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Fourth order resonances in the AGS-Booster lattice

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FOURTH ORDER RESONANCES IN THE AGS-BOOSTER LATTICE

AD
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OCTOBER 14, 1986

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FOURTH ORDER RESONANCES
in the AGS BOOSTER LATTICE

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Z. PARSA, S. TEPIKIAN, E. COURANT

I. Introduction:

We have studied the non-linear effects on the dynamical systems including the perturbation of tune; emittance growth; Hamiltonian Resonance strength; generating function resonance strengths; fixed points, Chirikov criteria; Island width, etc. We have incorporated some of our results into an algorithm ("NONLIN") which we used to study the structure resonances in the Booster lattice due to I) eddy current sextupoles and II) the chromaticity correcting sextupoles. For the present Booster Lattice [1] we emphasize the fourth order resonances which are important in the light of the space charge tune shift.

II. Theory:

The Hamiltonian we used to describe the dynamics of a beam in an accelerator is

$$H_o = \frac{1}{2}\left(p_x^2 + p_z^2\right) + \left(\frac{1}{\rho^2(s)} - K(s)\right)\frac{x^2}{2} + K(s)\frac{z^2}{2}$$
$$+ \frac{S(s)}{6}\left(x^3 - 3xz^2\right) + \frac{O(s)}{24}\left(x^4 - 6x^2z^2 + z^4\right) \qquad (1)$$

where x and z are the particle (transverse) position with p_x and p_z being the conjugate momentum, $\rho(s)$ is the bending angle of the dipole field, K(s) is the gradient and S(s) and O(s) are the strengths of the sextupoles and octupoles respectively. The length s the particle travels is the "time variable" of the Hamiltonian.

To find the resonance strengths, we make two series of canonical transformations of the Hamiltonian given in Eq.(1).

The first transformation is an action-angle transformation with angles Φ_x, Φ_z and conjugate momenta J_x, J_z (note, the conjugate momenta are proportional to the beam emittances $2J_x=E_x/\pi$ and $2J_z=E_z/\pi$). Thus, in an accelerator without sextupoles, octupoles, etc., the J_x and J_z are invariants of the motion. The generating function for this transformation is:

$$F\ (x,z,\Phi_x,\Phi_z,s) = -\frac{z^2}{2\beta_z(s)}\left[\tan\ \Phi_z - \frac{\beta_z'(s)}{2}\right] - \frac{x^2}{2\beta_x(s)}\left[\tan\ \Phi_x - \frac{\beta_x'(s)}{2}\right] \qquad (2)$$

where primes represent d/ds and the function $\beta_x(s)$ and $\beta_z(s)$ are solutions to

$$\frac{\beta_x\beta_x''}{2} - \frac{(\beta_x')^2}{4} + \left(\frac{1}{\rho^2} - K(s)\right)\beta_x^2 = 1 \qquad (3)$$

and

$$\frac{\beta_z\beta_z''}{2} - \frac{(\beta_z')^2}{4} + K\ (s)\ \beta_z^2 = 1. \qquad (4)$$

From the generating function we have

$$J_x = \frac{-\partial F}{\partial \Phi_x} \qquad (5)$$

$$J_z = \frac{-\partial F}{\partial \Phi_z} \qquad (6)$$

which implies that

$$x = \sqrt{2 J_x \beta_x(s)} \;\; \text{Cos}\, \phi_x \qquad (7)$$

$$z = \sqrt{2 J_z \beta_z(s)} \;\; \text{Cos}\, \phi_z \qquad (8)$$

The transformed Hamiltonian becomes

$$H_1 = H_o + \frac{\partial}{\partial s} F\,(x, z, \phi_x, \phi_z, s) \qquad (9)$$

or

$$H_1 = \frac{J_x}{\beta_x} + \frac{J_z}{\beta_z} + V\,(J_x, J_z, \phi_x, \phi_z, s) \qquad (10)$$

where the term V comes from the sextupoles and octupoles which is $O(J^{3/2})$.

Our goal in the next transformation is to find the invariant to Hamiltonian Equation (10) to $O(K^{5/2})$ (i.e., a Hamiltonian independent of the angle variables) where K_x and K_z are the (approximate) invariants [3]. The generating function for this transformation becomes [4].

$$\begin{aligned} G\,(K_x, K_z, \phi_x, \phi_z, s) &= \phi_x K_x + \phi_z K_z + K_x^{3/2} w_1(\phi_x, \phi_z, s) \\ &+ K_x^{1/2} K_z\, w_2(\phi_x, \phi_z, s) + K_x^2\, v_1(\phi_x, \phi_z, s) \\ &+ K_x K_z\, v_2(\phi_x, \phi_z, s) + K_z^2\, v_3(\phi_x, \phi_z, s) \end{aligned} \qquad (11)$$

This leads to a Hamiltonian of the form

$$H_2 = \frac{K_x}{\beta_x} + \frac{K_z}{\beta_z} + a(s)K_x^2 + b(s)K_x K_z + c(s)K_z^2 + O(K^{5/2}) \qquad (12)$$

Because of the periodicity of V in Eq.(10) the generating function in Eq.(11) can be expressed:

$$G\,(K_x, K_z, \phi_x, \phi_z, s) = K_x \phi_x + K_z \phi_z + \sum_k \frac{g_k(K_x, K_z, s)}{\text{Sin}\pi\,(n_{x_k}\nu_x + n_{z_k}\nu_z)} \text{Cos}\,(n_{x_k}\phi_x + n_{z_k}\phi_z + \Theta_k) \qquad (13)$$

where g_k (K_x, K_z, s) are the generating function resonance strengths whose magnitude shows to what extent J_x and J_z deviate from the invariants of the motion. The n_{x_k} and n_{z_k} are integers (defining a given resonance) and θ_k is the phase. From the Hamiltonian in Eq. (12) the perturbation on the tune due to sextupoles and octupoles can be found. This may be expressed as:

$$\nu_x = \nu_x^o + 2\alpha_{xx}K_x + 2\alpha_{xx}K_z \tag{14}$$

and

$$\nu_z = \nu_z^o + 2\alpha_{xz}K_x + 2\alpha_{zz}K_z \tag{15}$$

where $\overset{o}{\nu}_x$ and $\overset{o}{\nu}_z$ are the unperturbed tunes and $2K_x$, $2K_z$ are equal to the beam emittance divided by π just before the beam enters the accelerator.

Furthermore, emittance growth can be studied by using the generating function Eq. (13). Since J_x and J_z are related to emittance and

$$J_x = \frac{\partial G}{\partial \phi_x} \tag{16}$$

$$J_z = \frac{\partial G}{\partial \phi_z} \tag{17}$$

then

$$E_x \leqq 2\ \pi \left[K_x + \sum_k\ n_{x_k} \frac{g_k(K_x, K_z, s)}{\text{Sin}\ \pi\ (n_{x_k}\nu_x + n_{z_k}\nu_z)} \right] \tag{18}$$

$$E_z \leqq 2\ \pi \left[K_z + \sum_k\ n_{z_k} \frac{g_k(K_x, K_z, s)}{\text{Sin}\ \pi\ (n_{x_k}\nu_x + n_{z_k}\nu_z)} \right] \tag{19}$$

This estimates the upper limit that emittance may grow to as long as the tunes are far from any resonances.

III. Near Resonance:

The perturbation approach given in the previous section works well when $|n_x\nu_x + n_z\nu_z - p| >> 0$, (i.e. far from resonance), where n_x, n_z and p are integers. In this section, we study the behavior of a dynamical system that is near a given resonance but far from all others. In particular we will find the fixed points of the dynamical system [i.e., the distance from resonance at which there is no motion in a special reference frame where the Hamiltonian is an invariant].

Consider a general Hamiltonian in the form

$$H = \frac{2\pi}{C} J_x + \frac{2\pi}{C} J_z + V\ (J_x,J_z,\Phi_x,\Phi_z,s) \tag{20}$$

where (J_x, Φ_x) and (J_z, Φ_z) are the action angle variables; ν_x and ν_z are the tunes; C is the circumference and V is periodic in Φ_x, Φ_z and s. [Note, the Hamiltonian in Eq.(10) can easily be transformed into this form.]

Expanding V in a fourier series about Φ_x, Φ_z, and s, we find a term in which the argument of the Sine and Cosine term varies the slowest with s. Since this term gives the greatest contribution to the dynamics of the system, we only consider this term and neglect the others. This leads to the following Hamiltonian

$$H \cong \frac{2\pi}{c} \nu_x^o J_x + \frac{2\pi}{c} \nu_z^o J_z + I\ (J_x,J_z)$$

$$+ \frac{1}{c} A\ (J_x,J_z)\ \mathrm{Cos}\ (n_x\Phi_x + n_z\Phi_z - \frac{2\pi}{c} ps + \Theta) \tag{21}$$

Where A (J_x, J_z) is the Hamiltonian Resonance strength; θ is the constant phase and I (J_x, J_z) is the term that causes the perturbation of tune.

We can find the fixed points [4] of the system by defining the bandwidth

$$\delta \equiv n_x \nu_x + n_z \nu_z - p \tag{22}$$

which determines how far the tunes ν_x and ν_z are from the resonance (defined by the integers n_x, n_z and p). Then the system will be on a fixed point if

$$\delta = \pm \frac{1}{2\pi A(J_x, J_z)} \left(n_x \frac{\partial A(Jx, Jz)}{\partial J_x} + n_z \frac{\partial A(J_x, J_z)}{\partial J_z} \right) \tag{23}$$

at the action J_x and J_z.

Furthermore, if we are on a given resonance a criterion [4] (Chirikov criterion) determining whether a nearby resonance is important to the dynamics of the system (and should not have been neglected) can be found. Thus, if δ of a nearby resonance satisfies [4]

$$\delta >> \frac{4}{C} \sqrt{2\pi \left(n_x \frac{\partial I(J_x, J_z)}{\partial J_x} + n_z \frac{\partial I(J_x, J_z)}{\partial J_z} \right) A(J_x, J_z)} \tag{24}$$

then this resonance can be neglected. Otherwise, the resonance must be included to describe the behavior of the system properly. The detailed treatment of the above including stop bandwidths [4,5] and island widths are given in Reference [4].

IV. AGS Booster:

The AGS Booster has six superperiods with an operating point at ν_x=4.82 and ν_z=4.83. At injection, the space charge could cause the tune to shift, thus, crossing at least three structure resonances:

$$4\nu_x = 18$$

$$2\nu_x + 2\nu_z = 18$$

$$4\nu_z = 18$$

Depending on the nature of the space charge effect, the resonance $2\nu_x$-$2\nu_z$=0 may also be crossed. These resonances are excited by the eddy current sextupoles, chromaticity correcting sextupoles, etc.

To study these resonances, we calculated the resonance strengths; the fixed points; stop bandwidths; island widths and Chirikov Criterion. The perturbation to tune and the emittance growth are also calculated. The tunes were varied by changing the quadrupole gradients to simulate a space charge tune shift in order to study the effect of space charge (tune shift) on the stability of the beam. Some of our results are shown in Table I - III.

Table I gives the emittance growth and the perturbation to the betatron tune at the operating point of the Booster (i.e. ν_x=4.82, ν_z=4.83). In Table II the operating tune of the Booster has been changed to ν_x=4.501 and ν_z=4.511, which are the tunes near the fourth order resonances. At these (new) tunes we calculate the perturbation to tune, emittance growth and for the fourth order resonances we calculate the resonance strengths, fixed points, stop bandwidth, Chirikov criterion etc.

Finally, for completeness, we have included in Table III the third and sixth order resonances obtained at the operating tunes of ν_x=4.01 and ν_z=4.11 which are the tunes near the third and sixth order resonances.

V. Conclusion:

The fourth order sum resonance that may give the greatest problem is $4\nu_z=18$ with the largest stop bandwidth of .00924 (see Table II). To determine whether these resonances overlap and may lead to chaotic motion we used the Chirikov criterion [4,6]. Since the operating tunes differ only by .01 (i.e., $\nu_z-\nu_x=.01$) then the four fourth order resonances ($4\nu_z=18$, $2\nu_x-2\nu_z=0$, $2\nu_x+2\nu_z=18$ and $4\nu_x=18$) would be close together and differ in bandwidth by .02. Therefore, with the largest Chirikov criterion of .0019 (due to the $2\nu_x-2\nu_z=0$ resonance) which is smaller than the bandwidth of .02 [4] these resonances would not overlap (or overlap by a small amount).

The third and sixth order resonances are given in Table III. If the fourth order resonances are crossed (depending on the amount of space charge tune shift) we may also need to consider the third and sixth order resonances at the operating tunes of about $\nu_x=4.01$ and $\nu_z=4.11$.

Finally, the comparison of these results (using "NONLIN") with those obtained from Harmon is given in Reference [7].

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TABLE I-A

AGS BOOSTER LATTICE WITH CHROMATICITY SEXTUPOLES AND EDDY CURRENTS

Perturbation of tunes

```
vx = vx0 +    16.21610    * ex +  -6.796116    * ey
vy = vy0 +   -6.796116    * ex +   96.39642    * ey

where the unperturbed betatron tunes are
vx0 =    4.820000     and  vy0 =    4.829999
with circumference =   201.7800
and periodicity =            6

Given a beam with emittances
ex =  6.8300003E-05 and   ey =  6.8300003E-05 (*pi m-rad),
the perturbed tunes become
vx =    4.820643     and   vy =   4.836119

The emittance can grow to
exmax =  1.5738710E-04 at the          7 th element and
eymax =  1.4836394E-04 at the          7 th element.
```

The resonances are numbered as follows:

No.	Resonance		Strength	Stp bndw	Fix pts.	Width	Chirikov cr
1	0 vx +2 vy =	6	6.3595E-08	4.0173E-03	3.8864E-03	9.4313E-06	9.0590E-04
		12	1.6422E-07	9.6176E-03	5.9619E-03	1.4593E-05	1.4017E-03
2	0 vx +4 vy =	18	5.0017E-08	1.1717E-02	5.0466E-03	2.0134E-06	1.5471E-03
		24	1.1310E-07	2.6494E-02	1.2055E-02	3.0275E-06	2.3264E-03
3	1 vx -2 vy =	-6	2.1695E-06	9.5294E-02	9.5294E-02	0.0000E+00	0.0000E+00
		0	5.5394E-07	2.4331E-02	2.4331E-02	0.0000E+00	0.0000E+00
4	1 vx +0 vy =	0	2.7349E-06	4.0042E-02	1.0007E-01	0.0000E+00	0.0000E+00
		6	3.4107E-06	4.9936E-02	1.5680E-02	0.0000E+00	0.0000E+00
5	1 vx +2 vy =	12	4.9297E-07	3.6088E-02	3.6088E-02	0.0000E+00	0.0000E+00
		18	1.4475E-06	1.0596E-01	1.0596E-01	0.0000E+00	0.0000E+00
6	2 vx -4 vy =	-12	8.7514E-08	7.6879E-03	0.0000E+00	5.0498E-06	2.1586E-03
		-6	2.8816E-08	2.5315E-03	0.0000E+00	2.8977E-06	1.2387E-03
7	2 vx -2 vy =	-6	8.3129E-08	4.8684E-03	1.1233E-04	9.0739E-06	1.1411E-03
		0	2.7915E-07	1.6349E-02	4.5492E-04	1.6628E-05	2.0910E-03
8	2 vx +0 vy =	6	1.7416E-07	1.0200E-02	7.8753E-03	3.6640E-05	5.9204E-04
		12	1.0965E-07	6.4219E-03	1.3780E-02	2.9073E-05	4.6978E-04
9	2 vx +2 vy =	18	6.6229E-08	7.7574E-03	7.7574E-03	9.1436E-06	9.0218E-04
		24	1.2006E-07	1.4063E-02	1.4063E-02	1.2311E-05	1.2147E-03
10	2 vx +4 vy =	24	5.1020E-08	1.4940E-02	9.5616E-03	4.1260E-06	1.5402E-03
		30	2.7690E-08	8.1084E-03	5.1894E-03	3.0397E-06	1.1347E-03
11	3 vx +0 vy =	12	1.0024E-07	1.3209E-02	1.3209E-02	0.0000E+00	0.0000E+00
		18	2.5222E-07	3.3236E-02	3.3236E-02	0.0000E+00	0.0000E+00
12	4 vx -2 vy =	6	9.8755E-09	8.6754E-04	5.7836E-04	1.2797E-06	4.8060E-04
		12	7.7737E-09	6.8290E-04	4.5527E-04	1.1354E-06	4.2640E-04
13	4 vx +0 vy =	18	1.7241E-08	4.0388E-03	2.7327E-03	2.8820E-06	3.7255E-04
		24	2.9671E-08	6.9508E-03	2.7903E-03	3.7809E-06	4.8874E-04
14	4 vx +2 vy =	24	7.6268E-09	2.2333E-03	1.3400E-03	1.3333E-06	3.5625E-04
		30	3.6061E-09	1.0559E-03	6.3357E-04	9.1678E-07	2.4496E-04
15	6 vx +0 vy =	24	1.4463E-09	7.6234E-04	5.0823E-04	3.7100E-07	1.6186E-04
		30	7.0586E-10	3.7205E-04	2.4803E-04	2.5918E-07	1.1307E-04

TABLE I-B

The generating function resonance strength for the resonances:

Elements	Resonance numbers 1	2	3	4	5
7	4.83116E-07	9.47109E-08	2.89733E-06	8.44156E-06	1.05534E-06

Elements	Resonance numbers 6	7	8	9	10
7	2.13174E-07	3.37781E-07	6.01788E-07	1.10687E-07	3.26206E-08

Elements	Resonance numbers 11	12	13	14	15
7	3.62193E-07	2.00919E-08	5.62063E-08	6.42469E-09	2.38764E-09

TABLE II-A

AGS BOOSTER LATTICE WITH CHROMATICITY SEXTUPOLES AND EDDY CURRENTS

Perturbation of tunes

```
vx = vx0 +   10.75297    * ex +  -10.54684    * ey
vy = vy0 +  -10.54684    * ex +   83.98236    * ey
```

where the unperturbed betatron tunes are
vx0 = 4.500787 and vy0 = 4.511137
with circumference = 201.7800
and periodicity = 6

Given a beam with emittances
ex = 6.8300003E-05 and ey = 6.8300003E-05 (*pi m-rad),
the perturbed tunes become
vx = 4.500801 and vy = 4.516153

The emittance can grow to
exmax = 2.1763804E-04 at the 7 th element and
eymax = 1.6418037E-04 at the 7 th element.

The resonances are numbered as follows:

No.	Resonance		Strength	Stp bndw	Fix pts.	Width	Chirikov cr
1	0 vx +2 vy =	6	1.0474E-07	6.1341E-03	6.4135E-03	1.2486E-05	1.0449E-03
		12	1.4297E-07	8.3730E-03	5.0252E-03	1.4587E-05	1.2207E-03
2	0 vx +4 vy =	18	3.9424E-08	9.2355E-03	3.7297E-03	1.9151E-06	1.2821E-03
		24	1.0600E-07	2.4833E-02	1.1321E-02	3.1402E-06	2.1023E-03
3	1 vx -2 vy =	-6	2.3426E-06	1.0290E-01	1.0290E-01	0.0000E+00	0.0000E+00
		0	5.0439E-07	2.2155E-02	2.2155E-02	0.0000E+00	0.0000E+00
4	1 vx +0 vy =	0	2.7076E-06	3.9643E-02	1.0012E-01	0.0000E+00	0.0000E+00
		6	3.6077E-06	5.2822E-02	1.4688E-02	0.0000E+00	0.0000E+00
5	1 vx +2 vy =	12	5.2313E-07	3.8297E-02	3.8297E-02	0.0000E+00	0.0000E+00
		18	1.5332E-06	1.1224E-01	1.1224E-01	0.0000E+00	0.0000E+00
6	2 vx -4 vy =	-12	8.0065E-08	7.0335E-03	0.0000E+00	5.0731E-06	1.9658E-03
		-6	2.5632E-08	2.2517E-03	0.0000E+00	2.8704E-06	1.1122E-03
7	2 vx -2 vy =	-6	7.9474E-08	4.6544E-03	6.6012E-05	9.2610E-06	1.0689E-03
		0	2.5317E-07	1.4827E-02	7.9767E-04	1.6529E-05	1.9077E-03
8	2 vx +0 vy =	6	1.5711E-07	9.2011E-03	7.6746E-03	4.2736E-05	4.5790E-04
		12	1.0600E-07	6.2079E-03	1.3137E-02	3.5103E-05	3.7612E-04
9	2 vx +2 vy =	18	5.2226E-08	6.1173E-03	6.1173E-03	9.4154E-06	6.9089E-04
		24	1.1415E-07	1.3370E-02	1.3370E-02	1.3920E-05	1.0214E-03
10	2 vx +4 vy =	24	4.7999E-08	1.4055E-02	8.9955E-03	4.4390E-06	1.3468E-03
		30	4.2864E-08	1.2552E-02	8.0332E-03	4.1948E-06	1.2728E-03
11	3 vx +0 vy =	12	1.0208E-07	1.3451E-02	1.3451E-02	0.0000E+00	0.0000E+00
		18	2.6010E-07	3.4274E-02	3.4274E-02	0.0000E+00	0.0000E+00
12	4 vx -2 vy =	6	1.1453E-08	1.0061E-03	6.7073E-04	1.4545E-06	4.9039E-04
		12	6.9568E-09	6.1114E-04	4.0743E-04	1.1336E-06	3.8220E-04
13	4 vx +0 vy =	18	1.5485E-08	3.6275E-03	2.6153E-03	3.3542E-06	2.8751E-04
		24	2.9079E-08	6.8120E-03	2.7054E-03	4.5964E-06	3.9399E-04
14	4 vx +2 vy =	24	7.6555E-09	2.2417E-03	1.3450E-03	1.6796E-06	2.8386E-04
		30	5.7814E-09	1.6929E-03	1.0158E-03	1.4596E-06	2.4668E-04
15	6 vx +0 vy =	24	5.5785E-10	2.9403E-04	1.9602E-04	2.8295E-07	8.1856E-05
		30	1.6100E-09	8.4861E-04	5.6574E-04	4.8069E-07	1.3906E-04

TABLE II-B

The generating function resonance strength for the resonances:

Elements	Resonance numbers 1	2	3	4	5
7	3.95571E-07	7.18872E-08	3.25194E-06	9.45146E-06	6.95426E-07

Elements	Resonance numbers 6	7	8	9	10
7	1.86428E-07	3.19985E-07	5.20099E-07	7.16975E-08	4.61389E-08

Elements	Resonance numbers 11	12	13	14	15
7	3.27805E-07	1.80824E-08	2.35686E-08	9.45285E-09	3.92694E-09

TABLE III-A

AGS BOOSTER LATTICE WITH CHROMATICITY SEXTUPOLES AND EDDY CURRENTS

Perturbation of tunes

```
vx = vx0 +  -15.06851    * ex +  -27.55171    * ey
vy = vy0 +  -27.55171    * ex +   63.27081    * ey

where the unperturbed betatron tunes are
vx0 =    4.009855     and  vy0 =    4.110233
with circumference =   201.7800
and periodicity =            6

Given a beam with emittances
ex =  6.8300003E-05 and  ey =  6.8300003E-05 (*pi m-rad),
the perturbed tunes become
vx =    4.006944     and  vy =   4.112673

The emittance can grow to
exmax =  1.3557221E-04 at the          7 th element and
eymax =  1.0188754E-04 at the         13 th element.
```

The resonances are numbered as follows:

No.	Resonance		Strength	Stp bndw	Fix pts.	Width	Chirikov cr
1	0 vx +2 vy =	6	4.8429E-07	2.8363E-02	3.3819E-02	3.0932E-05	1.9501E-03
		12	2.3815E-07	1.3947E-02	1.2224E-02	2.1691E-05	1.3675E-03
2	0 vx +4 vy =	12	3.6595E-08	8.5728E-03	5.3493E-03	2.1257E-06	1.0721E-03
		18	1.6903E-07	3.9596E-02	2.6900E-02	4.5685E-06	2.3042E-03
3	1 vx -2 vy =	-6	2.6343E-06	1.1571E-01	1.1571E-01	0.0000E+00	0.0000E+00
		0	4.3900E-07	1.9283E-02	1.9283E-02	0.0000E+00	0.0000E+00
4	1 vx +0 vy =	0	2.7207E-06	3.9834E-02	1.0185E-01	0.0000E+00	0.0000E+00
		6	3.9276E-06	5.7506E-02	1.1941E-02	0.0000E+00	0.0000E+00
5	1 vx +2 vy =	12	5.6479E-07	4.1346E-02	4.1346E-02	0.0000E+00	0.0000E+00
		18	1.6836E-06	1.2325E-01	1.2325E-01	0.0000E+00	0.0000E+00
6	2 vx -4 vy =	-12	8.3445E-08	7.3304E-03	0.0000E+00	5.4730E-06	1.8990E-03
		-6	2.5668E-08	2.2549E-03	0.0000E+00	3.0354E-06	1.0532E-03
7	2 vx -2 vy =	-6	1.3219E-07	7.7415E-03	4.5655E-03	1.2647E-05	1.3019E-03
		0	3.0870E-07	1.8079E-02	2.1878E-03	1.9327E-05	1.9895E-03
8	2 vx +0 vy =	6	2.8459E-07	1.6667E-02	3.5616E-02	4.8588E-05	7.2954E-04
		12	5.7359E-07	3.3592E-02	6.6191E-02	6.8979E-05	1.0357E-03
9	2 vx +2 vy =	12	1.4958E-07	1.7521E-02	1.7521E-02	5.2052E-05	3.5794E-04
		18	1.7315E-07	2.0282E-02	2.0282E-02	5.6003E-05	3.8511E-04
10	2 vx +4 vy =	24	6.0754E-08	1.7790E-02	1.1386E-02	7.7084E-06	9.8169E-04
		30	1.9002E-07	5.5642E-02	3.5611E-02	1.3632E-05	1.7361E-03
11	3 vx +0 vy =	12	1.0322E-07	1.3602E-02	1.3602E-02	0.0000E+00	0.0000E+00
		18	2.8548E-07	3.7618E-02	3.7618E-02	0.0000E+00	0.0000E+00
12	4 vx -2 vy =	6	2.1119E-07	1.8553E-02	1.2369E-02	7.6355E-06	1.7226E-03
		12	3.5281E-08	3.0994E-03	2.0662E-03	3.1208E-06	7.0406E-04
13	4 vx +0 vy =	12	5.5297E-07	1.2954E-01	1.2237E-01	1.6932E-05	2.0339E-03
		18	1.2508E-07	2.9302E-02	1.6212E-02	8.0530E-06	9.6732E-04
14	4 vx +2 vy =	24	4.8961E-08	1.4337E-02	8.6022E-03	3.7777E-06	8.0714E-04
		30	1.3221E-07	3.8716E-02	2.3230E-02	6.2079E-06	1.3264E-03
15	6 vx +0 vy =	24	2.3212E-08	1.2235E-02	8.1566E-03	1.5418E-06	6.2506E-04
		30	6.5139E-08	3.4334E-02	2.2889E-02	2.5828E-06	1.0471E-03

TABLE III-B

The generating function resonance strength for the resonances:

Elements	Resonance numbers 1	2	3	4	5
7	3.98750E-07	2.48892E-07	3.76435E-06	1.09678E-05	5.31317E-07
13	7.56020E-07	2.59314E-07	3.39057E-06	9.20502E-06	6.02314E-07

Elements	Resonance numbers 6	7	8	9	10
7	1.72638E-07	4.27212E-07	9.83616E-07	2.66968E-07	7.05816E-08
13	1.68880E-07	4.08064E-07	1.21693E-06	2.58807E-07	8.81565E-08

Elements	Resonance numbers 11	12	13	14	15
7	6.31989E-07	2.91695E-07	1.24378E-06	4.63640E-08	2.27003E-08
13	6.52584E-07	2.69279E-07	1.06482E-06	5.52193E-08	2.43489E-08