + \alpha_I x_0/\beta_I, \tag{47}$$

we see that since x_0 and x_0' are constant, $x_c' + \alpha_I x_c/\beta_I$ must also be constant. Since the power supplies are monopolar one must choose x_c and x_c' so that the currents required in the dipoles do not pass through zero as the injection bump collapses. This will be the case if, for each x_c, we choose

$$x_c' + \alpha_I x_c/\beta_I = 0, \tag{48}$$

where $x_c \geq 0$ and the values of α_I and β_I are those listed in Table IV at the inflector exit. Table V lists the values of ϕ_j required for various values of x_c and x_c' which satisfy this condition. (The subscripts 1–4 refer to C1, C3, C7, and D1 respectively.) The values of x_c listed in the table are those obtained from the first of Eqs. (36) with the values of η_n given in Table II, $x_s = 48.5$ mm, $\beta_I = 10.96$ meters, and $\epsilon = 1.0 \times 10^{-6}$ meter radians.

Table V: Kick Angles (mrad) for Various x_c and x'_c						
η_n	x_c (mm)	x'_c (mrad)	ϕ_1	ϕ_2	ϕ_3	ϕ_4
0.30000	47.5	7.52	5.96	3.49	6.71	−0.70
2.33053	40.8	6.46	5.12	3.00	5.76	−0.60
3.82661	35.8	5.67	4.49	2.64	5.06	−0.52
5.23564	31.2	4.94	3.91	2.29	4.41	−0.46
6.63109	26.5	4.20	3.32	1.95	3.74	−0.38
8.02653	21.9	3.47	2.74	1.61	3.09	−0.32
9.42198	17.3	2.74	2.17	1.27	2.44	−0.25
10.81743	12.7	2.01	1.59	0.93	1.79	−0.19
12.21287	8.1	1.28	1.02	0.59	1.14	−0.12
13.60832	3.4	0.54	0.43	0.25	0.48	−0.05

3.3 Survival of Beam Injected into the Booster

The limiting aperture, A_L, in the booster is 50.8 mm (2.0 inches) at the horizontal beta max of 13.9 meters, and the incoming beam ellipse nominally has an emittance of 1π mm milliradians ($\epsilon = 1.0 \times 10^{-6}$ meter radians). Using these numbers in (37) we find that ρ_n must be less than 13.6. For the ideal case in which the septum thickness is zero and the beam ellipse parameters and tune are programed so that equations (28–29), (33–34), and (40–41) are satisfied, Table I then shows that 14 phase space layers are possible and a total of 102 turns can be injected without loss. For the case of injection into the booster, the parameter, b, and the tune, Q, are fixed, and since the septum is actually 1 mm thick and $\beta_I = 10.96$ meters, the parameter τ is 0.3 for a 1 π mm mrad beam ellipse. Figure 3 is a plot of the number of phase space layers possible versus the value of b for the case in which $\tau = 0.3$ and the maximum ρ_n is 13.6. Here we see that one obtains the most layers when $0.1 < b \leq 0.3$. In order to keep the beam envelope from becoming too large in the last quadrupole doublet of the HTB line, we choose $b = 0.3$. Table II lists the layer parameters for this case, and Table V lists the corresponding positions and angles of the bumped closed orbit at the inflector exit. Using $b = 0.3$, $\alpha_I = -1.736$, and $\beta_I = 10.96$ meters in (28) and (32) we find that the beam ellipse parameters at the exit of the inflector must be $\alpha = -0.521$ and $\beta = 3.29$ meters.

Figure 4 is a plot of the total number of turns which survive injection with no loss on the septum versus the fixed value of the horizontal tune for the case in which $b = 0.3$, $\tau = 0.3$, and the maximum ρ_n is 13.6. Figure 5 shows the same for the case in which $\tau = 0.6$. Since the plots are symmetric about the tune $Q = 0.5$—i.e. the number of turns indicated for tune $Q \leq 0.5$ is equal to the number of turns for tune $1 - Q$—only the number of turns for tunes between 0 and 0.5 are shown. Both figures show some rather narrow spikes and a number of flat regions where the number of turns is independent of the tune. Since the tune may drift during injection, it is desirable to choose a tune centered on one of the flat regions. For the nominal horizontal tune of 4.8 in the Booster, the relevant flat region is the one centered on $Q = 0.2$. Here one finds that 37 turns can be injected without loss on the septum. Table II lists the number of turns injected into each layer in this case. It may also be desirable to inject heavy ions into the booster at a tune which is less than 4.5. If we choose a tune of 4.4, then the relevant flat region is the one centered on 0.4 in Figure 4. Here we see that 38 turns can be injected without loss on the septum. Comparing figures 4 and 5 we see that increasing the septum thickness substantially reduces the number of turns that can be injected without loss on the septum.

Figure 6 is a plot of the total number of turns which survive injection versus the value of the fixed tune for the case in which beam is lost on the septum (as described in section 2.3.2), $b = 0.3$, $\tau = 0.3$, and the maximum ρ_n is 13.6. Here we see that the maximum survival occurs at a tune of approximately 0.1, which corresponds to a horizontal tune of 4.9 in the Booster. Figure 7 shows the total number of turns injected versus the fixed tune under the same conditions, and Figure 8 is a plot of the injection efficiency versus tune. Table III lists the number of turns injected and the number surviving for each layer when $Q = 0.1$. A total of 79 turns are injected in this case, and of these 60.4 survive. The total injection efficiency is therefore $60.4/79 = 0.76$.

4 Appendix I

Here we wish to find the conditions under which the ellipse defined by equation (24) is the smallest such ellipse which contains the beam ellipse defined by equations (19–20). To this end we define new coordinates

$$\mathbf{Z} = \mathbf{N}(\mathbf{X} - \mathbf{X}_c) = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{Z}_0 = \mathbf{N}(\mathbf{X}_0 - \mathbf{X}_c) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{49}$$

where the transformation

$$\mathbf{N} = \begin{pmatrix} 1/\sqrt{\beta_I} & 0 \\ \alpha_I/\sqrt{\beta_I} & \sqrt{\beta_I} \end{pmatrix} \tag{50}$$

is such that the ellipse defined by equation (24) becomes a circle of radius $\sqrt{\epsilon_I}$. In terms of the new coordinates the two ellipses become

$$(\mathbf{Z}^\dagger - \mathbf{Z}_0^\dagger)\mathbf{F}^{-1}(\mathbf{Z} - \mathbf{Z}_0) = \epsilon, \quad \mathbf{Z}^\dagger\mathbf{Z} = \epsilon_I, \tag{51}$$

where

$$\mathbf{F} = \mathbf{N}\mathbf{E}\mathbf{N}^\dagger = \begin{pmatrix} b & -a \\ -a & g \end{pmatrix}, \tag{52}$$

and

$$a = \alpha - \alpha_I\beta/\beta_I, \quad b = \beta/\beta_I, \quad g = (1 + a^2)/b. \tag{53}$$

These ellipses are shown in Figure A1. Using (49–50) and (52–53) in (51) we find

$$g(u - u_0)^2 + 2a(u - u_0)(v - v_0) + b(v - v_0)^2 = \epsilon, \tag{54}$$

and

$$u^2 + v^2 = \epsilon_I, \tag{55}$$

where

$$u = (x - x_c)/\sqrt{\beta_I}, \quad v = \alpha_I(x - x_c)/\sqrt{\beta_I} + \sqrt{\beta_I}(x' - x_c'), \tag{56}$$

$$u_0 = (x_0 - x_c)/\sqrt{\beta_I}, \quad v_0 = \alpha_I(x_0 - x_c)/\sqrt{\beta_I} + \sqrt{\beta_I}(x_0' - x_c'). \tag{57}$$

In terms of the new coordinates, we wish to find the smallest circle (55) which contains the transformed beam ellipse (54). Since this ellipse occupies the region for which $u > u_s$, where

$$u_s = (x_s - x_c)/\sqrt{\beta_I}, \tag{58}$$

it is clear from Figure A1 that in order to make ϵ_I as small as possible, the ellipse should be upright—i.e.

$$a = 0, \tag{59}$$

and should be centered such that

$$u_0 = u_s + \sqrt{\epsilon b}, \quad v_0 = 0. \tag{60}$$

In terms of the original coordinates and parameters these conditions become

$$\alpha = \alpha_I \beta / \beta_I, \tag{61}$$

and

$$x_0 = x_s + \sqrt{\epsilon \beta}, \quad x'_0 - x'_c = -\alpha_I (x_0 - x_c)/\beta_I. \tag{62}$$

Using (59–60) in (54), the equation for the beam ellipse becomes

$$b^2 v^2 = -u^2 + 2u(u_s + \sqrt{\epsilon b}) - u_s^2 - 2u_s\sqrt{\epsilon b}, \tag{63}$$

where

$$u_s \le u \le u_s + 2\sqrt{\epsilon b}. \tag{64}$$

Now, for each point (u, v) on the beam ellipse (63) let us calculate $\epsilon_I = u^2 + v^2$. Introducing dimensionless parameters

$$\rho^2 = \epsilon_I/\epsilon, \quad \eta = u_s/\sqrt{\epsilon}, \quad \mu = u/\sqrt{\epsilon} \tag{65}$$

we find

$$b^2 \rho^2 = (b^2 - 1)\mu^2 + 2\mu(\eta + \sqrt{b}) - \eta^2 - 2\eta\sqrt{b}, \tag{66}$$

where

$$\eta \le \mu \le \eta + 2\sqrt{b}. \tag{67}$$

The largest value of $\rho\sqrt{\epsilon}$ obtained as μ varies in the interval (67) is then the radius of the smallest circle (55) which contains the transformed beam ellipse (54). If the maximum ρ occurs for some $\mu < \eta + 2\sqrt{b}$, then the derivative of ρ^2 with respect to μ will be zero at this point. Setting this derivative equal to zero one finds

$$\mu = \frac{\eta + \sqrt{b}}{1 - b^2}. \tag{68}$$

This is less than $\eta + 2\sqrt{b}$ provided $b^{3/2}(\eta + 2\sqrt{b}) < 1$. Using (68) in (66) one finds that the maximum ρ^2 is then

$$\rho^2 = \frac{1 + b\eta^2 + 2b^{3/2}\eta}{b(1 - b^2)}. \tag{69}$$

If $b^{3/2}(\eta + 2\sqrt{b}) \geq 1$, then the function $\rho^2(\mu)$ given by (66) is monotonically increasing in the interval (67), and the maximum ρ^2 therefore occurs when $\mu = \eta + 2\sqrt{b}$. In this case the maximum ρ is

$$\rho = \eta + 2\sqrt{b}. \tag{70}$$

For fixed values of b and η, these equations give the radius of the smallest circle (55) which contains the transformed beam ellipse (54). If we allow b to vary and examine the zeros of the derivative of equation (69) with respect to b, we find that ρ^2 reaches a minimum of

$$\rho^2 = \frac{1+3b^2}{4b^3} \tag{71}$$

when

$$1 - 3b^2 = 2\eta b^{3/2}. \tag{72}$$

5 Appendix II

In terms of the new coordinates (49), the transfer matrix for m turns around the machine is

$$\mathbf{R} = \mathbf{N}\mathbf{M}_I^m\mathbf{N}^{-1} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}, \tag{73}$$

where

$$C = \cos 2\pi m Q, \quad S = \sin 2\pi m Q, \tag{74}$$

$\mathbf{M}_I$ is given by (18), and Q is the horizontal tune. After m turns, the beam ellipse given by the first of equations (51) becomes

$$(\mathbf{Z}_m^\dagger - \mathbf{Z}_{0m}^\dagger)\mathbf{H}^{-1}(\mathbf{Z}_m - \mathbf{Z}_{0m}) = \epsilon, \tag{75}$$

where

$$\mathbf{Z}_m = \mathbf{R}\mathbf{Z} = \begin{pmatrix} u_m \\ v_m \end{pmatrix}, \quad \mathbf{Z}_{0m} = \mathbf{R}\mathbf{Z}_0 = \begin{pmatrix} u_{0m} \\ v_{0m} \end{pmatrix}, \tag{76}$$

and

$$\mathbf{H} = \mathbf{R}\mathbf{F}\mathbf{R}^\dagger = \begin{pmatrix} B & -A \\ -A & G \end{pmatrix}. \tag{77}$$

Using (49), (52), (59–60), and (73) in (76) and (77) we find

$$u_{0m} = C(u_s + \sqrt{\epsilon b}), \quad v_{0m} = -S(u_s + \sqrt{\epsilon b}), \tag{78}$$

and

$$A = SC(b - 1/b), \quad B = C^2 b + S^2/b, \quad G = (1 + A^2)/B \tag{79}$$

Now if m is the smallest integer (greater than or equal to 1) for which

$$u_{0m} + \sqrt{\epsilon B} > u_s - t/\sqrt{\beta_I}, \tag{80}$$

then beam will not be lost on the septum until the mth turn around the machine and one can inject m turns before having to move the closed orbit. In terms of the dimensionless parameters (65), equation (80) becomes

$$C(\eta + \sqrt{b}) + \sqrt{C^2 b + S^2/b} > \eta - \tau, \tag{81}$$

where $\tau = t/\sqrt{\epsilon\beta_I}$.

6 Appendix III

Consider the ellipse

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon, \tag{82}$$

and suppose that $x \geq s$, where $0 \leq s \leq \sqrt{\epsilon\beta}$. Then the fraction of the ellipse area for which $x \geq s$ is given by

$$f = (\arccos|d| - |d|\sqrt{1 - d^2})/\pi, \tag{83}$$

where $d = s/\sqrt{\epsilon\beta}$. This result is obtained by transforming x and x' to coordinates for which the ellipse is a circle of radius $\sqrt{\epsilon}$ and then applying the formulae given in Ref. 6. For the case of the beam ellipse, the fraction of the beam which is lost on the septum on the mth turn is then given by (83) for $0 < d < 1$ and by $1 - f$ for $-1 < d < 0$, where

$$d = (u_s - t/\sqrt{\beta_I} - u_{0m})/\sqrt{\epsilon B}. \tag{84}$$

7 References

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3. C. J. Gardner, 'Booster Inflector Specifications', Booster Tech. Note No. 159, February 28, 1990

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6. W. H. Beyer, CRC Standard Mathematical Tables, CRC Press, 1979, p. 143.

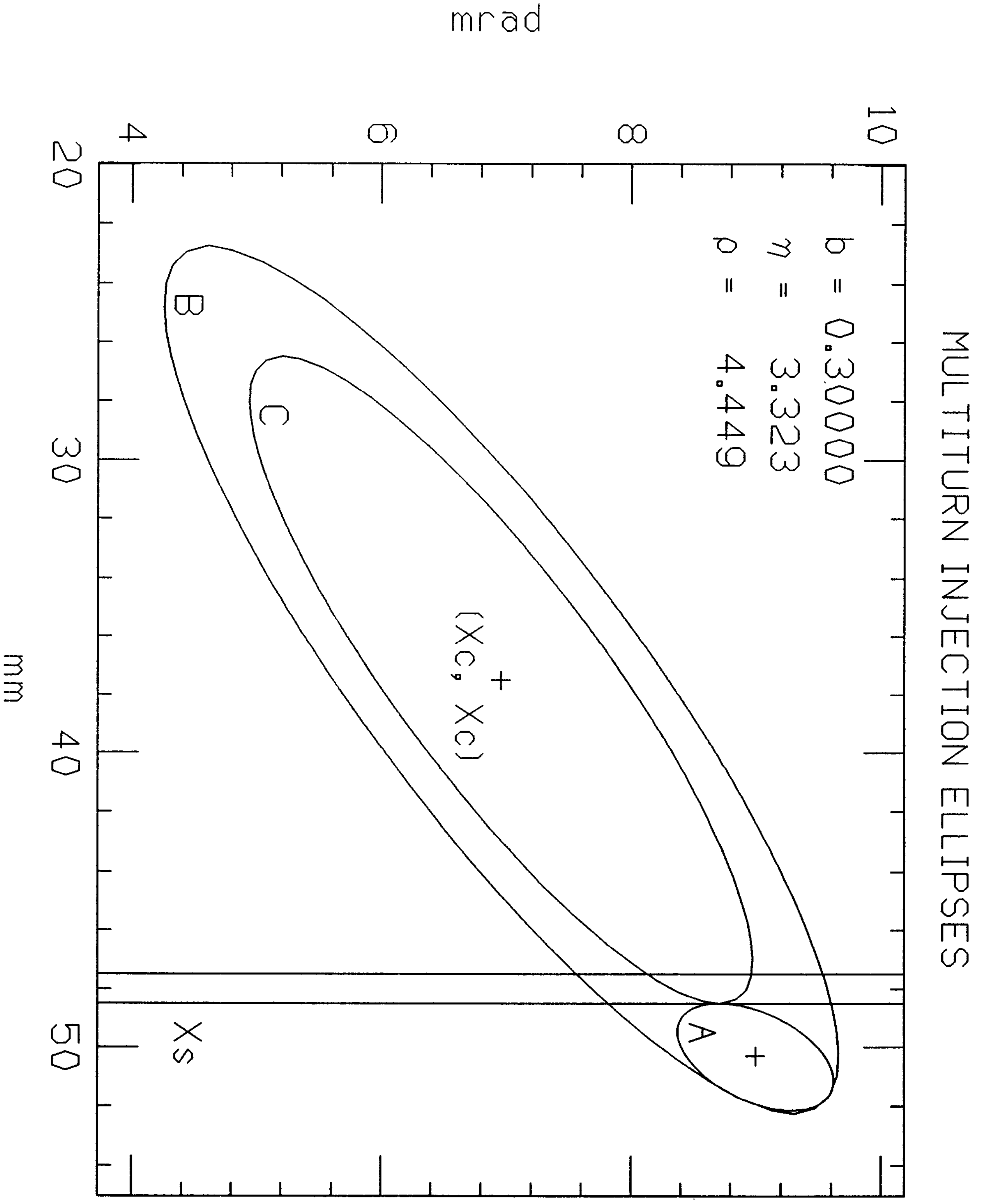

MULTITURN INJECTION ELLIPSES
b = 0.30000
η = 3.323
ρ = 4.449
mrad
mm
10
8
6
4
20
30
40
50
A
B
C
(Xc, Xc)
Xs

Fig. 1

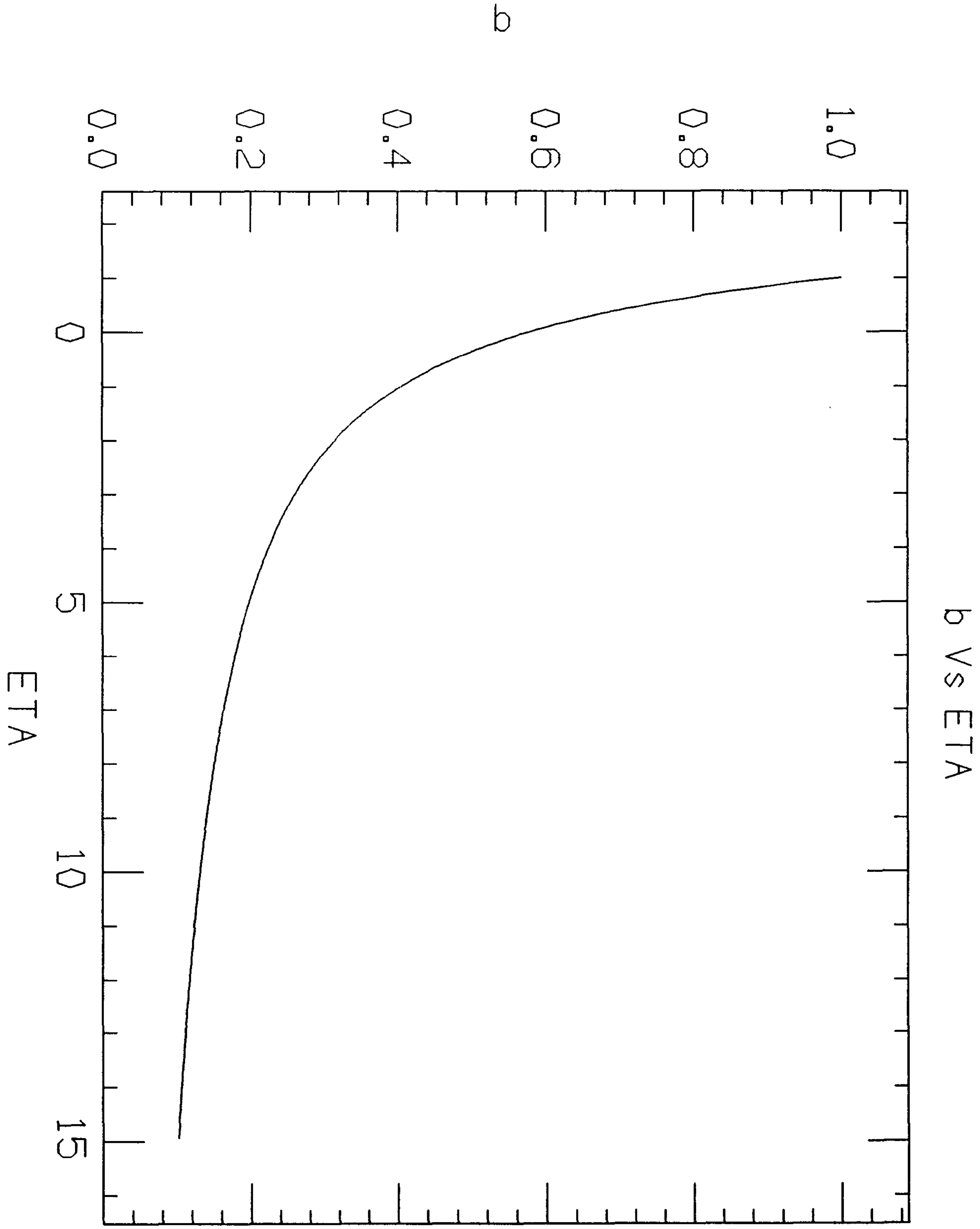


Fig. 2

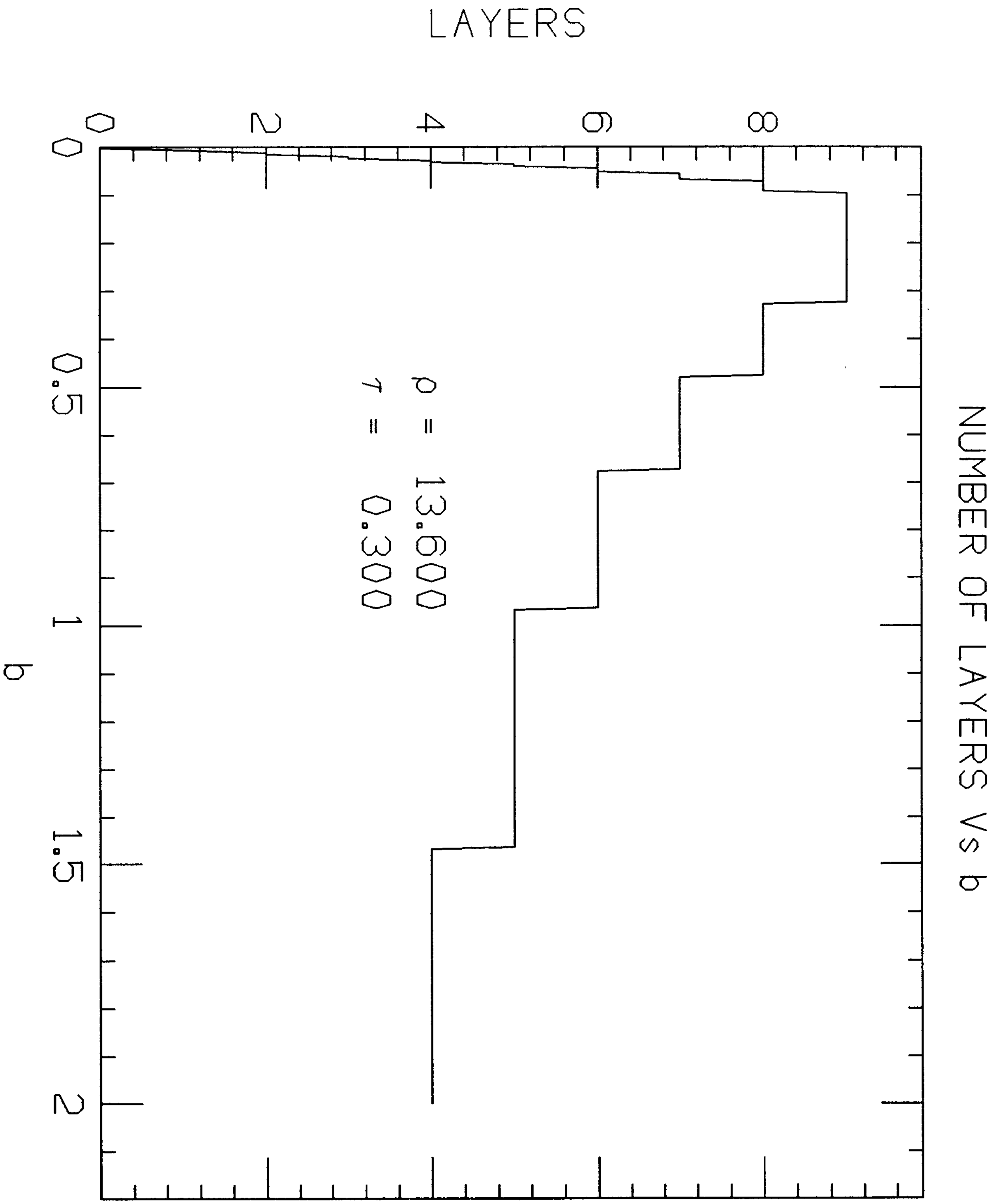
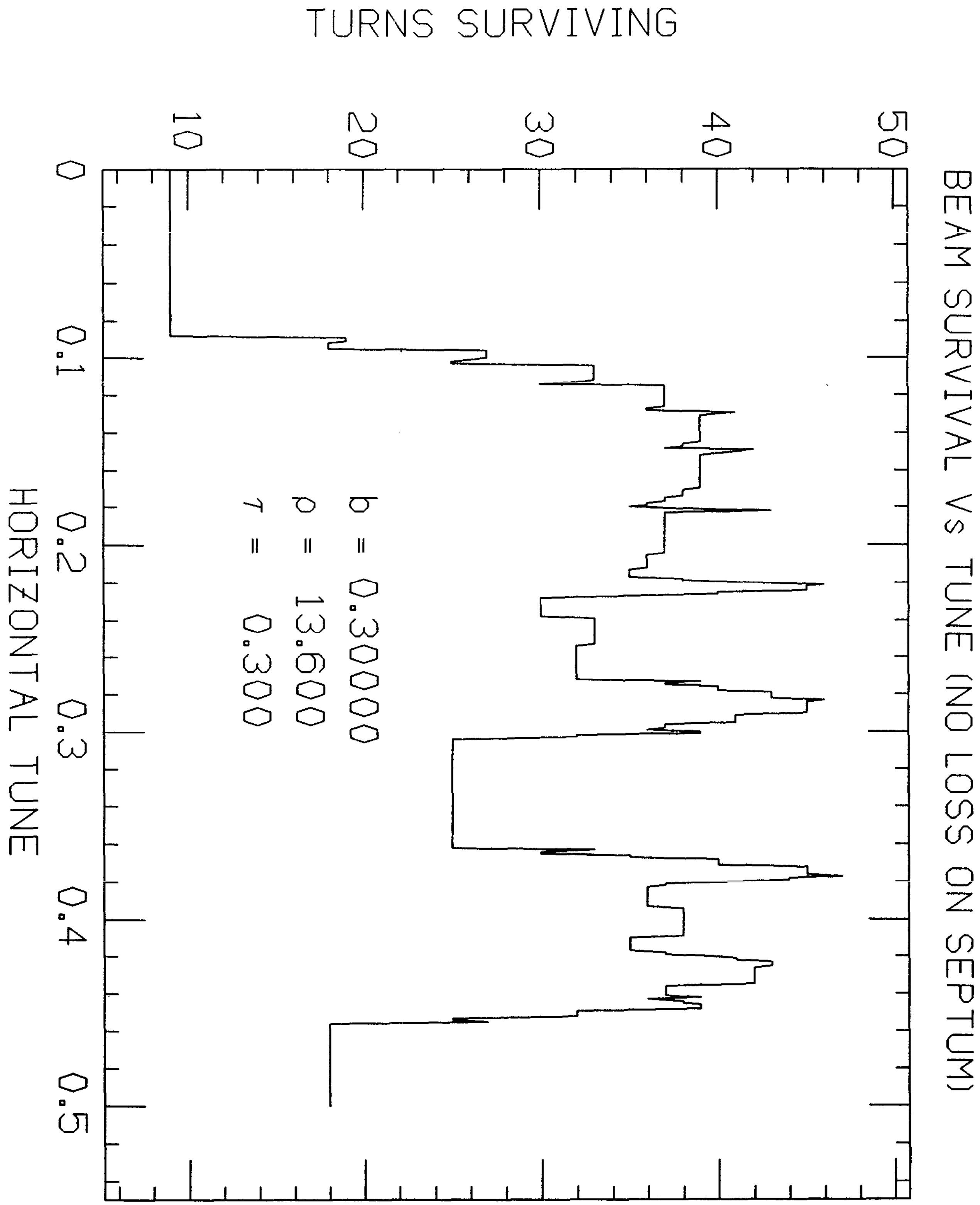
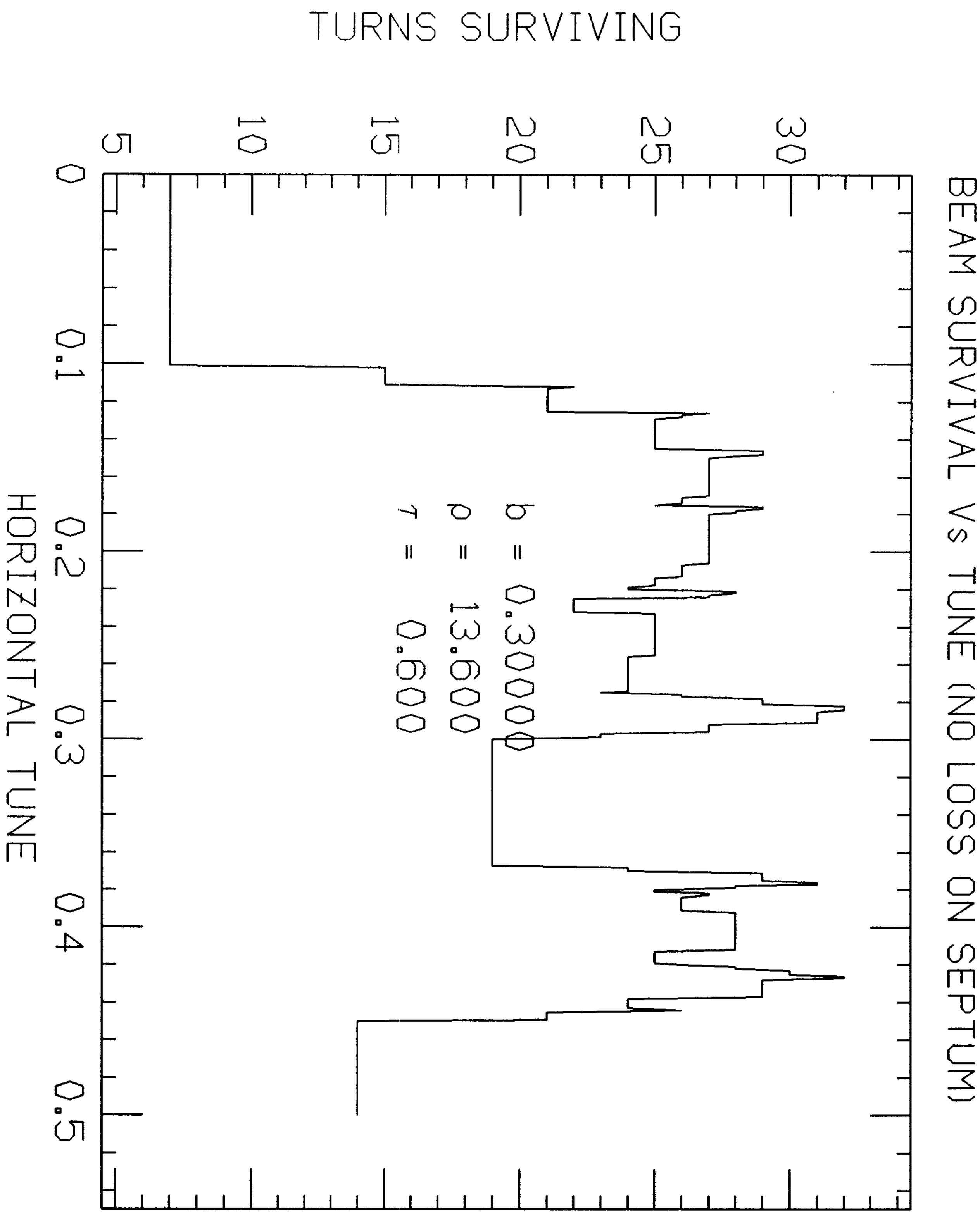


Fig. 3



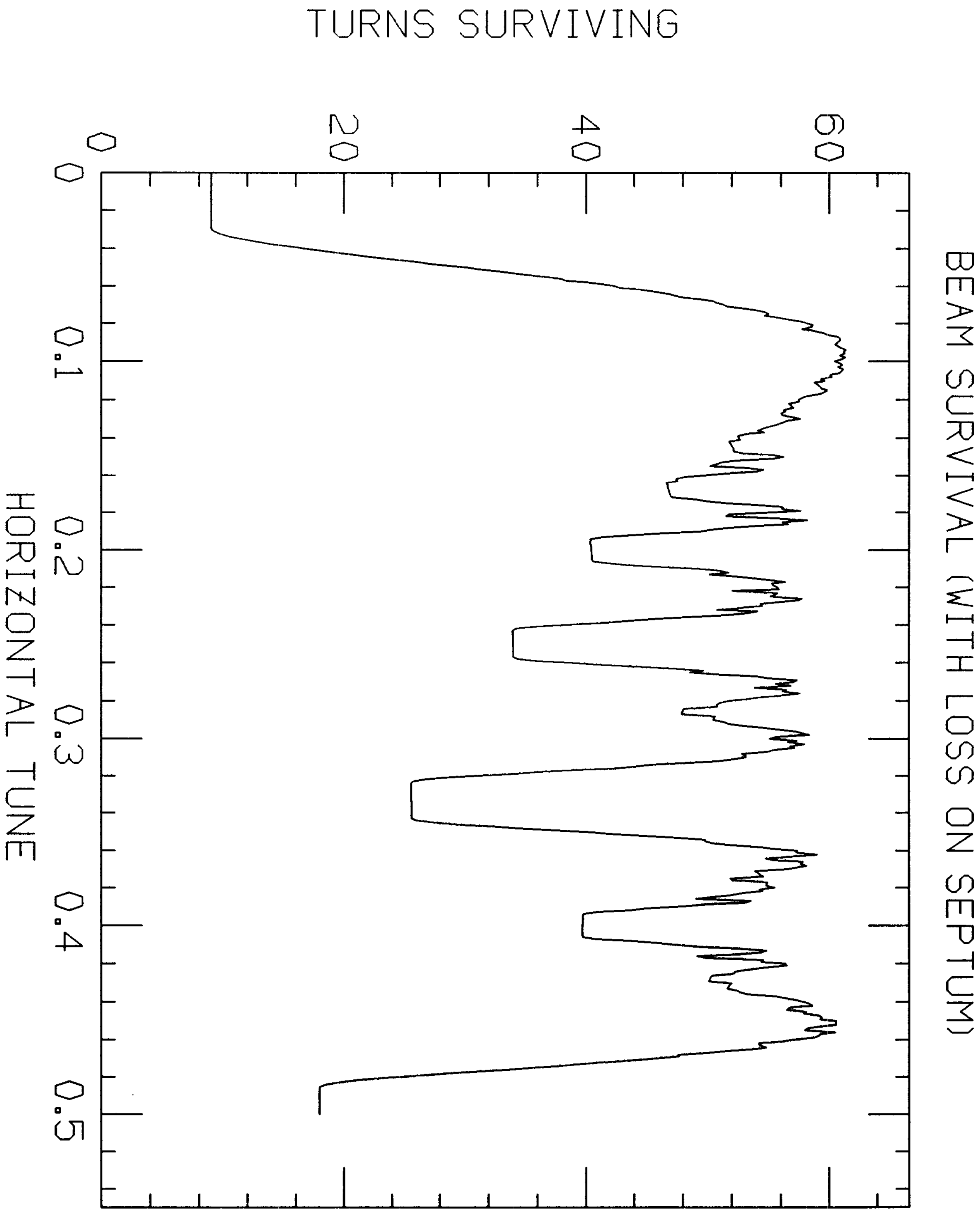
BEAM SURVIVAL Vs TUNE (NO LOSS ON SEPTUM)
TURNS SURVIVING
HORIZONTAL TUNE
b = 0.30000
ρ = 13.600
τ = 0.300

Fig. 4



BEAM SURVIVAL Vs TUNE (NO LOSS ON SEPTUM)
TURNS SURVIVING
HORIZONTAL TUNE
b = 0.3000
ρ = 13.600
τ = 0.600
5
10
15
20
25
30
0
0.1
0.2
0.3
0.4
0.5

Fig. 5



BEAM SURVIVAL (WITH LOSS ON SEPTUM)
TURNS SURVIVING
0
20
40
60
0
0.1
0.2
0.3
0.4
0.5
HORIZONTAL TUNE

Fig. 6

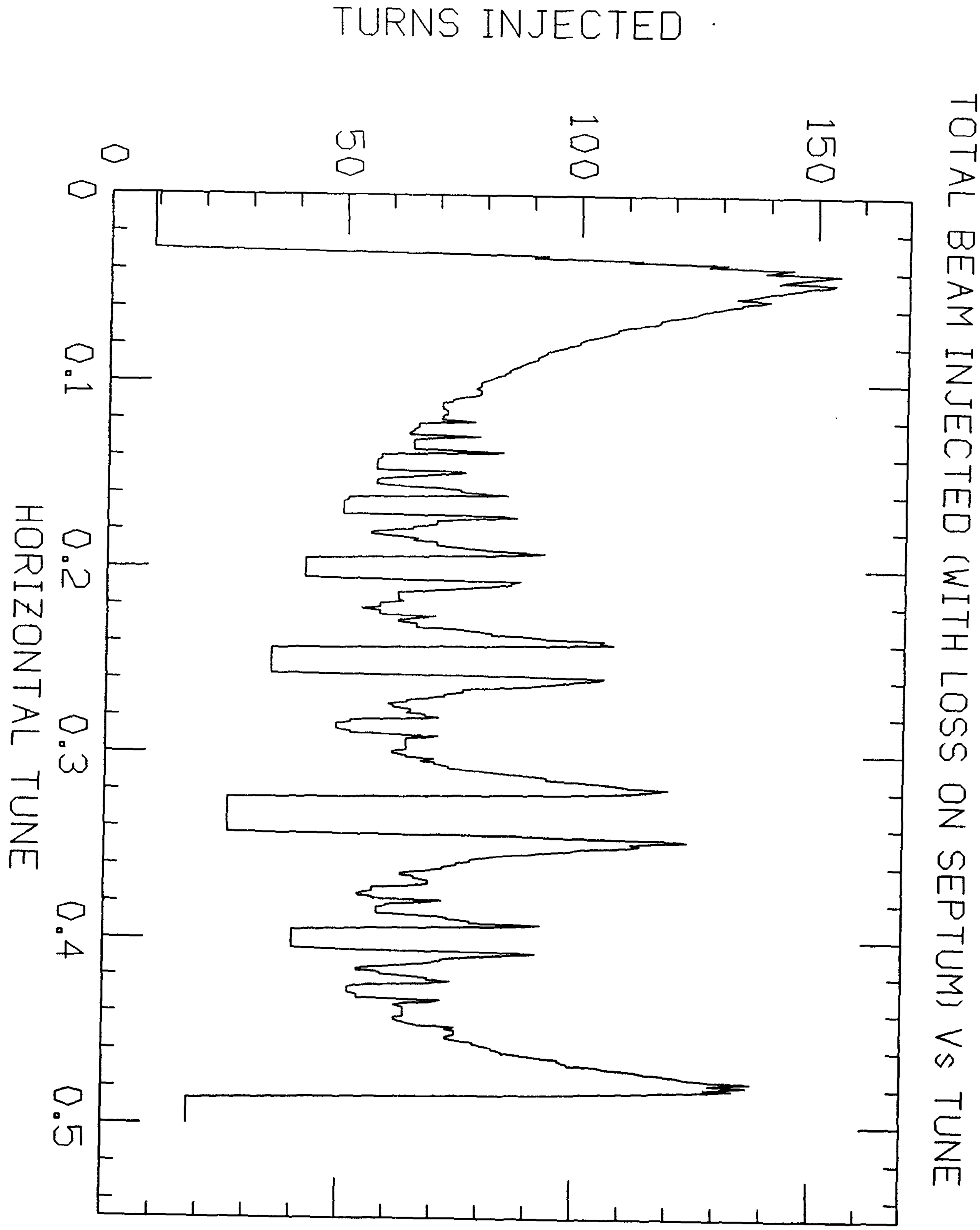
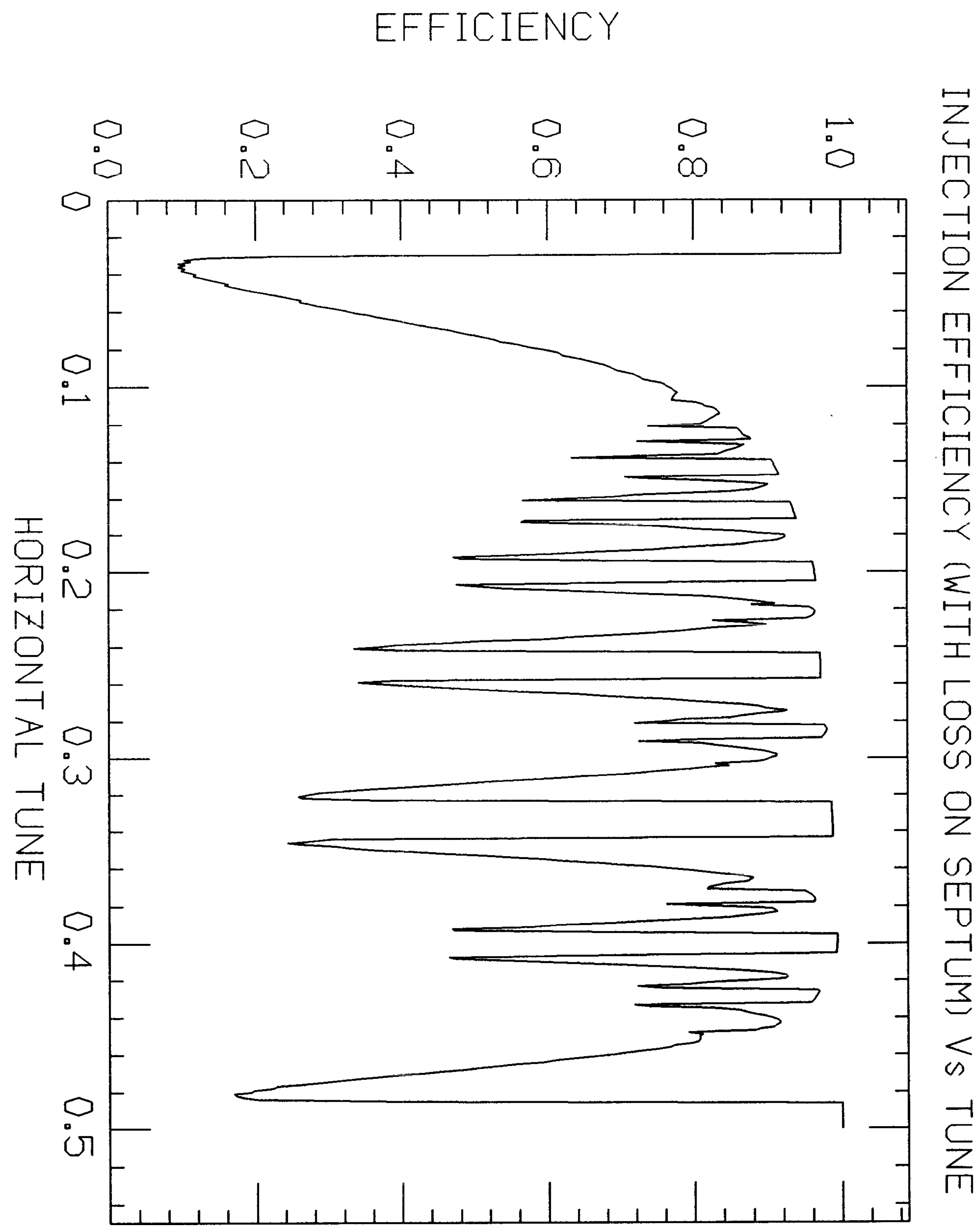

TOTAL BEAM INJECTED (WITH LOSS ON SEPTUM) Vs TUNE
TURNS INJECTED
0
50
100
150
HORIZONTAL TUNE
0
0.1
0.2
0.3
0.4
0.5

Fig. 7



INJECTION EFFICIENCY (WITH LOSS ON SEPTUM) Vs TUNE
EFFICIENCY
HORIZONTAL TUNE
0.0
0.2
0.4
0.6
0.8
1.0
0
0.1
0.2
0.3
0.4
0.5

Fig. 8

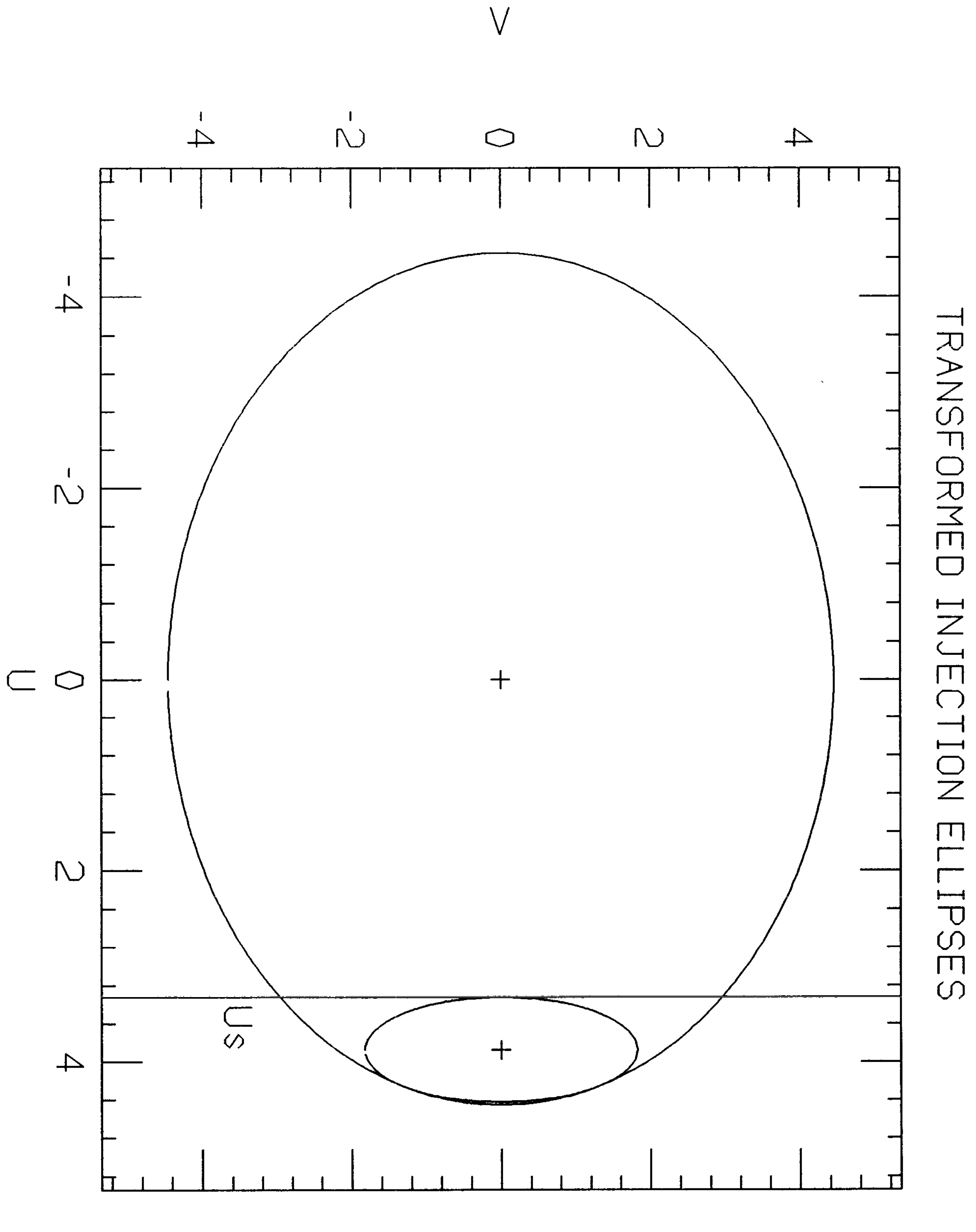


Fig. A1