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Effective Placement of Stopband Correction Elements in an AGS Lattice

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1 Abstract

Following is the development of some formulae useful in determining the effectiveness of various configurations of correction elements used to eliminate components of imperfections which can excite certain transverse resonances. Specifically, formulae for the correction of the $2Q_x = p$, $2Q_y = p$, $3Q_x = p$, $Q_x + 2Q_y = p$, $3Q_y = p$, $2Q_x + Q_y = p$, resonances are developed and applied to the AGS and to the Booster.

2 Excitation Coefficients

In the papers of G. Guignard [1,2,3] on the theory of sum and difference resonances, it is shown that if the tunes are near a particular resonance then this resonance will be excited whenever the excitation coefficient, κ, is nonzero.

2.1 Resonances $2Q_x = p$ and $2Q_y = p$

For the $2Q_x = p$ and $2Q_y = p$ resonances the excitation coefficients are respectively

$$\kappa_x = C \int_0^{2\pi r} k(s)\beta_x(s)e^{i\psi_x} ds, \quad \kappa_y = -C \int_0^{2\pi r} k(s)\beta_y(s)e^{i\psi_y} ds \qquad (1)$$

where, $2\pi r$ is the circumference of the Equilibrium Orbit (E.O.), $s = r\theta$ is the distance along the E.O. measured from a fixed reference point,

$$k(s) = \frac{e}{cP}\left(\frac{\partial B_y}{\partial x}\right) \tag{2}$$

is proportional to the quadrupole strength on the E.O. ($x = 0$, $y = 0$) and inversely proportional to the momentum, P,

$$\psi_x = 2\mu_x + (p - 2Q_x)\theta, \quad \psi_y = 2\mu_y + (p - 2Q_y)\theta, \tag{3}$$

$$\mu_x(s) = \int_0^s \frac{ds'}{\beta_x(s')}, \quad \mu_y(s) = \int_0^s \frac{ds'}{\beta_y(s')} \tag{4}$$

are the betatron phase advances,

$$C = \frac{1}{8\pi r}, \tag{5}$$

and Q_x and Q_y are the unperturbed horizontal and vertical tunes. These resonances can produce unlimited growth in the amplitudes of the betatron oscillations whenever the tunes are such that

$$-\frac{W}{2} < 2Q_{x,y} - p < \frac{W}{2} \tag{6}$$

where,

$$W = 4r\,|\kappa_{x,y}| \tag{7}$$

is the stopband width.

2.2 Resonances $3Q_x = p$ and $Q_x + 2Q_y = p$

For the $3Q_x = p$ and $Q_x + 2Q_y = p$ resonances the excitation coefficients are respectively

$$\kappa_x = C\int_0^{2\pi r} k(s)\beta_x^{3/2}e^{i\psi_x}ds, \quad \kappa_{xy} = -3C\int_0^{2\pi r} k(s)\beta_x^{1/2}\beta_y e^{i\psi_{xy}}ds, \tag{8}$$

where,

$$k(s) = \frac{e}{cP}\left(\frac{\partial^2 B_y}{\partial x^2}\right) \tag{9}$$

is proportional to the sextupole strength on the E.O.,

$$\psi_x = 3\mu_x + (p - 3Q_x)\theta, \quad \psi_{xy} = \mu_x + 2\mu_y + (p - Q_x - 2Q_y)\theta, \tag{10}$$

$$C = \frac{1}{2\pi r}\frac{1}{2\sqrt{2}}\frac{1}{6}. \tag{11}$$

The stopband widths for these resonances are

$$W_x = \frac{18}{\sqrt{2\pi}}\, r\, |\kappa_x|\, \sqrt{\epsilon_x}, \tag{12}$$

$$W_{xy} = \sqrt{\frac{2}{\pi}}\, r\, |\kappa_{xy}|\, \frac{1}{\sqrt{\epsilon_x}}(\epsilon_y + 4\epsilon_x), \tag{13}$$

where ϵ_x, ϵ_y are the initial emittances.

2.3 Resonances $3Q_y = p$ and $2Q_x + Q_y = p$

The excitation coefficients and stopband widths for the $3Q_y = p$ and $2Q_x + Q_y = p$ resonances may be obtained from equations (8), (10), and (12–13) by interchanging x and y and replacing $k(s)$ with

$$k(s) = \frac{e}{cP}\left(\frac{\partial^2 B_x}{\partial x^2}\right) \tag{14}$$

which is proportional to the skew sextupole strength on the E.O.

2.4 Comments

Careful inspection of the equations for the excitation coefficients, κ, shows that each κ is essentially proportional to the pth harmonic in the azimuthal variation of $k(s)$ around the machine. The real and imaginary parts of κ are then the *cos* and *sin* components of this harmonic. The resonances discussed in sections 2.1–2.3 are therefore excited by the pth harmonic in the azimuthal variations of the quadrupole, sextupole, and skew sextupole fields around the machine.

Gaussian units (cm, gram, second, erg, gauss, statcoulomb) are employed in the equations for $k(s)$ given in sections 2.1–2.3. Thus if the momentum, P, is expressed in eV/c then $cP/e = 3335.641$ gauss-cm per MeV, or $cP/e = 3.335641$ tesla-m per GeV.

3 Correction Schemes

Any naturally occuring fields in the machine, or fields due to imperfections, which produce nonzero values of the excitation coefficients, κ, can excite resonances resulting in beam loss. We call each κ produced by these fields an intrinsic excitation coefficient of the machine and denote it by κ_0. To cancel each κ_0, so that the resonances can not be excited, correction elements located at various positions, s_j, in the ring are excited with currents, I_j, in such a way that they produce a κ equal to $-\kappa_0$. When this is done we say that the resonances have been corrected.

3.1 Correction of resonances $2Q_x = p$ and $2Q_y = p$

Suppose there are N identical correction quadrupoles located at positions, s_j, and excited with currents I_j. If the integrated strength of each quadrupole is Q gauss/amp, then the set of quadrupoles will produce excitation coefficients

$$\kappa_x = C\left(\frac{eQ}{cP}\right)\sum_{j=1}^{N}\beta_{xj}I_j e^{i\psi_{xj}}, \quad \kappa_y = -C\left(\frac{eQ}{cP}\right)\sum_{j=1}^{N}\beta_{yj}I_j e^{i\psi_{yj}}, \tag{15}$$

in which

$$\beta_{xj} = \beta_x(s_j), \quad \beta_{yj} = \beta_y(s_j),$$
$$\psi_{xj} = 2\mu_x(s_j) + (p - 2Q_x)\theta_j, \quad \psi_{yj} = 2\mu_y(s_j) + (p - 2Q_y)\theta_j,$$

and equations (1–4) have been employed in the thin lens approximation. Generally it is necessary to correct both resonances simultaneously since some particles in the beam may be near the $2Q_x = p$ resonance while others are near the $2Q_y = p$ resonance. This is especially true near injection where the beam is spread over a large region of tune space due to space charge detuning. In general, then, the positions, s_j, of the quadrupoles must be chosen so that it is always possible to find a set of currents, I_j, which produce the values of κ_x and κ_y required to correct both resonances at the same time. Since each κ has a real and an imaginary part, we see from equations (15) that four correction elements are needed.

3.1.1 Correction of one resonance only

Before considering the general case let us consider the special case in which it is necessary to correct only one of the resonances, say $2Q_x = p$. Then only two correction elements are needed, and taking $N = 2$ the first of equations (15) becomes

$$\begin{pmatrix} CX \\ SX \end{pmatrix} = C\left(\frac{eQ}{cP}\right)\begin{pmatrix} C_1 & C_2 \\ S_1 & S_2 \end{pmatrix}\begin{pmatrix} \beta_{x1}I_1 \\ \beta_{x2}I_2 \end{pmatrix} \tag{16}$$

where CX and SX are respectively the cos (real) and sin (imaginary) parts of κ_x, $C_j = cos(\psi_{xj})$, and $S_j = sin(\psi_{xj})$. Solving for I_1 and I_2 we find

$$\begin{pmatrix} \beta_{x1}I_1 \\ \beta_{x2}I_2 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eQ}\right)\frac{1}{S_{12}}\begin{pmatrix} S_2 & -C_2 \\ -S_1 & C_1 \end{pmatrix}\begin{pmatrix} CX \\ SX \end{pmatrix} \tag{17}$$

where $S_{12} = C_1S_2 - S_1C_2 = sin(\psi_{x2} - \psi_{x1})$. Here we see that the amount of current required to produce a given κ_x is proportional to $1/S_{12}$ which becomes infinite whenever the phase difference, $\psi_{x2} - \psi_{x1}$, is an integral multiple of π. If we define

$$\phi_{xj} = \phi_x(s_j) = \mu_x(s_j)/Q_x, \quad \phi_{yj} = \phi_y(s_j) = \mu_y(s_j)/Q_y \tag{18}$$

then near the $2Q_x = p$ resonance the phase difference

$$\psi_{x2} - \psi_{x1} = p\phi_x(s_2) - p\phi_x(s_1) = p(\phi_{x2} - \phi_{x1}).$$

Thus the effectiveness of the currents, I_1 and I_2, in producing the desired corrections is proportional to $|S_{12}| = |sin(p\phi_{x2} - p\phi_{x1})|$ and we see that one must avoid positions for which $p(\phi_{x2} - \phi_{x1})$ is an integral multiple of π. The optimum positions—i.e. those for which the least amount of current is required to produce the desired corrections—are those for which $p(\phi_{x2} - \phi_{x1})$ is an odd multiple of $\pi/2$, and β_{x1} and β_{x2} are beta maximums.

3.1.2 Correction of both resonances simultaneously

Let us now return to the general problem of correcting both resonances simultaneously. In this case four correction elements are needed, and with $N = 4$ equations (15) become

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C\left(\frac{eQ}{cP}\right)\mathrm{M}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \tag{19}$$

where,
$$\mathbf{M} = \begin{pmatrix} \beta_{x1}C_{x1} & \beta_{x2}C_{x2} & \beta_{x3}C_{x3} & \beta_{x4}C_{x4} \\ \beta_{x1}S_{x1} & \beta_{x2}S_{x2} & \beta_{x3}S_{x3} & \beta_{x4}S_{x4} \\ -\beta_{y1}C_{y1} & -\beta_{y2}C_{y2} & -\beta_{y3}C_{y3} & -\beta_{y4}C_{y4} \\ -\beta_{y1}S_{y1} & -\beta_{y2}S_{y2} & -\beta_{y3}S_{y3} & -\beta_{y4}S_{y4} \end{pmatrix}, \tag{20}$$

CX and SX are the *cos* and *sin* parts of κ_x, CY and SY are the *cos* and *sin* parts of κ_y, $C_{xj} = cos(\psi_{xj})$, $S_{xj} = sin(\psi_{xj})$, $C_{yj} = cos(\psi_{yj})$, and $S_{yj} = sin(\psi_{yj})$. The currents which produce the desired corrections are then

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eQ}\right)\mathbf{M}^{-1}\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}. \tag{21}$$

Here we see that the effectiveness of the correctors in producing the desired corrections depends on the inverse of the matrix $\mathbf{M}$ which is, in general, rather complicated. However, if one makes some assumptions about the machine lattice and the placement of the correctors, then both $\mathbf{M}$ and its inverse become much simpler. The determination of the effectiveness of the correctors then becomes rather straight forward. The conditions under which the following simplifing assumptions are valid will be discussed in section 3.4.

In the previous section we found that the $2Q_x = p$ resonance is most effectively corrected when the two correction elements are located at horizontal beta maximums. Thus, if we wish also to correct the $2Q_y = p$ resonance, two correction elements should be placed at vertical beta maximums. We therefore take positions s_1 and s_2 to be vertical beta maximums and positions s_3 and s_4 to be horizontal beta maximums. We shall also take $s_1 = 0$ and assume that

$$\phi_x(s_j) = \phi_y(s_j) = \phi(s_j) = \phi_j,$$

where $\phi_x(s_j)$ and $\phi_y(s_j)$ are the normalized betatron phase advances defined in equations (18). Then near the resonances we have

$$\psi_{xj} = \psi_{yj} = p\phi_j$$

and therefore

$$C_{xj} = C_{yj} = C_j = cos(p\phi_j), \quad S_{xj} = S_{yj} = S_j = sin(p\phi_j).$$

We shall also assume that

$$\beta_{x1} = \beta_{x2} = a, \quad \beta_{x3} = \beta_{x4} = A,$$

$$\beta_{y1} = \beta_{y2} = B, \quad \beta_{y3} = \beta_{y4} = b,$$

where (a, b) and (A, B) are respectively values of beta minima and beta maxima in the machine lattice. With these assumptions the matrix, $\mathbf{M}$, becomes

$$\mathbf{M} = \begin{pmatrix} aC_1 & aC_2 & AC_3 & AC_4 \\ aS_1 & aS_2 & AS_3 & AS_4 \\ -BC_1 & -BC_2 & -bC_3 & -bC_4 \\ -BS_1 & -BS_2 & -bS_3 & -bS_4 \end{pmatrix}. \tag{22}$$

Introducing

$$\mathbf{m} = \begin{pmatrix} C_1 & C_2 \\ S_1 & S_2 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} C_3 & C_4 \\ S_3 & S_4 \end{pmatrix}$$

we then have

$$\mathbf{M} = \begin{pmatrix} a\mathbf{m} & A\mathbf{n} \\ -B\mathbf{m} & -b\mathbf{n} \end{pmatrix} = \begin{pmatrix} a & A \\ -B & -b \end{pmatrix} \begin{pmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{n} \end{pmatrix}, \tag{23}$$

and

$$\mathbf{M}^{-1} = \frac{1}{AB - ab} \begin{pmatrix} \mathbf{m}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{n}^{-1} \end{pmatrix} \begin{pmatrix} -b & -A \\ B & a \end{pmatrix} \tag{24}$$

where

$$\mathbf{m}^{-1} = \frac{1}{S_{12}} \begin{pmatrix} S_2 & -C_2 \\ -S_1 & C_1 \end{pmatrix}, \quad \mathbf{n}^{-1} = \frac{1}{S_{34}} \begin{pmatrix} S_4 & -C_4 \\ -S_3 & C_3 \end{pmatrix},$$

$$S_{12} = C_1 S_2 - S_1 C_2 = sin(p\phi_2 - p\phi_1),$$

$$S_{34} = C_3 S_4 - S_3 C_4 = sin(p\phi_4 - p\phi_3).$$

Putting (24) into (21) we obtain

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{-g}{(AB - ab)S_{12}} \begin{pmatrix} S_2 & -C_2 \\ -S_1 & C_1 \end{pmatrix} \begin{pmatrix} bCX + ACY \\ bSX + ASY \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} I_3 \\ I_4 \end{pmatrix} = \frac{g}{(AB - ab)S_{34}} \begin{pmatrix} S_4 & -C_4 \\ -S_3 & C_3 \end{pmatrix} \begin{pmatrix} BCX + aCY \\ BSX + aSY \end{pmatrix}$$

where

$$g = \frac{1}{C}\left(\frac{cP}{eQ}\right).$$

Here we see that the effectiveness of currents I_1 and I_2 in producing the desired corrections is proportional to $|S_{12}| = |sin(p\phi_2 - p\phi_1)|$. Likewise the effectiveness of currents I_3 and I_4 is proportional to $|S_{34}| = |sin(p\phi_4 - p\phi_3)|$. Thus, corrector positions for which either $p(\phi_2 - \phi_1)$ or $p(\phi_4 - \phi_3)$ is an integral multiple of π must be avoided. The optimum positions are those for which $p(\phi_2 - \phi_1)$ and $p(\phi_4 - \phi_3)$ are odd multiples of $\pi/2$. We also see that the effectiveness of the currents is proportional to $AB - ab$, which is zero when $AB = ab$. This is consistent with our earlier assumption that two correction elements should be placed at horizontal beta maximums and two at vertical beta maximums.

3.2 Correction of resonances $3Q_x = p$ and $Q_x + 2Q_y = p$

As in section 3.1 we suppose that there are N identical correction elements—sextupoles in this case—located at positions, s_j, and excited with currents I_j. If the integrated strength of each sextupole is S gauss/cm per amp, then the set of sextupoles will produce excitation coefficients

$$\kappa_x = C\left(\frac{eS}{cP}\right)\sum_{j=1}^{N}\beta_{xj}^{3/2}I_j e^{i\psi_{xj}}, \quad \frac{1}{3}\kappa_{xy} = -C\left(\frac{eS}{cP}\right)\sum_{j=1}^{N}\beta_{xj}^{1/2}\beta_{yj}I_j e^{i\psi_{yj}} \quad (26)$$

in which

$$\beta_{xj} = \beta_x(s_j), \quad \beta_{yj} = \beta_y(s_j),$$
$$\psi_{xj} = 3\mu_x(s_j) + (p - 3Q_x)\theta_j,$$
$$\psi_{yj} = \mu_x(s_j) + 2\mu_y(s_j) + (p - Q_x - 2Q_y)\theta_j,$$

and equations (8-11) have been employed in the thin lens approximation. As with the half–integer resonances it is generally necessary to correct the $3Q_x = p$ and $Q_x + 2Q_y = p$ resonances simultaneously. Since each κ has a real and an imaginary part we see from equations (26) that four correction elements are needed. Taking $N = 4$ equations (26) become

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathrm{M}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \quad (27)$$

where

$$\mathbf{M} = \begin{pmatrix} \beta_{x1}^{3/2} C_{x1} & \beta_{x2}^{3/2} C_{x2} & \beta_{x3}^{3/2} C_{x3} & \beta_{x4}^{3/2} C_{x4} \\ \beta_{x1}^{3/2} S_{x1} & \beta_{x2}^{3/2} S_{x2} & \beta_{x3}^{3/2} S_{x3} & \beta_{x4}^{3/2} S_{x4} \\ -\beta_{x1}^{1/2} \beta_{y1} C_{y1} & -\beta_{x2}^{1/2} \beta_{y2} C_{y2} & -\beta_{x3}^{1/2} \beta_{y3} C_{y3} & -\beta_{x4}^{1/2} \beta_{y4} C_{y4} \\ -\beta_{x1}^{1/2} \beta_{y1} S_{y1} & -\beta_{x2}^{1/2} \beta_{y2} S_{y2} & -\beta_{x3}^{1/2} \beta_{y3} S_{y3} & -\beta_{x4}^{1/2} \beta_{y4} S_{y4} \end{pmatrix}, \tag{28}$$

CX and SX are the *cos* and *sin* parts of κ_x, CY and SY are the *cos* and *sin* parts of $\kappa_{xy}/3$, $C_{xj} = cos(\psi_{xj})$, $S_{xj} = sin(\psi_{xj})$, $C_{yj} = cos(\psi_{yj})$, and $S_{yj} = sin(\psi_{yj})$. The currents which produce the desired corrections are then

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \frac{1}{C} \left(\frac{cP}{eS} \right) \mathbf{M}^{-1} \begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}. \tag{29}$$

We now make some assumptions, as before, which simplify the form of $\mathbf{M}$ and make the determination of the effectiveness of a given set of correction elements straight forward. The conditions under which these assumptions are valid will be discussed in section 3.4. As in section 3.1.2 we take positions s_1 and s_2 to be vertical beta maximums and positions s_3 and s_4 to be horizontal beta maximums. We also take $s_1 = 0$ and assume that

$$\phi_x(s_j) = \phi_y(s_j) = \phi(s_j) = \phi_j,$$

where $\phi_x(s_j)$ and $\phi_y(s_j)$ are the normalized betatron phase advances defined in equations (18). Then near the resonances we have

$$\psi_{xj} = \psi_{yj} = p\phi_j$$

and therefore

$$C_{xj} = C_{yj} = C_j = cos(p\phi_j), \quad S_{xj} = S_{yj} = S_j = sin(p\phi_j).$$

We also assume that

$$\beta_{x1} = \beta_{x2} = a, \quad \beta_{x3} = \beta_{x4} = A,$$

$$\beta_{y1} = \beta_{y2} = B, \quad \beta_{y3} = \beta_{y4} = b,$$

where (a, b) and (A, B) are respectively values of beta minima and beta maxima in the machine lattice. With these assumptions the matrix, $\mathbf{M}$, becomes

$$\mathbf{M} = \begin{pmatrix} a^{3/2}C_1 & a^{3/2}C_2 & A^{3/2}C_3 & A^{3/2}C_4 \\ a^{3/2}S_1 & a^{3/2}S_2 & A^{3/2}S_3 & A^{3/2}S_4 \\ -a^{1/2}BC_1 & -a^{1/2}BC_2 & -A^{1/2}bC_3 & -A^{1/2}bC_4 \\ -a^{1/2}BS_1 & -a^{1/2}BS_2 & -A^{1/2}bS_3 & -A^{1/2}bS_4 \end{pmatrix}. \tag{30}$$

Introducing

$$\mathbf{m} = \begin{pmatrix} C_1 & C_2 \\ S_1 & S_2 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} C_3 & C_4 \\ S_3 & S_4 \end{pmatrix}$$

we then have

$$\mathbf{M} = \begin{pmatrix} a^{3/2} & A^{3/2} \\ -a^{1/2}B & -A^{1/2}b \end{pmatrix} \begin{pmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{n} \end{pmatrix}, \tag{31}$$

and

$$\mathbf{M}^{-1} = \frac{1}{a^{1/2}A^{1/2}(AB - ab)} \begin{pmatrix} \mathbf{m}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{n}^{-1} \end{pmatrix} \begin{pmatrix} -A^{1/2}b & -A^{3/2} \\ a^{1/2}B & a^{3/2} \end{pmatrix} \tag{32}$$

where

$$\mathbf{m}^{-1} = \frac{1}{S_{12}} \begin{pmatrix} S_2 & -C_2 \\ -S_1 & C_1 \end{pmatrix}, \quad \mathbf{n}^{-1} = \frac{1}{S_{34}} \begin{pmatrix} S_4 & -C_4 \\ -S_3 & C_3 \end{pmatrix},$$

$$S_{12} = C_1S_2 - S_1C_2 = sin(p\phi_2 - p\phi_1),$$

$$S_{34} = C_3S_4 - S_3C_4 = sin(p\phi_4 - p\phi_3).$$

Putting (32) into (29) we obtain

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{-g}{a^{1/2}(AB - ab)S_{12}} \begin{pmatrix} S_2 & -C_2 \\ -S_1 & C_1 \end{pmatrix} \begin{pmatrix} bCX + ACY \\ bSX + ASY \end{pmatrix}, \tag{33}$$

$$\begin{pmatrix} I_3 \\ I_4 \end{pmatrix} = \frac{g}{A^{1/2}(AB - ab)S_{34}} \begin{pmatrix} S_4 & -C_4 \\ -S_3 & C_3 \end{pmatrix} \begin{pmatrix} BCX + aCY \\ BSX + aSY \end{pmatrix},$$

where

$$g = \frac{1}{C}\left(\frac{cP}{eS}\right).$$

Here we see, as before, that the effectiveness of currents I_1 and I_2 in producing the desired corrections is proportional to $|S_{12}| = |sin(p\phi_2 - p\phi_1)|$. Likewise the effectiveness of currents I_3 and I_4 is proportional to $|S_{34}| = |sin(p\phi_4 - p\phi_3)|$. Corrector positions for which either $p(\phi_2 - \phi_1)$ or $p(\phi_4 - \phi_3)$ is an integral multiple of π must therefore be avoided. The optimum positions are those for which $p(\phi_2 - \phi_1)$ and $p(\phi_4 - \phi_3)$ are odd multiples of $\pi/2$. We also see that the effectiveness of the currents is proportional to $AB - ab$, which is zero when $AB = ab$. This is consistent with our assumption that two correction elements should be placed at horizontal beta maximums and two at vertical beta maximums.

3.3 Correction of resonances $3Q_y = p$ and $2Q_x + Q_y = p$

The formulae for the correction of the $3Q_y = p$ and $2Q_x + Q_y = p$ resonances may be obtained from the formulae of section 3.2 by interchanging x and y and replacing the sextupole strength with the skew sextupole strength.

3.4 Comments

We have seen in sections 3.1 and 3.2 that by making some assumptions about the machine lattice and the placement of correction elements, the task of determining the effectiveness of a given set of correctors becomes rather straight forward. Here we discuss the conditions under which these assumptions are valid.

Consider first the case in which the lattice is composed of N identical FODO cells, and let $\phi_x = \mu_x/Q_x$ and $\phi_y = \mu_y/Q_y$ be the normalized betatron phase advances in the x and y planes between two horizontal beta maximums, two vertical beta minimums, or between a beta minimum and a beta maximum. Then in each case $\phi_x = \phi_y$. Furthermore any two horizontal beta maximums (or minimums) in the lattice are equal, any two vertical beta maximums (or minimums) are equal, and the beta maximums in one plane occur at the same locations as the beta minimums in the opposite plane. Thus, if two correction elements are placed at horizontal beta maximums and two at vertical beta maximums, then all of the assumptions made in sections 3.1 and 3.2 are valid.

Now in general not all of the FODO cells in an AGS lattice are identical. However there are usually symmetries which imply that the assumptions made in sections 3.1 and 3.2 are valid for some set of points in the lattice. In each superperiod of the Brookhaven AGS, for example, we have

$$\beta_x(s_5 - t) = \beta_x(s_5 + t) = \beta_y(s_{15} - t) = \beta_y(s_{15} + t), \tag{34}$$
$$\beta_y(s_5 - t) = \beta_y(s_5 + t) = \beta_x(s_{15} - t) = \beta_x(s_{15} + t),$$

and

$$\phi_x(s_{15} + t) - \phi_x(s_{15} - t) = \phi_y(s_5 + t) - \phi_y(s_5 - t), \tag{35}$$
$$\phi_x(s_5 + t) - \phi_x(s_5 - t) = \phi_y(s_{15} + t) - \phi_y(s_{15} - t),$$
$$\phi_x(s_{15} - t) - \phi_x(s_5 + t) = \phi_y(s_{15} - t) - \phi_y(s_5 + t),$$

where s_5 and s_{15} are respectively the distances from the beginning of a superperiod to the middle of the number 5 and number 15 straight sections, $0 < t < s_5$, and ϕ_x and ϕ_y are the normalized betatron phase advances defined in equation (18). Adding equations (35) we have also

$$\phi_x(s_{15} + t) - \phi_x(s_5 - t) = \phi_y(s_{15} + t) - \phi_y(s_5 - t). \tag{36}$$

Thus in each superperiod we have

$$\beta_{x1} = \beta_{x9} = \beta_{y11} = \beta_{y19}, \quad \beta_{x3} = \beta_{x7} = \beta_{y13} = \beta_{y17}, \tag{37}$$
$$\beta_{y1} = \beta_{y9} = \beta_{x11} = \beta_{x19}, \quad \beta_{y3} = \beta_{y7} = \beta_{x13} = \beta_{x17},$$
$$\beta_{x5} = \beta_{y15}, \quad \beta_{y5} = \beta_{x15},$$

and

$$\phi_{x19} - \phi_{x1} = \phi_{y19} - \phi_{y1}, \quad \phi_{x17} - \phi_{x3} = \phi_{y17} - \phi_{y3}, \tag{38}$$
$$\phi_{x15} - \phi_{x5} = \phi_{y15} - \phi_{y5}, \quad \phi_{x13} - \phi_{x7} = \phi_{y13} - \phi_{y7},$$
$$\phi_{x11} - \phi_{x9} = \phi_{y11} - \phi_{y9},$$

where the numbers 1–19 correspond to straight sections 1–19. (Note that beta minima and maxima occur only in the odd numbered straight sections of the AGS). It is, of course, also true that $\beta_{x1} = \beta_{x2}$, $\beta_{y1} = \beta_{y2}$, and $\phi_{x2} - \phi_{x1} = \phi_{y2} - \phi_{y1}$ for any two points, 1 and 2, separated by one or more superperiods in the AGS.

The assumptions of sections 3.1 and 3.2 are therefore valid if two correction elements are placed in any one pair of the following pairs of straight sections: (1,19), (3,17), (5,15), (7,13), (9,11), and another two are placed in the same straight sections of another superperiod.

In addition to the relations (34–38), which are nearly exact, we have the following approximate relations due to the shortening of magnets 1, 2, 9, 10, 11, 12, 19, and 20 in each superperiod [4]:

$$\beta_{x5} \approx \beta_{x9} \approx \beta_{x13}, \quad \beta_{y5} \approx \beta_{y9} \approx \beta_{y13}, \tag{39}$$
$$\beta_{x7} \approx \beta_{x11} \approx \beta_{x15}, \quad \beta_{y7} \approx \beta_{y11} \approx \beta_{y15}.$$

It follows that the five FODO cells in each superperiod are approximately equivalent so that the results stated above for the case of a lattice composed of identical FODO cells are approximately true. Thus, if correctors are placed at any two horizontal beta maximums and at any two vertical beta maximums in the AGS, then the assumptions of sections 3.1–3.2 are always at least approximately valid and one may use the formulae developed in these sections to estimate the effectiveness of the correctors.

In the AGS Booster each of the six superperiods is composed of four FODO cells which are to first order identical. Therefore, if one places correctors at any two horizontal beta maximums and at any two vertical beta maximums in the booster lattice, the assumptions of sections 3.1–3.2 are valid.

4 Application to the AGS

The correction schemes discussed in the following sections (4.1–4.3) were first worked out by E. Raka [5,6]. We re-derive his results here using the formulae developed in section 3.

4.1 Correction of resonances $2Q_x = 17$ and $2Q_y = 17$

In Raka's scheme for the correction of these resonances one first considers four correction quadrupoles located in the C3, F3, C17, and F17 straight sections. We shall take positions s_1, s_2, s_3, s_4 to be the locations of the quads in straight sections C3, F3, C17, and F17 respectively, with $s_1 = 0$. Then using equations (34–36) and the superperiod symmetry we have

$$\beta_{x1} = \beta_{x2} = \beta_{y3} = \beta_{y4} = b, \quad \beta_{y1} = \beta_{y2} = \beta_{x3} = \beta_{x4} = B, \tag{40}$$

and

$$\phi_1 = \phi_x(C3) = \phi_y(C3) = 0, \quad \phi_2 = \phi_x(F3) = \phi_y(F3), \tag{41}$$

$$\phi_3 = \phi_x(C17) = \phi_y(C17), \quad \phi_4 = \phi_x(F17) = \phi_y(F17).$$

Now the normalized betatron phase advance between two points separated by three superperiods in the AGS is $\pi/2$, and the normalized phase advance between the number 3 and number 17 straight sections of a superperiod is very nearly $2\pi/17$. Thus we have

$$\phi_1 = 0, \quad \phi_2 = \frac{\pi}{2}, \quad \phi_3 = \frac{2\pi}{17}, \quad \phi_4 = \frac{\pi}{2} + \frac{2\pi}{17}. \tag{42}$$

Using (40–42) and $p = 17$ in the equations of section 3.1.2 we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CX_1 \\ SX_1 \\ CY_1 \\ SY_1 \end{pmatrix} = C\left(\frac{eQ}{cP}\right)\mathbf{M}_1 \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{43}$$

where

$$\mathbf{M}_1 = b\begin{pmatrix} 1 & R \\ -R & -1 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{44}$$

$$\mathbf{M}_1^{-1} = \frac{1}{b(R^2-1)}\begin{pmatrix} -1 & -R \\ R & 1 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad R = B/b. \tag{45}$$

Now, to insure that the correction scheme does not introduce any 9θ harmonic components, additional quadrupoles at E3, H3, E17, H17 are excited with the same currents as the quads at C3, F3, C17, F17 respectively. Since the additional quads are two superperiods away from the first set of quads, we have

$$\begin{aligned} \beta_{x,y}(E3) = \beta_{x,y}(C3), &\quad \beta_{x,y}(H3) = \beta_{x,y}(F3), \\ \beta_{x,y}(E17) = \beta_{x,y}(C17), &\quad \beta_{x,y}(H17) = \beta_{x,y}(F17), \\ \phi_x(E3) = \phi_y(E3) = \phi_1 + \omega, &\quad \phi_x(H3) = \phi_y(H3) = \phi_2 + \omega, \\ \phi_x(E17) = \phi_y(E17) = \phi_3 + \omega, &\quad \phi_x(H17) = \phi_y(H17) = \phi_4 + \omega, \end{aligned}$$

where ϕ_1–ϕ_4 are given by (42), and $\omega = \pi/3$ is the normalized betatron phase advance for two superperiods. The excitation coefficients produced by the additional quads are therefore

$$\begin{pmatrix} CX_2 \\ SX_2 \\ CY_2 \\ SY_2 \end{pmatrix} = C\left(\frac{eQ}{cP}\right)\mathbf{M}_2 \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{46}$$

where

$$\mathbf{M}_2 = \begin{pmatrix} \eta & \mathbf{0} \\ \mathbf{0} & \eta \end{pmatrix} \mathbf{M}_1, \quad \eta = \begin{pmatrix} cos(p\omega) & -sin(p\omega) \\ sin(p\omega) & cos(p\omega) \end{pmatrix}, \quad (p\omega = 17\pi/3). \tag{47}$$

The excitation coefficients produced by both sets of quadrupoles are given by the sum of equations (43) and (46).

In addition to insuring that no 9θ harmonic components are introduced by the correction scheme it is necessary to insure that no 0θ components are introduced. (Any 0θ components would alter the machine tunes). This is accomplished by adding a second group of eight quadrupoles to the scheme. These eight quads, at I3, L3, I17, L17, K3, B3, K17, and B17, are excited with currents opposite to those in the first group at C3, F3, C17, F17, E3, H3, E17, and H17 respectively. Using the fact that the second group of quads are a normalized betatron phase advance of π away from the first group, the equations of section 3.1.2 with $p = 17$ show that the second group produces the same excitation coefficients as the first group. Thus the excitation coefficients produced by all 16 quads are

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C \left(\frac{eQ}{cP}\right) \mathbf{M} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{48}$$

where

$$\mathbf{M} = 2(\mathbf{M}_1 + \mathbf{M}_2) = 2 \begin{pmatrix} \mathbf{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega} \end{pmatrix} \mathbf{M}_1, \tag{49}$$

$$\mathbf{M}^{-1} = \frac{1}{2} \mathbf{M}_1^{-1} \begin{pmatrix} \mathbf{\Omega}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^{-1} \end{pmatrix},$$

$$\mathbf{\Omega} = \mathbf{I} + \eta = \frac{1}{2} \begin{pmatrix} 3 & \sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \quad \mathbf{\Omega}^{-1} = \frac{2}{12} \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

Using (44–45) in (49) we then have

$$\mathbf{M} = 2b \begin{pmatrix} \mathbf{\Omega} & R\mathbf{\Omega} \\ -R\mathbf{\Omega} & -\mathbf{\Omega} \end{pmatrix} = b \begin{pmatrix} 3 & \sqrt{3} & 3R & \sqrt{3}R \\ -\sqrt{3} & 3 & -\sqrt{3}R & 3R \\ -3R & -\sqrt{3}R & -3 & -\sqrt{3} \\ \sqrt{3}R & -3R & \sqrt{3} & -3 \end{pmatrix}, \tag{50}$$

$$\mathbf{M}^{-1} = \frac{1}{12b(R^2-1)} \begin{pmatrix} -3 & \sqrt{3} & -3R & \sqrt{3}R \\ -\sqrt{3} & -3 & -\sqrt{3}R & -3R \\ 3R & -\sqrt{3}R & 3 & -\sqrt{3} \\ \sqrt{3}R & 3R & \sqrt{3} & 3 \end{pmatrix}. \tag{51}$$

The currents which produce the desired corrections are then

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eQ}\right)\mathbf{M}^{-1}\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}. \tag{52}$$

4.2 Correction of resonances $3Q_x = 26$ and $Q_x + 2Q_y = 26$

In Raka's scheme [5,6] for the correction of these resonances one first considers four correction sextupoles located in the C7, E7, C13, and E13 straight sections. We shall take positions s_1, s_2, s_3, s_4 to be the locations of the sextupoles in straight sections C7, E7, C13, and E13 respectively, with $s_1 = 0$. Then using equations (34–36) and the superperiod symmetry we have

$$\beta_{x1} = \beta_{x2} = \beta_{y3} = \beta_{y4} = b, \quad \beta_{y1} = \beta_{y2} = \beta_{x3} = \beta_{x4} = B, \tag{53}$$

and

$$\begin{aligned} \phi_1 = \phi_x(C7) = \phi_y(C7) = 0, \quad \phi_2 = \phi_x(E7) = \phi_y(E7), \\ \phi_3 = \phi_x(C13) = \phi_y(C13), \quad \phi_4 = \phi_x(E13) = \phi_y(E13). \end{aligned} \tag{54}$$

Now the normalized betatron phase advance between two points separated by two superperiods in the AGS is $\pi/3$, and the normalized phase advance between the number 7 and number 13 straight sections of a superperiod is $\phi \approx \pi/20$. Thus we have

$$\phi_1 = 0, \quad \phi_2 = \pi/3, \quad \phi_3 = \phi, \quad \phi_4 = \pi/3 + \phi \tag{55}$$

Using (53–55) and $p = 26$ in the equations of section 3.2 we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CX_1 \\ SX_1 \\ CY_1 \\ SY_1 \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{56}$$

where

$$\mathbf{M} = b^{3/2} \begin{pmatrix} C_1 & C_2 & R^{3/2}C_3 & R^{3/2}C_4 \\ S_1 & S_2 & R^{3/2}S_3 & R^{3/2}S_4 \\ -RC_1 & -RC_2 & -R^{1/2}C_3 & -R^{1/2}C_4 \\ -RS_1 & -RS_2 & -R^{1/2}S_3 & -R^{1/2}S_4 \end{pmatrix}, \tag{57}$$

and

$$\mathbf{M}^{-1} = Gb^{-3/2} \begin{pmatrix} -R^{1/2}S_2 & R^{1/2}C_2 & -R^{3/2}S_2 & R^{3/2}C_2 \\ R^{1/2}S_1 & -R^{1/2}C_1 & R^{3/2}S_1 & -R^{3/2}C_1 \\ RS_4 & -RC_4 & S_4 & -C_4 \\ -RS_3 & RC_3 & -S_3 & C_3 \end{pmatrix}, \tag{58}$$

$$R = B/b, \quad G = \frac{1}{R^{1/2}(R^2-1)}\frac{2}{\sqrt{3}}$$

$$C_1 = 1, \quad C_2 = -1/2, \quad C_3 = cos\,\omega, \quad C_4 = cos(\omega + 2\pi/3),$$

$$S_1 = 0, \quad S_2 = \sqrt{3}/2, \quad S_3 = sin\,\omega, \quad S_4 = sin(\omega + 2\pi/3),$$

and $\omega = 26\phi \approx 26\pi/20$.

Now, to insure that the correction scheme does not introduce any 0θ harmonic components—which would alter the machine chromaticities—additional sextupoles at F7, H7, F13, and H13 are excited with currents opposite to those at C7, E7, C13, and E13 respectively. Since the additional sextupoles are three superperiods away from the first set of sextupoles, the normalized betatron phase advance between the two sets is $\pi/2$, and hence with $p = 26$ the equations of section 3.2 show that the additional set produces the same excitation coefficients as the first set. To insure that no odd harmonics in θ, and in particular no 9θ or 17θ harmonics, are produced, a second group of eight sextupoles is added to the scheme. These eight sextupoles, at I7, K7, I13, K13, L7, B7, L13, and B13 are excited with the same currents as those at C7, E7, C13, E13, F7, H7, F13, and H13 respectively. Using the fact that the second group of sextupoles is a normalized betatron phase advance of π away from the first group, the equations of section 3.2 with $p = 26$ show that the second group produces the same excitation coefficients as the first group. Thus the excitation coefficients produced by all 16 sextupoles are four times those produced by the original set of four, and are therefore

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = 4C\left(\frac{eS}{cP}\right)\mathbf{M}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{59}$$

where M is given by (57). The currents which produce the desired corrections are then

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \frac{1}{4C} \left(\frac{cP}{eS} \right) \mathbf{M}^{-1} \begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}, \tag{60}$$

where $\mathbf{M}^{-1}$ is given by (58).

4.3 Correction of resonances $3Q_y = 26$ and $2Q_x + Q_y = 26$

These resonances are currently corrected with four air-core skew sextupoles located in straight sections E15, F15, I5, and K5. However, during the 1989 summer shutdown the skew sextupole in E15 will be removed and it will no longer be possible to correct both resonances simultaneously. During the 1990 summer shutdown the remaining air-core skew sextupoles will be removed and four new iron-core units will be installed. A number of correction schemes using these new skew sextupoles have been considered, and based on constraints imposed by the straight section committee (Willem van Asselt) and the vacuum group (Kimo Welch) two sets of locations for the magnets are currently recognized as possibilities. One set of locations would consist of straight sections 7 and 13 in one superperiod and the same straight sections two superperiods away. The other set would consist of straight sections 1 and 19 in one superperiod and the same straight sections two superperiods away. In each case the two pairs of straight sections are separated by two superperiods so that additional magnets may be added, (if necessary) as in the scheme discussed in the previous section, to insure that no harmful harmonic components are produced.

Here we consider the more general case in which skew sextupoles are placed in straight sections i and j of superperiod M and in the same straight sections of superperiod N, where (i, j) is any one of the pairs (19, 1), (3, 17), (15, 5), (7, 13), or (11, 9), and M and N are n superperiods apart. (Note that the first number of each pair corresponds to a vertical beta maximum and the second to a horizontal beta maximum). We shall take positions s_1, s_2, s_3, and s_4 to be the locations of the skew sextupoles in straight sections Mi, Ni, Mj, and Nj respectively, with $s_1 = 0$. Then using equations (34–36) and the superperiod symmetry we have

$$\beta_{x1} = \beta_{x2} = \beta_{y3} = \beta_{y4} = b, \quad \beta_{y1} = \beta_{y2} = \beta_{x3} = \beta_{x4} = B, \tag{61}$$

and

$$\phi_1 = \phi_x(Mi) = \phi_y(Mi) = 0, \quad \phi_2 = \phi_x(Ni) = \phi_y(Ni), \tag{62}$$
$$\phi_3 = \phi_x(Mj) = \phi_y(Mj), \quad \phi_4 = \phi_x(Nj) = \phi_y(Nj).$$

Now the normalized betatron phase advance between two points separated by n superperiods in the AGS is $n\pi/6$, and the normalized phase advance between the i and j straight sections of a superperiod is $\phi \approx (j-i)\pi/120$. Thus we have

$$\phi_1 = 0, \quad \phi_2 = n\pi/6, \quad \phi_3 = \phi, \quad \phi_4 = \phi + n\pi/6. \tag{63}$$

Using (61–63) and $p = 26$ in the equations of section 3.2 with x and y interchanged we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CY \\ SY \\ CX \\ SX \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \tag{64}$$

where CY and SY are the cos and sin parts of κ_y, CX and SX are the cos and sin parts of $\kappa_{yx}/3$, and

$$\mathbf{M} = b^{3/2}\begin{pmatrix} R^{3/2}C_1 & R^{3/2}C_2 & C_3 & C_4 \\ R^{3/2}S_1 & R^{3/2}S_2 & S_3 & S_4 \\ -R^{1/2}C_1 & -R^{1/2}C_2 & -RC_3 & -RC_4 \\ -R^{1/2}S_1 & -R^{1/2}S_2 & -RS_3 & -RS_4 \end{pmatrix}, \tag{65}$$

and

$$\mathbf{M}^{-1} = Gb^{-3/2}\begin{pmatrix} -RS_2 & RC_2 & -S_2 & C_2 \\ RS_1 & -RC_1 & S_1 & -C_1 \\ R^{1/2}S_4 & -R^{1/2}C_4 & R^{3/2}S_4 & -R^{3/2}C_4 \\ -R^{1/2}S_3 & R^{1/2}C_3 & -R^{3/2}S_3 & R^{3/2}C_3 \end{pmatrix}, \tag{66}$$

$$R = B/b, \quad G = \frac{-1}{R^{1/2}(R^2-1)sin(n\pi/3)},$$

$$C_1 = 1, \quad C_2 = cos(n\pi/3), \quad C_3 = cos\,\omega, \quad C_4 = cos(\omega + n\pi/3),$$

$$S_1 = 0, \quad S_2 = sin(n\pi/3), \quad S_3 = sin\,\omega, \quad S_4 = sin(\omega + n\pi/3),$$

and $\omega = 26\phi \approx 26(j-i)\pi/120$. The currents which produce the desired corrections are then

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eS}\right)\mathbf{M}^{-1}\begin{pmatrix} CY \\ SY \\ CX \\ SX \end{pmatrix}, \tag{67}$$

where $\mathbf{M}^{-1}$ is given by (66).

5 Application to the Booster

The booster lattice [7] consists of six superperiods—labled A, B, C, D, E, and F—each containing four FODO cells which are, to first order, identical. The positions of the horizontal beta maximums in each superperiod are s_1, s_3, s_5, s_7, and those of the vertical beta maximums are s_0, s_2, s_4, s_6, s_8, with $s_j > s_i$ if $j > i$. We shall take s_0 in superperiod A to be the point of zero betatron phase. The normalized betatron phase advance (defined by equation 18) for each superperiod is $\pi/3$. Assuming the four FODO cells in each superperiod are identical we have

$$\phi_i = \phi_{xi} = \phi_{yi} = i\pi/24 \tag{68}$$

and

$$\begin{aligned} \beta_{x0} = \beta_{x2} = \beta_{x4} = \beta_{x6} = \beta_{x8} = a, \\ \beta_{y0} = \beta_{y2} = \beta_{y4} = \beta_{y6} = \beta_{y8} = B, \\ \beta_{x1} = \beta_{x3} = \beta_{x5} = \beta_{x7} = A, \\ \beta_{y1} = \beta_{y3} = \beta_{y5} = \beta_{y7} = b, \end{aligned} \tag{69}$$

where ϕ_{xi} and ϕ_{yi} are the normalized betatron phase advances at the positions s_i, and $\beta_{xi} = \beta_x(s_i)$, $\beta_{yi} = \beta_y(s_i)$. To first order we also have

$$a = b, \quad A = B. \tag{70}$$

Although equations (68–70) are not exact—because the booster dipoles do not occupy the same positions in each FODO cell—they are good enough for estimating the currents required for various correction schemes in the booster and therefore will be used in the following sections.

The correction elements for the schemes discussed in the following sections are located at or near the positions s_i in each superperiod and are excited with currents I_{ji} where $j = 1, 2, 3, 4, 5,$ and 6 corresponds to superperiods A, B, C, D, E, and F respectively. Tepikian has shown [8, 9] that by choosing

$$I_{ji} = f_j I_i, \quad f_j = cos\left[p(j-1)\pi/3\right], \tag{71}$$

one can correct the $mQ_x + nQ_y = p$ resonances without introducing unwanted harmonics.

5.1 Correction of resonances $2Q_x = 9$ and $2Q_y = 9$

Using equation (68) we find that for $p = 9$ the phase differences, $p\phi_6 - p\phi_2$ and $p\phi_5 - p\phi_1$ are odd multiples of $\pi/2$ which, as we have shown in section 3, gives the most effective correction of the resonances. Thus, for the correction of the $2Q_x = 9$ and $2Q_y = 9$ resonances we consider the four correction quadrupoles located in superperiod A at s_1, s_2, s_5, and s_6. Using (68–70) and $p = 9$ in the equations of section 3.1.2 we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CX_a \\ SX_a \\ CY_a \\ SY_a \end{pmatrix} = C\left(\frac{eQ}{cP}\right) \mathbf{M}_a \begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix}, \tag{72}$$

where

$$\mathbf{M}_a = b \begin{pmatrix} C_2 & C_6 & RC_1 & RC_5 \\ S_2 & S_6 & RS_1 & RS_5 \\ -RC_2 & -RC_6 & -C_1 & -C_5 \\ -RS_2 & -RS_6 & -S_1 & -S_5 \end{pmatrix}, \tag{73}$$

$$C_j = cos(3\pi j/8), \quad S_j = sin(3\pi j/8), \quad R = B/b,$$

and

$$\mathbf{M}_a^{-1} = \frac{1}{b(R^2-1)} \begin{pmatrix} S_6 & -C_6 & RS_6 & -RC_6 \\ -S_2 & C_2 & -RS_2 & RC_2 \\ -RS_5 & RC_5 & -S_5 & C_5 \\ RS_1 & -RC_1 & S_1 & -C_1 \end{pmatrix}. \tag{74}$$

If we now excite the correctors at s_8, s_4, s_7, and s_3 with currents $I_8 = I_2$, $I_4 = -I_6$, $I_7 = I_1$, and $I_3 = -I_5$ we find that the excitation coefficients produced by these quadrupoles are

$$\begin{pmatrix} CX_b \\ SX_b \\ CY_b \\ SY_b \end{pmatrix} = C \left(\frac{eQ}{cP} \right) \mathbf{M}_b \begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix}, \tag{75}$$

where

$$\mathbf{M}_b = \begin{pmatrix} \eta & \mathbf{0} \\ \mathbf{0} & \eta \end{pmatrix} \mathbf{M}_a, \quad \eta = \begin{pmatrix} cos(\pi/4) & -sin(\pi/4) \\ sin(\pi/4) & cos(\pi/4) \end{pmatrix}. \tag{76}$$

The excitation coefficients produced by all eight quadrupoles in superperiod A are then

$$\begin{pmatrix} CX_1 \\ SX_1 \\ CY_1 \\ SY_1 \end{pmatrix} = C \left(\frac{eQ}{cP} \right) \mathbf{M}_1 \begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix}, \tag{77}$$

where

$$\mathbf{M}_1 = \mathbf{M}_a + \mathbf{M}_b = \begin{pmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{pmatrix} \mathbf{M}_a, \tag{78}$$

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 + cos(\pi/4) & -sin(\pi/4) \\ sin(\pi/4) & 1 + cos(\pi/4) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 + \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 2 + \sqrt{2} \end{pmatrix}.$$

Now, in the scheme proposed by S. Tepikian [8] for the correction of the $2Q_x = 9$ and $2Q_y = 9$ resonances, the quadrupoles in the remaining superperiods are excited with currents given by (71). This insures that no 10θ, 5θ, 4θ, or 0θ harmonic components are produced by the scheme. The excitation coefficients produced by the quadrupoles in superperiod j are then

$$\begin{pmatrix} CX_j \\ SX_j \\ CY_j \\ SY_j \end{pmatrix} = C \left(\frac{eQ}{cP} \right) f_j \mathbf{M}_j \begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix}, \tag{79}$$

where

$$\mathbf{M}_j = \begin{pmatrix} \eta_j & \mathbf{0} \\ \mathbf{0} & \eta_j \end{pmatrix} \mathbf{M}_1, \quad \eta_j = \begin{pmatrix} cos\, p(j-1)\pi/3 & -sin\, p(j-1)\pi/3 \\ sin\, p(j-1)\pi/3 & cos\, p(j-1)\pi/3 \end{pmatrix}, \tag{80}$$

and $\mathbf{M}_1$ is given by (78) and (73). Summing equation (79) over j we find that the excitation coefficients produced by the quadrupoles in all six superperiods are

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C\left(\frac{eQ}{cP}\right)\mathbf{M}\begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix}, \tag{81}$$

where, for $p = 9$,

$$\mathbf{M} = \sum_{j=1}^{6} f_j \mathbf{M}_j = 6\mathbf{M}_1.$$

The currents which produce the desired corrections are then

$$\begin{pmatrix} I_2 \\ I_6 \\ I_1 \\ I_5 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eQ}\right)\mathbf{M}^{-1}\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}. \tag{82}$$

5.2 Correction of resonances $3Q_x = 14$ and $Q_x + 2Q_y = 14$

Using equation (68) we find that for $p = 14$ the phase differences, $p\phi_8 - p\phi_2$ and $p\phi_7 - p\phi_1$ are odd multiples of $\pi/2$ which, as we have shown in section 3.2, gives the most effective correction of the resonances. Thus for the correction of the $3Q_x = 14$ and $Q_x + 2Q_y = 14$ resonances we consider the four correction sextupoles located in superperiod A at s_1, s_2, s_7, and s_8. Using (68–70) and $p = 14$ in the equations of section 3.2 we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CX_a \\ SX_a \\ CY_a \\ SY_a \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_a\begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{83}$$

where

$$\mathbf{M}_a = b^{3/2}\begin{pmatrix} C_2 & C_8 & R^{3/2}C_1 & R^{3/2}C_7 \\ S_2 & S_8 & R^{3/2}S_1 & R^{3/2}S_7 \\ -RC_2 & -RC_8 & -R^{1/2}C_1 & -R^{1/2}C_7 \\ -RS_2 & -RS_8 & -R^{1/2}S_1 & -R^{1/2}S_7 \end{pmatrix}, \tag{84}$$

and

$$R = B/b, \quad C_j = cos(7\pi j/12), \quad S_j = sin(7\pi j/12).$$

If we now excite the correctors at s_4, s_6, s_3, and s_5 with currents $I_4 = -I_2$, $I_6 = -I_8$, $I_3 = -I_1$, and $I_5 = -I_7$ we find that the excitation coefficients produced by these sextupoles are

$$\begin{pmatrix} CX_b \\ SX_b \\ CY_b \\ SY_b \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_b \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{85}$$

where

$$\mathbf{M}_b = -b^{3/2}\begin{pmatrix} C_4 & C_6 & R^{3/2}C_3 & R^{3/2}C_5 \\ S_4 & S_6 & R^{3/2}S_3 & R^{3/2}S_5 \\ -RC_4 & -RC_6 & -R^{1/2}C_3 & -R^{1/2}C_5 \\ -RS_4 & -RS_6 & -R^{1/2}S_3 & -R^{1/2}S_5 \end{pmatrix}. \tag{86}$$

The excitation coefficients produced by all eight sextupoles in superperiod A are then

$$\begin{pmatrix} CX_1 \\ SX_1 \\ CY_1 \\ SY_1 \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_1 \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{87}$$

where

$$\mathbf{M}_1 = \mathbf{M}_a + \mathbf{M}_b, \tag{88}$$

$$\mathbf{M}_1 = 2b^{3/2}cos(\pi/12)\begin{pmatrix} C_2^+ & C_8^- & R^{3/2}C_1^+ & R^{3/2}C_7^- \\ S_2^+ & S_8^- & R^{3/2}S_1^+ & R^{3/2}S_7^- \\ -RC_2^+ & -RC_8^- & -R^{1/2}C_1^+ & -R^{1/2}C_7^- \\ -RS_2^+ & -RS_8^- & -R^{1/2}S_1^+ & -R^{1/2}S_7^- \end{pmatrix},$$

$$C_j^+ = cos(p\phi_j + \pi/12), \quad S_j^+ = sin(p\phi_j + \pi/12),$$

$$C_j^- = cos(p\phi_j - \pi/12), \quad S_j^- = sin(p\phi_j - \pi/12),$$

$$(p\phi_j = 14\phi_j = 7\pi j/12).$$

Now, in the scheme proposed by S. Tepikian [9] for the correction of the $3Q_x = p$ and $Q_x + 2Q_y = p$ resonances, the sextupoles in the remaining superperiods are excited with currents given by (71). For $p = 14$ this insures that no 13θ, 9θ, 5θ, or 0θ harmonics are produced. (The scheme does produce 10θ and 4θ harmonic components which are potentially harmful. However, for the tune spreads and operating point expected in the booster this should not be a problem). The excitation coefficients produced by the sextupoles in superperiod j are then

$$\begin{pmatrix} CX_j \\ SX_j \\ CY_j \\ SY_j \end{pmatrix} = C\left(\frac{eS}{cP}\right) f_j \mathbf{M}_j \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{89}$$

where

$$\mathbf{M}_j = \begin{pmatrix} \eta_j & \mathbf{0} \\ \mathbf{0} & \eta_j \end{pmatrix} \mathbf{M}_1, \quad \eta_j = \begin{pmatrix} cos\, p(j-1)\pi/3 & -sin\, p(j-1)\pi/3 \\ sin\, p(j-1)\pi/3 & cos\, p(j-1)\pi/3 \end{pmatrix},$$

and $\mathbf{M}_1$ is given by (88). Summing equation (89) over j we find that the excitation coefficients produced by the sextupoles in all six superperiods are

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C\left(\frac{eS}{cP}\right) \mathbf{M} \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{90}$$

where, for $p = 14$,

$$\mathbf{M} = \sum_{j=1}^{6} f_j \mathbf{M}_j = 3\mathbf{M}_1.$$

The currents which produce the desired corrections are then

$$\begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix} = \frac{1}{C}\left(\frac{cP}{eS}\right) \mathbf{M}^{-1} \begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix}. \tag{91}$$

5.3 Correction of resonances $3Q_x = 13$ and $Q_x + 2Q_y = 13$

For the correction of these resonances we again consider the four correction sextupoles located in superperiod A at s_1, s_2, s_7, and s_8. Using (68–70) and $p = 13$ in the equations of section 3.2 we find that the excitation coefficients produced by the four correctors are

$$\begin{pmatrix} CX_a \\ SX_a \\ CY_a \\ SY_a \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_a \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{92}$$

where

$$\mathbf{M}_a = b^{3/2}\begin{pmatrix} C_2 & C_8 & R^{3/2}C_1 & R^{3/2}C_7 \\ S_2 & S_8 & R^{3/2}S_1 & R^{3/2}S_7 \\ -RC_2 & -RC_8 & -R^{1/2}C_1 & -R^{1/2}C_7 \\ -RS_2 & -RS_8 & -R^{1/2}S_1 & -R^{1/2}S_7 \end{pmatrix}, \tag{93}$$

and

$$R = B/b, \quad C_j = cos(13\pi j/24), \quad S_j = sin(13\pi j/24).$$

(Note that for $p = 13$ the phase differences, $p\phi_8 - p\phi_2$ and $p\phi_7 - p\phi_1$, are $2\pi + 5\pi/4$. The effectiveness of the correctors is proportional to the sin of this phase, as discussed in section 3.2).

Now, as before, we excite the correctors at s_4, s_6, s_3, and s_5 with currents $I_4 = -I_2$, $I_6 = -I_8$, $I_3 = -I_1$, and $I_5 = -I_7$. The excitation coefficients produced by these sextupoles are

$$\begin{pmatrix} CX_b \\ SX_b \\ CY_b \\ SY_b \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_b \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{94}$$

where

$$\mathbf{M}_b = -b^{3/2}\begin{pmatrix} C_4 & C_6 & R^{3/2}C_3 & R^{3/2}C_5 \\ S_4 & S_6 & R^{3/2}S_3 & R^{3/2}S_5 \\ -RC_4 & -RC_6 & -R^{1/2}C_3 & -R^{1/2}C_5 \\ -RS_4 & -RS_6 & -R^{1/2}S_3 & -R^{1/2}S_5 \end{pmatrix}. \tag{95}$$

The excitation coefficients produced by all eight sextupoles in superperiod A are then

$$\begin{pmatrix} CX_1 \\ SX_1 \\ CY_1 \\ SY_1 \end{pmatrix} = C\left(\frac{eS}{cP}\right)\mathbf{M}_1 \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{96}$$

where

$$\mathbf{M}_1 = \mathbf{M}_a + \mathbf{M}_b, \tag{97}$$

$$\mathbf{M}_1 = 2b^{3/2}cos(\pi/24)\begin{pmatrix} C_2^+ & C_8^- & R^{3/2}C_1^+ & R^{3/2}C_7^- \\ S_2^+ & S_8^- & R^{3/2}S_1^+ & R^{3/2}S_7^- \\ -RC_2^+ & -RC_8^- & -R^{1/2}C_1^+ & -R^{1/2}C_7^- \\ -RS_2^+ & -RS_8^- & -R^{1/2}S_1^+ & -R^{1/2}S_7^- \end{pmatrix},$$

$$C_j^+ = cos(p\phi_j + \pi/24), \quad S_j^+ = sin(p\phi_j + \pi/24),$$

$$C_j^- = cos(p\phi_j - \pi/24), \quad S_j^- = sin(p\phi_j - \pi/24),$$

$$(p\phi_j = 13\phi_j = 13\pi j/24).$$

Now, as before, we excite the sextupoles in the remaining superperiods with currents given by (71). For $p = 13$ this insures that no 14θ, 10θ, 9θ, 4θ, or 0θ harmonics are produced. (The scheme does produce 5θ harmonic components which are potentially harmful. However, for the tune spreads and operating point expected in the booster this should not be a problem). The excitation coefficients produced by the sextupoles in superperiod j are then

$$\begin{pmatrix} CX_j \\ SX_j \\ CY_j \\ SY_j \end{pmatrix} = C\left(\frac{eS}{cP}\right) f_j \mathbf{M}_j \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{98}$$

where

$$\mathbf{M}_j = \begin{pmatrix} \eta_j & 0 \\ 0 & \eta_j \end{pmatrix} \mathbf{M}_1, \quad \eta_j = \begin{pmatrix} cos\, p(j-1)\pi/3 & -sin\, p(j-1)\pi/3 \\ sin\, p(j-1)\pi/3 & cos\, p(j-1)\pi/3 \end{pmatrix},$$

and $\mathbf{M}_1$ is given by (97). Summing equation (98) over j we find that the excitation coefficients produced by the sextupoles in all six superperiods are

$$\begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} = C\left(\frac{eS}{cP}\right) \mathbf{M} \begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix}, \tag{99}$$

where, for $p = 13$,

$$\mathbf{M} = \sum_{j=1}^{6} f_j \mathbf{M}_j = 3\mathbf{M}_1.$$

The currents which produce the desired corrections are then

$$\begin{pmatrix} I_2 \\ I_8 \\ I_1 \\ I_7 \end{pmatrix} = \frac{1}{C} \left(\frac{cP}{eS} \right) \mathbf{M}^{-1} \begin{pmatrix} CX \\ SX \\ CY \\ SY \end{pmatrix} . \tag{100}$$

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