

## The linear effect of the space charge force

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THE LINEAR EFFECT OF THE SPACE CHARGE FORCE

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# The linear effect of the space charge force

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## ABSTRACT

Linear effect of the space charge is studied to understand the particle motion. We analyze the modulation of the betatron functions due to the space charge. We simulate the realistic machine with misalignment error of magnets, dipole field strength error, and dipole rotation error. Because of the space charge tune shift, the particles in the bunch will experience the half-integer and integer resonances. We found that the half-integer resonance at 4.5 for the booster is rather narrow and correctable by quadrupole harmonic correctors. The integer resonance at 4.0 may pose the beam intensity limitation for the booster.

## 1.) Introduction

It is known that the space charge force<sup>1</sup> is important to the low energy machine. To reach the proposed intensity at the Brookhaven AGS Booster, space charge tune shift is of the order of  $\Delta Q \approx 0.7^2$ , where

$$\Delta Q = \frac{3r_0 FN}{4B_f \beta^2 \gamma^3 \epsilon}$$

with  $r_0 = 1.535 \times 10^{-18}$  m,  $B_f, F, N$  and  $\epsilon$  are bunching factor, form factor, number of particles in the accelerator and the emittance respectively. It is therefore useful to understand the effect of space-charge on the beam particles. Recently, tracking calculations have incorporated the space charge force to estimate the maximum number of particles acceptable to the accelerator<sup>3</sup>. Tracking results should offer interesting guidance to the design of the accelerator. However it is interesting as well to obtain some analytic information on the performance of the particles in the accelerator in the presence of large space charge tune shift. Present study analyzes the linear tune shift effect on the beam particles. The linear effect should be understood before nonlinear problem can be resolved.

Section 2 studies the linear space charge tune shift on a perfect machine. Section 3 analyzes the similar effect on a realistic machine with errors. The remanent effect after the orbit correction shall be studied. Section 4 studies the half integer stop-band width. Conclusion is given in section 5.

## 2. Equation of motion in the perfect machine.

Equation of motion for particles in the accelerator is given by<sup>4</sup>

$$x'' + \left( \frac{1}{\rho} - K \right) x = 0 \quad (1)$$

$$y'' + K y = 0 \quad (2)$$

where  $\rho$  is the radius of curvature,  $K$  is the focusing strength of an element,  $K = -(\partial B_y / \partial x) / B\rho$ . In the strong focusing machine, eq.(1) can be expressed as

$$x'' + \bar{b} x = 0 \quad (3)$$

where  $\bar{b} = (Q/R)^2$  represents the average focusing strength on the particles. The presence of space charge force will naturally decrease the focusing force of the machine. Let us represent the average space charge defocusing strength by  $-b_s$ , i.e.

$$x'' + (\bar{b} - b_s) x = 0 \quad (4)$$

where  $b_s$  is proportional to the number of particles per bunch and depends on the beam geometric profile of the bunch. The space charge force shall decrease the focusing strength of the accelerator. If the accelerator is limited by the admittance of

the magnet aperture and the vacuum chamber, the space charge shall induce the effect of beam limitation due to a larger requirement of the beam pipe. The ratio  $b_{sc}/b$  is a measure of relative strength between the space charge force and quadrupole focusing force. For a constant  $b_{sc}/b$ , the number of particles per bunch is proportional to the the average focusing strength of the machine. The advantage of a smaller machine in the space charge consideration is that (1)The average focusing strength is larger and (2)The corresponding space charge tune shift is smaller for the same number of particles in the accelerator. These fact should be obtained from our calculation in the following.

We have observed that the effect of the space charge force is equivalent to a defocusing quadrupole lens to particles in the bunch. The defocusing force can be approximated by a delta-function kick as the following



where the defocusing force due to the space charge is divided into 4 equal kicks located in the cell shown in the above figure. The matrix can be expressed as

$$M = \begin{bmatrix} 1 & 0 \\ \frac{\delta_{sc}-\delta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta_{sc} & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta_{sc}+\delta & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta_{sc} & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\delta_{sc}-\delta}{2} & 1 \end{bmatrix} \quad (5)$$

Let us define the focusing strength as  $\xi = 1/\delta = 2\sin(\mu_0/2)$  and the relative strength between the space charge and the quadrupoles as  $\eta = \delta_{sc}/\delta$ . We shall obtain

$$M_{11} = 1 + 5\xi\eta - 5\xi^2(1-\eta-7.5\eta^2) - 25\xi^3\eta(1-\eta-3.5\eta^2) - \frac{1}{32}\xi^4\eta^2(1-\eta-2\eta^2) \\ = 1.2 \sin^2 \mu/2 \quad (6)$$

or

$$\sin^2 \frac{\mu}{2} = -5\eta \cdot \sin^2 \mu_0/2 + (1-2\eta-7.5\eta^2)\sin^2 \mu_0/2 + \eta(1-\eta-3.5\eta^2)\sin^3 \mu_0/2 \\ + .25\eta^2(1-\eta-2\eta^2)\sin^4 \mu_0/2 \quad (7)$$

Thus given a relative strength between the space charge force and the quadrupole focusing strength,  $\eta$ , we can calculate the tune shift of the machine or vice versa. Once the phase advance is obtained, the betatron amplitude can be obtained as

$$M_{12} = 1 \cdot (2+2(1+2.25\eta)\sin \mu_0/2 + 4\eta(.5+.875\eta)\sin^2 \mu_0/2 + .5\eta^2(1+\eta)\sin^3 \mu_0/2) \\ = \beta \cdot \sin \mu \quad (8)$$

Once  $\beta$  is obtained,  $\Delta\beta/\beta = (\beta - \beta_0)/\beta_0$  can be calculated. Figure 1

shows the tune shift and the  $\beta$ -function modulation as a function of  $\eta$  for the Booster. Similar figure for AOS is shown in Fig.2.

The above result is valid for a perfect machine. In reality, the machine has imperfection, e.g. misalignment of magnets, gradient error of quadrupoles, rotation angle error of dipoles etc. These imperfection is then translated into half-integer and integer stop-bands for the accelerator. The particles in the linear machine would encounter the stop-bands at half-integer and integer tune respectively. We shall address the question in next section.

### 3. Closed orbit error

Fig. 3 shows the rms.closed orbit distortion due to various sources of the error. We observe that the orbit is most sensitive to misalignment error of quadrupoles. The dipole field strength error  $\Delta B/B$  (in  $10^{-4}$ ) and the rotation angle error  $\Delta\theta$  (in  $10^{-4}$  rad) gives approximately similar sensitivity. For the booster lattice, these sensitivity calculated from TEAPOT<sup>5</sup> with one random seed can be expressed as

$$\sigma_{x,y} \simeq 7.5 \sigma_{\Delta x, \Delta y} \quad (9)$$

$$\sigma_x [\text{m}] \simeq 3.7 \sigma_{\Delta B/B} \quad (10)$$

$$\sigma_y [\text{m}] \simeq 2.7 \sigma_{\Delta\theta} \text{ (rad)} \quad (11)$$

The effect of the misalignment errors can be expressed in the following equation<sup>4</sup>,

$$d^2u/ds^2 + K(s)u = \Delta B(s)/E\rho \quad (12)$$

Here  $\Delta B(s)$  is the field error and  $E\rho$  is the magnetic rigidity of the particles. Making the transformation,

$$v = \beta^{-1/2} u \quad (13)$$

$$v' = \alpha\beta^{-1/2}u + \beta^{1/2}u' = dv/d\phi \quad (14)$$

$$\phi = \int ds/Q\beta \quad (15)$$

we obtain,

$$d^2v/d\phi^2 + Q^2v = Q^2\beta^{3/2}\Delta B(s)/E\rho \equiv f(\phi) \quad (16)$$

The closed orbit becomes,

$$\begin{aligned} v(\phi) &= \frac{v}{2\sin \pi Q} \int_{\phi}^{\phi+2\pi} f(\psi) \cos Q(\pi+\phi-\psi) d\psi \\ &= \sum_k \frac{Q^2 f_k \exp(ik\phi)}{Q^2 - k^2} \end{aligned} \quad (17)$$

with

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \exp(-ik\phi) d\phi \quad (18)$$

When the detailed behavior of the error field is unknown, statistical prediction of the closed orbit error can be evaluated. The invariant  $V(\phi) = v^2 + (v'/Q)^2$ , can be calculated to be,

$$V(\phi) = \frac{Q^2}{4\sin^2 \pi Q} \int_{\phi}^{\phi+2\pi} \int_{\phi}^{\phi+2\pi} (\psi) f(\chi) \cos Q(\chi-\psi) d\psi d\chi \quad (19)$$

The expectation value is given by,

$$\langle V(\phi) \rangle = \frac{Q^2}{4\sin^2 \pi Q} \int_{\phi}^{\phi+2\pi} \int_{\phi}^{\phi+2\pi} \langle f(\psi) f(\chi) \rangle \cos Q(\chi-\psi) d\psi d\chi. \quad (20)$$

By the hypothesis of randomness, these errors in different magnets are uncorrelated. Hence  $\langle f(\psi) f(\chi) \rangle = 0$  unless  $\psi - \chi \approx 0$  in the same magnet. If the length of the magnet is short compared to the betatron wavelength, then  $\cos Q(\psi - \chi) \approx 1$ . The double integral is then transformed into the simple sum over magnets,

$$\langle V(\phi) \rangle = \frac{1}{4\sin^2 \pi Q} \sum_k \beta_k (\Delta B_k / B\rho)^2 L_k^2 \quad (21)$$

The closed orbit distortion on the  $i$ -th magnet is  $u_i = (\beta_i V)^{1/2}$ , therefore the mean square becomes,

$$\langle u_i^2 \rangle = \frac{\beta_i}{4\sin^2 \pi Q} \sum_k \beta_k (\Delta B_k / B\rho)^2 L_k^2 \quad (22)$$

There are three possible sources of closed orbit distortions:

(a) Misalignment of quadrupoles: When a quadrupole is misaligned with displacement  $\delta$ , it produces a field error on the equilibrium orbit,  $\Delta B = KB\rho\delta$ , where  $K$  is the quadrupole strength. The random misalignment error would follow a statistical distribution with rms width  $\sigma_{\Delta x}$ , eq. (22) becomes,

$$\begin{aligned} \sigma_u^2 &= \frac{\langle \beta \rangle}{4\sin^2 \pi Q} N_c \frac{L}{\sin \mu} K^2 L_k^2 \sigma_{\Delta x}^2 \\ &= \frac{\tan(\mu/2)}{\pi Q \sin^2 \pi Q} N_c^2 \sigma_{\Delta x}^2 \end{aligned} \quad (23)$$

For the booster, the rms of closed orbit distortion  $\sigma_u \approx 9 \sigma_{\Delta x}$ . The result agrees very well with the numerical simulation of eq. (9). The result may also vary when different random seed is used.



(b) Effective dipole field error: When the effective length of dipole magnets vary from one to the other. The field error can be expressed as  $\Delta B/B$ . We assume that the error also follows a statistical distribution with rms error as  $\sigma_{\Delta B/B}$ . The rms closed orbit distortion becomes,

$$\sigma_{\Delta x}^2 = \frac{\langle \beta \rangle^2}{4 \sin^2 \pi Q} N_d \vartheta^2 \sigma_{\Delta B/B}^2 \quad (24)$$

where  $N_d$  and  $\vartheta$  are the number of dipoles and the bending angle per dipole respectively. Eq.(24) indicates that the rms closed orbit distortion  $\sigma_{\Delta x} \propto 1/\sqrt{N_d}$ . For the booster, we obtain  $\sigma_{\Delta x} [\text{m}] \approx 6\sigma_{\Delta B/B}$ . It agrees also reasonably well with ec.(10) of numerical simulation.

(c) Dipole rotation: When the dipole is rotated with an angle  $\alpha$  with respect to the axis, it creates a horizontal dipole field. The horizontal field error becomes  $\Delta B/B = \sin \alpha \approx \alpha$ . Let the rms dipole rotation be  $\sigma_\alpha$ , the rms closed orbit becomes,

$$\sigma_{\Delta y}^2 = \frac{\langle \beta \rangle^2}{4 \sin^2 \pi Q} N_d \vartheta^2 \sigma_\alpha^2 \quad (25)$$

Thus the sensitivity of the vertical closed orbit due to the dipole rotation is about the same as the horizontal closed orbit due to the dipole error. Eqs.(10) and (11) indicate that the sensitivity is about the same in these two cases. In general, all the errors can happen in the accelerator construction. Since there is no correlation between different types of error, the rms closed orbit becomes rms of the sum of square of each closed orbit error. The distribution function is exponential in  $\langle Y \rangle$  or Gaussian in  $\Delta x$  and  $\Delta y$ , the probability to have  $2.5 \sigma$  would be approximately 5%. Typically, the misalignment error is of the order of <sup>6</sup>

$$\begin{aligned} \Delta X = \Delta Y &\leq 1.4 \cdot 10^{-4} \text{ m} \\ \Delta B/B &\leq 5 \cdot 10^{-4} \\ \Delta \vartheta &\leq 2.4 \cdot 10^{-4} \text{ rad} \end{aligned}$$

We expect therefore the rms of closed orbit would be approximately 2.3 mm before orbit correction. The orbit correction scheme such as MICADO in SYNCH<sup>7</sup> can effectively decrease the closed error by a factor of 5  $\sim$  10. Figs.4 and 5 show an example of the orbit correction. The closed orbit is reduced to about .2mm by 11 and 9 correctors respectively.

Once the orbit is corrected by the MICADO method in SYNCH, the space charge tune depression is obtained by a set of defocusing quadrupoles around the accelerator. The amount of space charge tune shift depends on the current intensity in the accelerator. Fig. 6 shows the behavior of betatron function as a function of tune of the particles. The unperturbed tune of the machine is kept at 4.82. We observe that the betatron functions experience singularity at 4.5 and 4.0 due to the gradient error. The closed orbit functions has also singularity at 4.0. These resonances are important to the beam dynamic. The width of the integer resonance at 4.0 is a factor of 5 larger than that of the half-integer resonance(Figs. 7 and 8). It would be difficult for the particle to survive at a larger space charge tune-shift. Note that the closed orbit functions becomes very large at the integer resonance(Fig. 8).

#### 4. Half-integer stop band width

The half integer resonance is due to the gradient error of quadrupoles, with the resonance width  $J_p$  given by<sup>4</sup>

$$J_p = \int_0^C \beta(s) k(s) \exp(-i p \phi(s)) ds \quad (26)$$

where  $k(s)$  is the gradient perturbation. Assuming that the random gradient error has a zero average with Gaussian distribution, we expect that the probability distribution for the resonance width to be,<sup>8</sup>

$$P(J) = \frac{1}{(2\pi)^{1/2} \sigma_J} \exp(-J^2/2\sigma_J^2) \quad (27)$$

The rms resonance width can therefore be expressed as

$$\begin{aligned} \sigma_J^2 &= \left| \iint_0^C \beta(s) k(s) \beta(s') k(s') \exp(-ip(\phi(s)-\phi(s'))) ds ds' \right| \\ &= \sum_1 \beta_1^2 k_1^2 L_1^2 = \frac{32 \sin^2(\mu/2)}{\sin \mu} N_c (1 + \sin^2(\mu/2)) \sigma_{\Delta G/G}^2 \quad (28) \end{aligned}$$

The sensitivity of the Booster to the gradient error is given by  $\sigma_J \approx 19.8 \sigma_{\Delta G/G}$ . Taking  $\sigma_{\Delta G/G} \approx 5 \cdot 10^{-4}$ , we obtain the half integer stop band width at 4.5 to be  $\sigma_J/(2\pi) \approx .0016$ . This estimation agrees rather well with the SYNCH<sup>7</sup> calculation in Fig.7. The resonance can be corrected by 9th harmonic quadrupole correctors.

Besides the gradient error, the booster has a relatively large eddy current sextupole strength<sup>9</sup>,  $B''/B \approx 1.56 \text{ m}^{-2}$ , which remain constant in the early ramping stage. The dipole has a length of 2.4m. Thus the integrated strength of the sextupole is about  $K_2 = B''L/B\rho \approx 0.27 \text{ m}^{-2}$ . The equation of motion of the particle

in the accelerator with the sextupole is given by

$$x'' + \left( \frac{1}{\rho} - K \right) x = K_2 (x^2 - y^2) \quad (29)$$

$$y'' + K y = -2K_2 xy \quad (30)$$

When there is a misalignment error for these dipoles, the eddy current sextupole would induce effects of random quadrupole field and random skew quadrupole field. The random quadrupole field is given by

$$\Delta K_x = 2K_2 \Delta x$$

$$\Delta K_y = -2K_2 \Delta x$$

while the amount of skew component is proportional to  $K_2 \Delta y$ . Here we shall estimate the effect of the randomly feed-down quadrupole to the half integer stopband width. Using eq.(26) and assuming a random Gaussian distribution for the error distribution, we obtain

$$\sigma_J^2 = \sum_i \beta_{x1}^2 \Delta K_{xi}^2 \simeq 4 \langle \beta \rangle^2 N_d K_2^2 \sigma_{\Delta x}^2$$

where  $N_d$  is the number of dipoles in the accelerator. For the Booster,  $\sigma_J \simeq 22 \sigma_{\Delta x}$  [m]. At a typical value of  $\sigma_{\Delta x} \simeq .14$  mm, we obtain  $\sigma_J \simeq .003$  or the stopband width of  $\sigma_J / (2\pi) \simeq .0005$ . This is a factor of two smaller than that due to the gradient error.

The consideration of half integer stopband width are important only at the injection and the early ramping stage when the space charge tune shift cover the half integer of 4.5. They should be corrected by the quadrupole harmonic correctors.

## 5. Conclusion.

We have studied the effect of the linear space charge tune shift on the particle in the perfect and imperfect machine. The sensitivity of the various types of machine error is examined. Analytic calculation agrees rather well with the numerical simulation of TEAPOT and SYNCH programs. The closed orbit error can be effectively corrected by MICADO or other harmonic correction schemes. A minimum set of 12 dipole correctors are needed to achieve orbit correction. Because of the gradient errors, the half integer resonance at 4.5 should be corrected as well. These quadrupole correctors shall correct the 9th harmonic in the injection period. Our calculation however shows that the integer resonance are far more important than the half integer resonance from the space charge tune-shift point of view. It suggests that the maximum intensity (or space charge tune shift) achievable in the booster should be limited by the integer resonance, where the closed orbit has a singularity due to imperfections. The designed intensity of the booster remains inside the limit of the integer resonance. Similar to AGS, the booster may need also sextupole correction for the third order resonance within the space charge tune spread. Fig.9 shows the tune diagram and resonance lines up to third order.

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## Figures captions:

- Fig.1. The  $\beta$ -function modulation as a function of space charge tune shift for a perfect Booster. We observe that the  $\Delta\beta/\beta$  is about 0.4 at a tune shift of 1.5. The booster is aimed to operate at a tune shift of 0.7 with intensity of  $1.5 \cdot 10^{13}$  ppp of protons.
- Fig.2. Similar to that of figure 1 is shown for AGS. We observe that a smaller  $\delta_{sc}/\delta$  is reflected to weaker focusing in the AGS than the booster. Presently, AGS is operating at  $1.8 \cdot 10^{13}$  ppp with the space charge tune shift of  $.6 \sim .7$ .
- Fig.3. rms closed orbit distortion is shown with different sources of imperfection error. The closed orbit is most sensitive to misalignment error in the quadrupoles. The sensitivity can be expressed in eqs.(9-11). The calculation is based on one random seed. Different random seed may have different sensitivity.
- Fig.4. MICADO method in SYNCH is used to correct the x-orbit.
- Fig.5. Same as that in Fig.4 for y motion.
- Fig.6. The behavior of the betatron functions in the presence of space charge tune shift with the unperturbed tune at 4.82. The resonances at 4.5 and 4.0 are clearly seen.
- Fig.7. Detailed structure of the resonance at 4.5 is shown. The width of the resonance is about .001.
- Fig.8. Detailed view of the resonance at 4.0 is shown. The closed orbit as well as the betatron function have singularity at 4.0.

# Linear Space Charge Effect in Booster

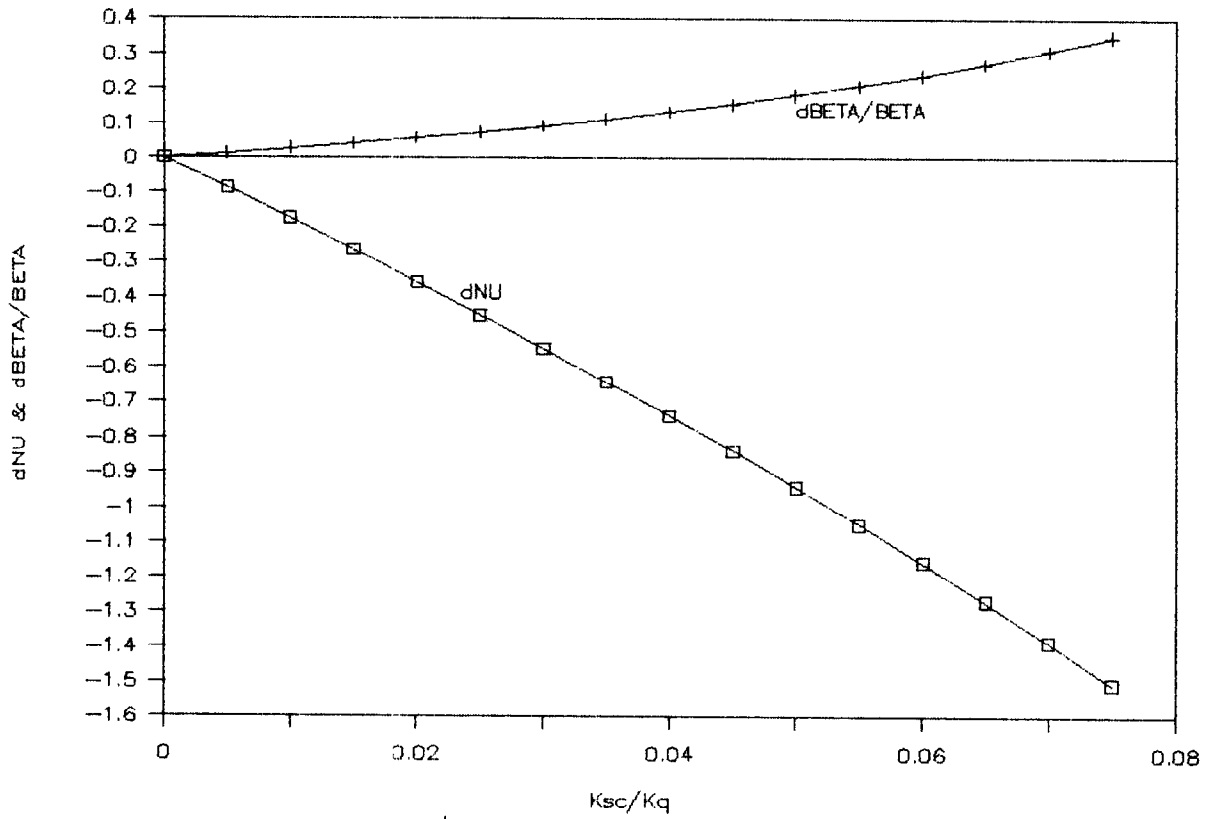


Fig 1

# LINEAR SPACE CHARGE EFFECT OF AGS

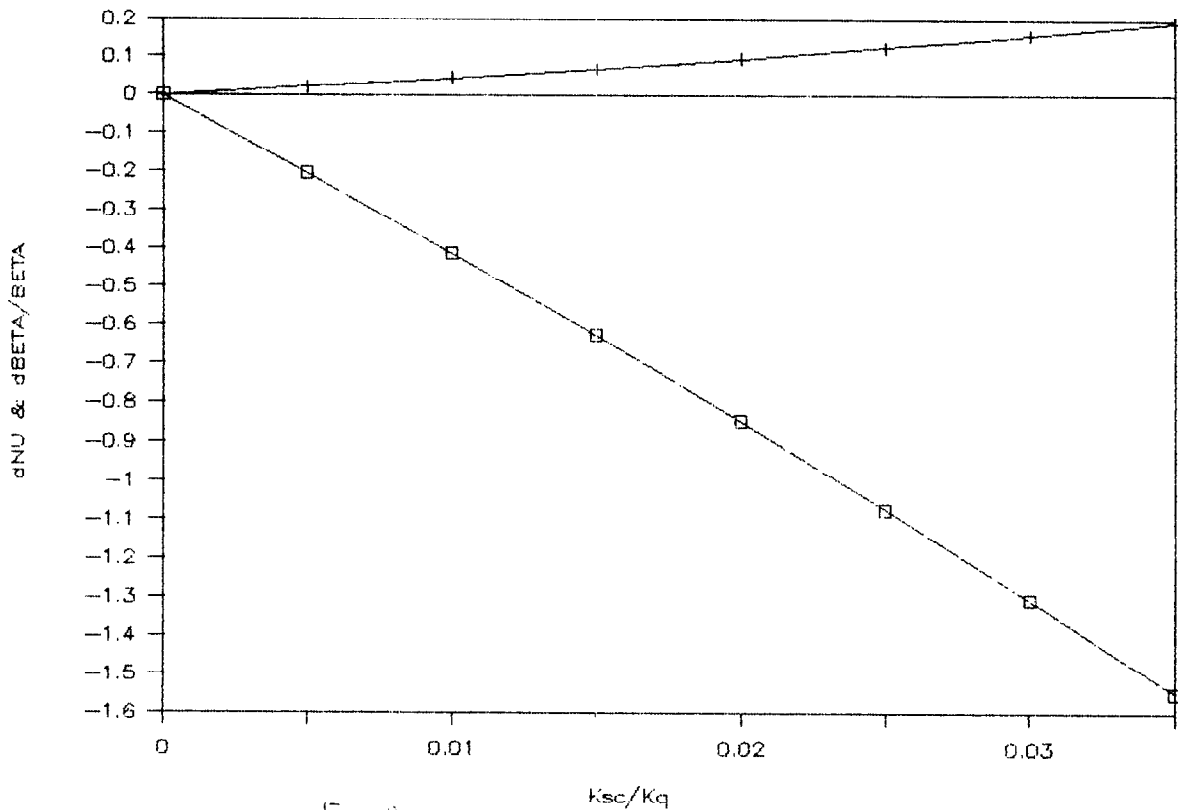


Fig. 2

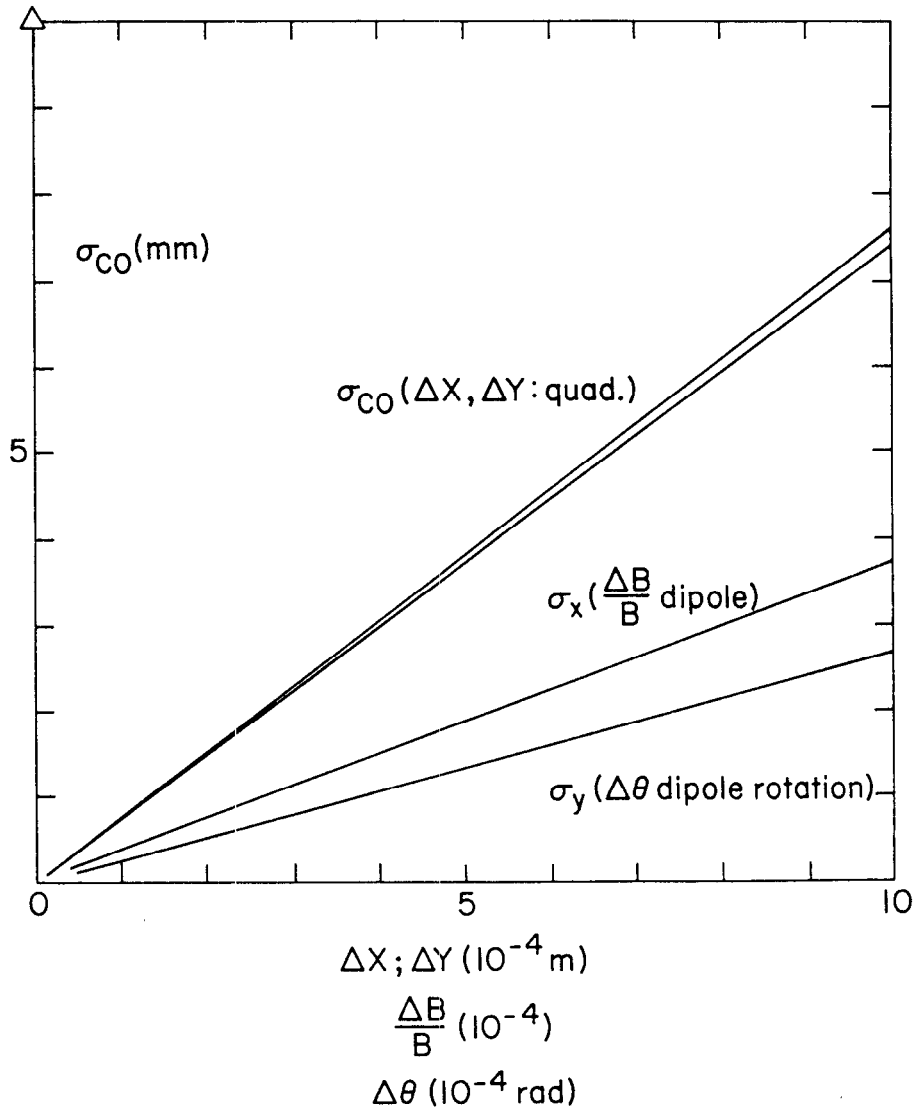


Fig. 3

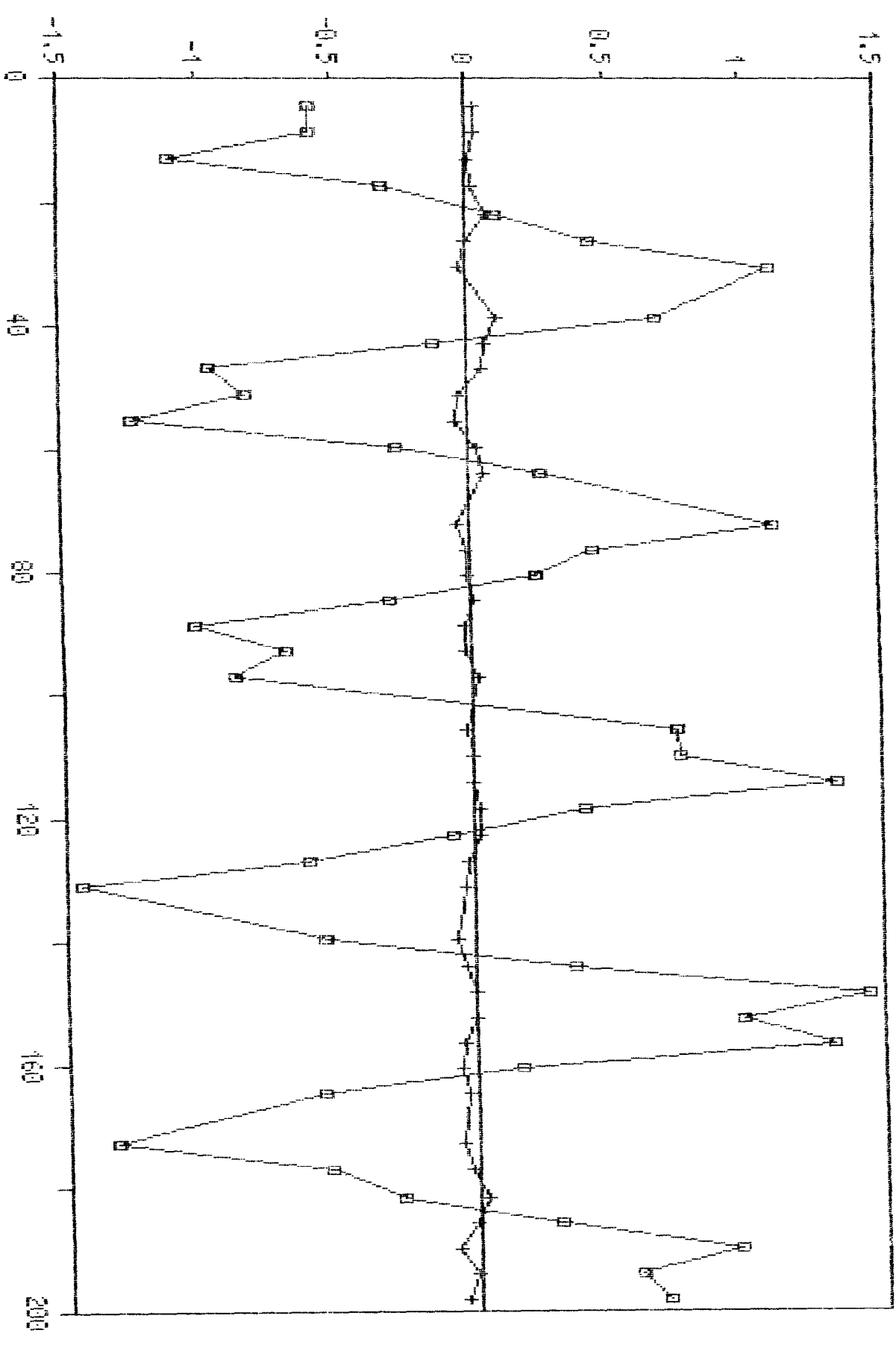


Fig 4

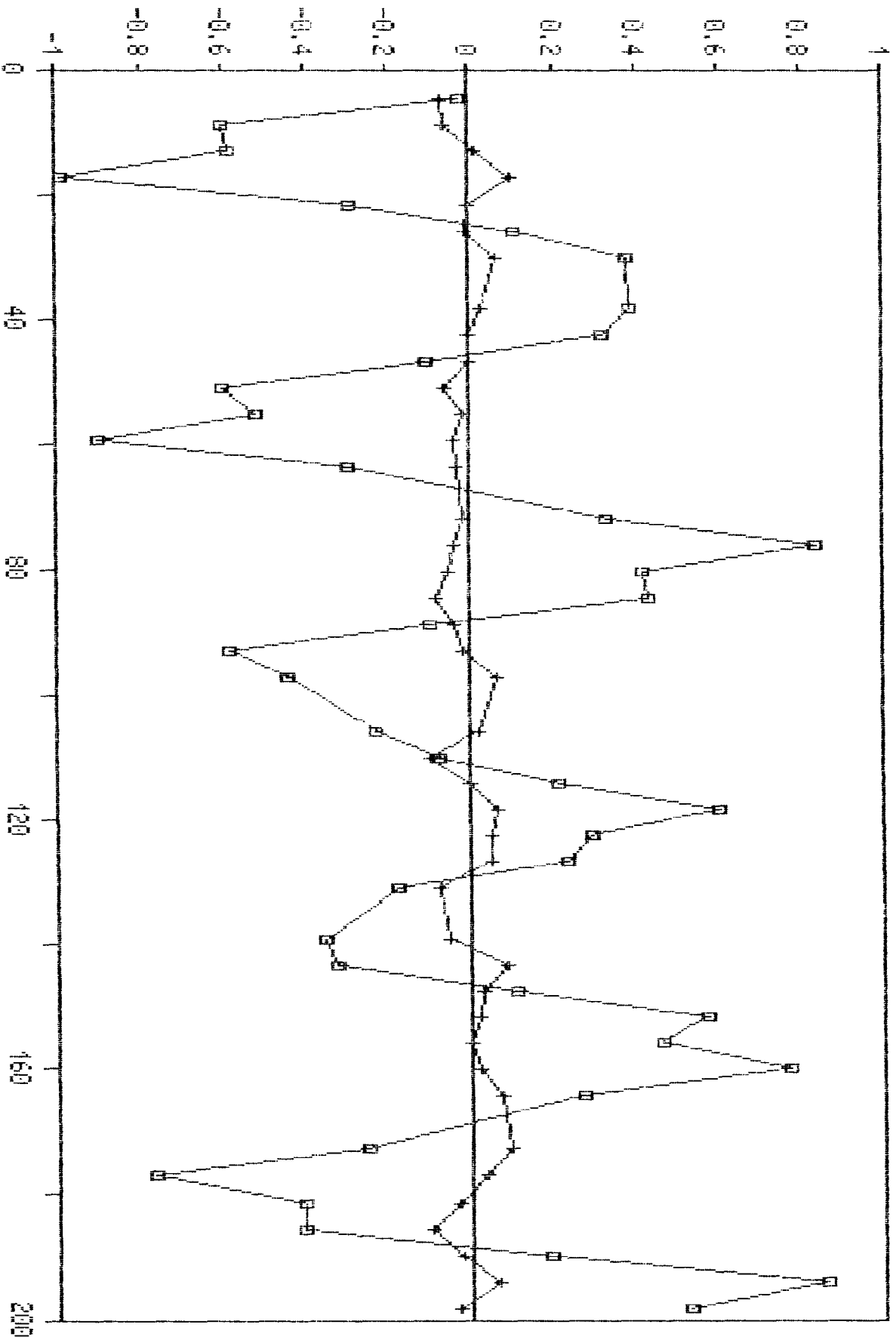


Fig. 5



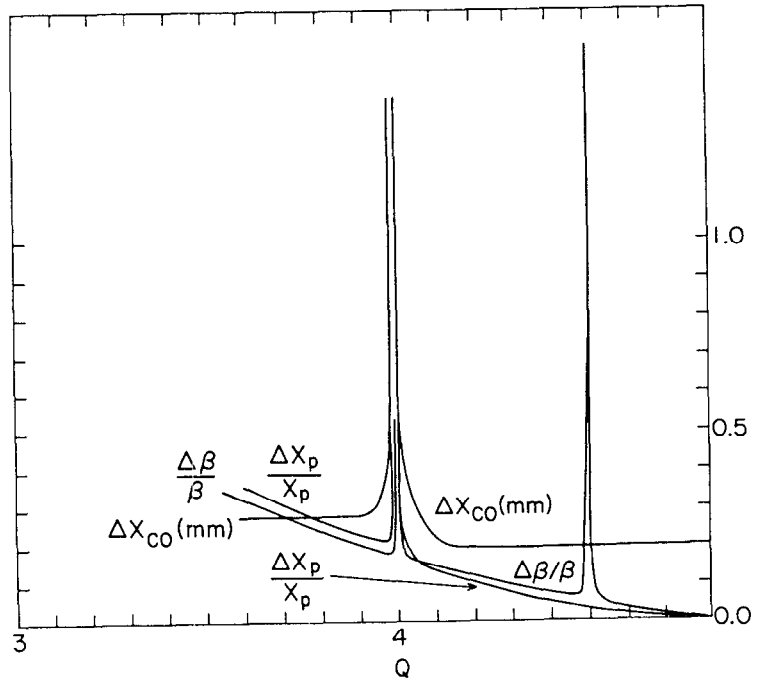


Fig 6

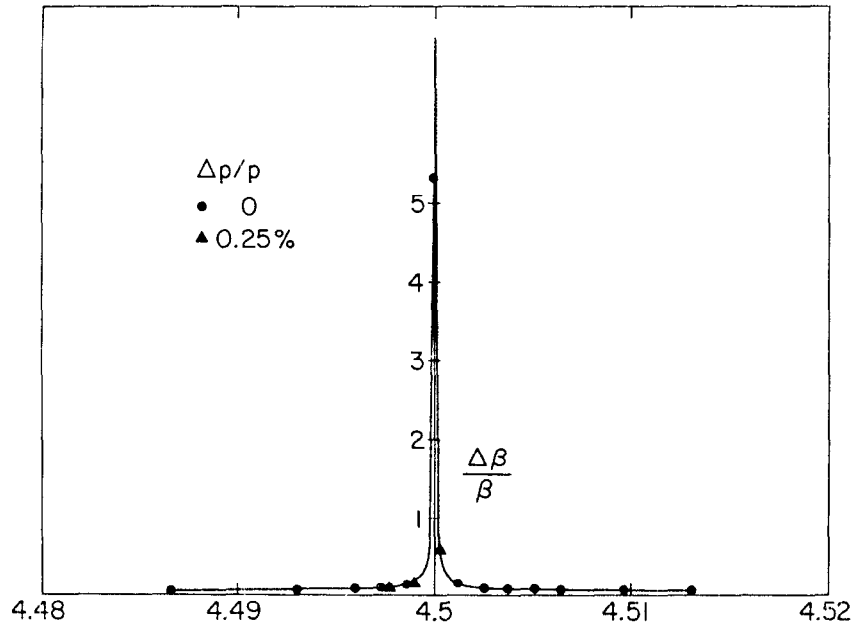


Fig 7

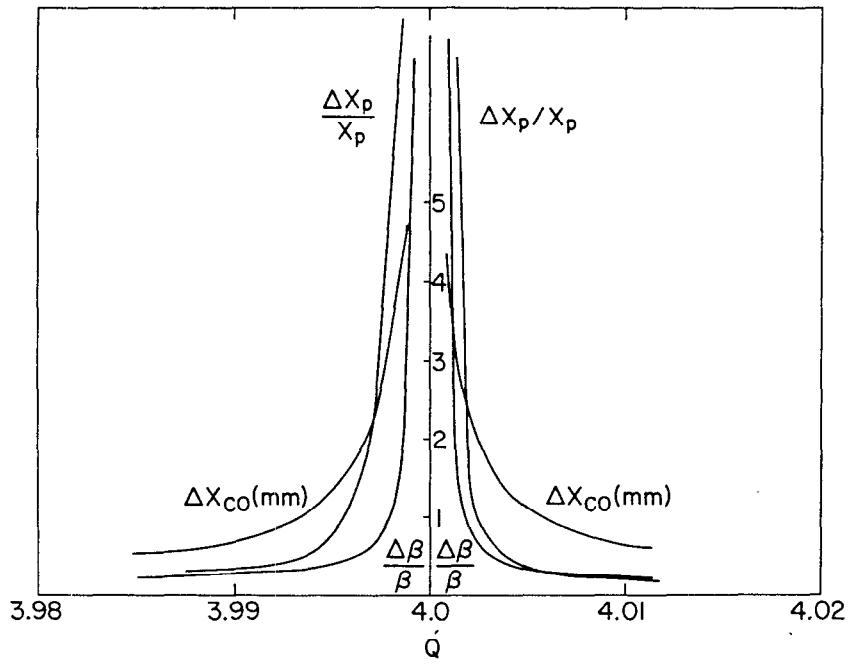


Fig 8

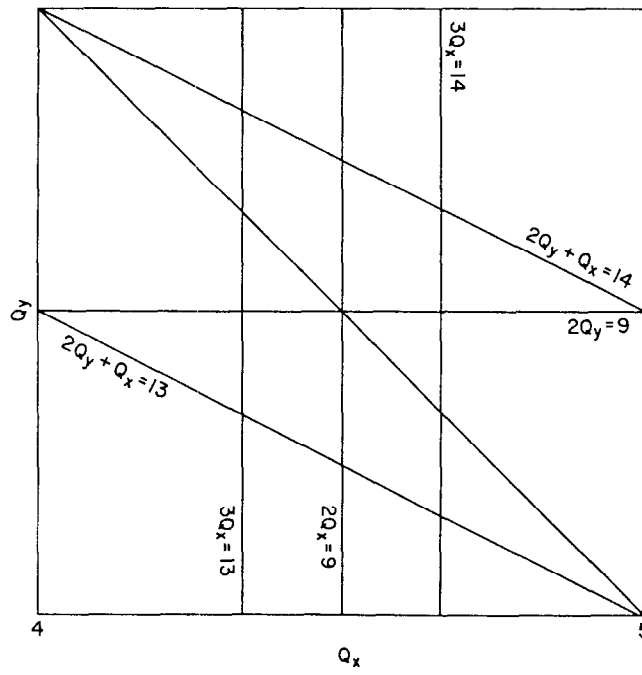


Fig. 9