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Comment on systematic resonances, space charge, and periodicity

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Collider Accelerator Department Brookhaven National Laboratory

## **U.S. Department of Energy**

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# COMMENTS ON SYSTAMATIC RESONANCES, SPACE CHARGE AND PERIODICITY

Booster Technical Note No. 48

> A. G. RUGGIERO JULY 1, 1986

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## COMMENTS ON SYSTAMATIC RESONANCES, SPACE CHARGE AND PERIODICITY

A. G. RUGGIERO JULY 1, 1986

BOOSTER AT INJECTION	PROTONS
Kinetic Energy	200 MeV
Betatron Acceptance (Vert)	50 π mm.mrad
Intensity	1.5 x 10 <sup>1</sup> 3 in 3 bunches

#### CONCERNS:

Space Charge Limit

Eddy Currents

Chromatic Sextupoles

Systematic Resonances

### "STANDARD" LATTICE:

Periodicity 6 (see later)

### $Q_{\rm H}$ = 4.82, $Q_{\rm V}$ = 4.83

### BEAM DIMENSIONS

$$\epsilon_{\rm H} = \epsilon_{\rm V} = 50~\pi$$
 mm.mrad (full)  
 $\Delta p/p = \pm 2.5~^0/_{00}$  (full)

	QF	QD
β <sub>H</sub>	13.865m	3.5754m
β <sub>v</sub>	3.7033	13.644
Х <sub>р</sub>	2.9515	0.54004



а	26.12mm	13.37mm
b	13.61	26.33
d	7.38	1.35
$\sqrt{a^2 + d^2}$	27.14	13.44

The beam is <u>"round"</u>

$\frac{N_{s.c.}}{\varepsilon_N} = (\beta \gamma^2) \frac{4B_f}{3r_0 F} \Delta Q$	ε <sub>N</sub> = (βΥ) ε
Kinetic Energy	200 MeV
β	0.56616
Ŷ	1.2132
	1 505 10-18
r.o	1.535 X 10 '' m
F	1
₿ <sub>f</sub>	0.5

(round beam)

SPACE CHARGE LIMIT

•

ΔQ N <sub>s</sub> .c./ε <sub>N</sub>	0.250 9 x 10 <sup>16</sup>	m-1	0.375 13.5 x 10	16 m <sup>-1</sup>
$\epsilon_{\rm H} = \epsilon_{\rm V}$ $\pi$ mm.mrad	50	75	50	75
N <sub>s.c.</sub>	1.0x10 <sup>13</sup>	1.5x10 <sup>1</sup> 3	1.5x10 <sup>13</sup>	2.25x10 <sup>13</sup>

	$Q_{\rm H} = 4.82$		Periodicity = $6$
	$Q_{v} = 4.83$		
		Р	$n Q_{H} + m Q_{V} - p$ n + m
	Q <sub>H</sub>	6	1.180
	3Q <sub>H</sub>	12	0.820
	$Q_{\rm H}$ + 2 $Q_{\rm V}$	12	0.827
	$Q_{\rm H}$ - 2 $Q_{\rm V}$	-6	0.387 .
•	2Q <sub>H</sub>	12	1.180
•	2Q <sub>v</sub>	12	1.170
•	4 <sub>QH</sub>	18	0.320 .
•	4Q <sub>v</sub>	18	0.330 .
•	$2Q_{H} + 2Q_{v}$	18	0.325 .
•	$2Q_{\rm H}$ - $2Q_{\rm V}$	0	0.005 *
	$4Q_{H} + 2Q_{V}$	30	0.177 +
	$4Q_{\rm H}$ - $2Q_{\rm V}$	12	0.397 .
	6Q <sub>H</sub>	30	0.180 +
	2Q <sub>H</sub> + 4Q <sub>V</sub>	30	0.173
	$2Q_{\rm H} - 4Q_{\rm V}$	<del>-</del> 12	0.387 .

#### CHROMATICITY, SEXTUPOLES AND EDDY CURRENTS

Natural Chromaticity (H,V) (without eddy currents)		<b>-</b> 5	
With Eddy Currents:	εH	+3	(+8)
	ε <sub>v</sub>	-13	(-8)

### SEXTUPOLES STRENGTH TO CANCEL CHROMATICITY WITH EDDY CURRENT

SF	~ 0.1
SD	~ 1.0

Very asymmetric

Very large average contribution (from E.C.) which enhances  $2{\tt Q}_{\rm H}$  -  $2{\tt Q}_{\rm V}$  = 0 resonance.

SUGGESTION: Compensation of Eddy Currents with pole face windings in Dipole Magnets.

## SPACE CHARGE

$$p = \frac{Nf(z)}{2\pi \sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

$$\sum 2 \phi = -4 \pi \rho$$

$$E_r = -\frac{\partial \phi}{\partial r} = -\frac{2N}{\sigma^2 r} f(z) \int_0^r ue^{-\frac{u^2}{2\sigma^2}} du$$

$$\phi = \frac{2N}{\sigma^2} f(z) \int_0^r \frac{dr^1}{r^1} \int_0^{r^1} ue^{-\frac{u^2}{2\sigma^2}} du$$
constraint  $\left\{x r^2 - \frac{r^4}{8\sigma^2} + \cdots \right\}_{r^4}$ 

$$r^4 = (x^2 + y^2)^2 = (x^4 + 2x^2y^2 + y^4)$$

ordinary octupole  $--------x^4 - 6x^2y^2 + y^4$ 

≈

## FRANK SACHERER (1971)

RMS Beam Envelope Equations 
$$(\tilde{x}, \tilde{y})$$
  

$$\begin{cases} \tilde{x}^{\prime\prime} + k_{X} \tilde{x} = \frac{E_{X}^{2}}{\tilde{x}^{3}} + \frac{g}{\tilde{x} + \tilde{y}} \\ \\ \tilde{y}^{\prime\prime} + k_{y} \tilde{y} = \frac{E_{y}^{2}}{\tilde{y}^{3}} + \frac{g}{\tilde{x} + \tilde{y}} \\ \\ \\ g = \frac{Nr_{O}}{2\pi R \ Bf \ \beta^{2} \gamma^{3}} \end{cases}$$

Floquet tranformation:

$$\eta = \frac{\tilde{x}}{\sqrt{\beta_x}} \qquad \zeta = \frac{\tilde{y}}{\sqrt{\beta_y}} \qquad ds = Q \beta_H d \phi$$

$$\eta'' + Q^{2} \eta = Q^{2} \beta_{X}^{3} {}^{2} \frac{E_{X}^{2}}{\tilde{x}^{3}} + Q^{2} \beta_{X}^{3} {}^{2} \frac{g}{\tilde{x} + \tilde{y}}$$

$$\zeta'' + Q^{2} \zeta = Q^{2} \beta_{Y}^{3} {}^{2} \frac{\frac{g}{\tilde{y}^{2}}}{\tilde{y}^{3}} + Q^{2} \beta_{Y}^{3} {}^{2} \frac{g}{\tilde{x} + \tilde{y}}$$

For 
$$g = 0$$
  $E_x = \frac{\tilde{x}^2}{\beta_x}$ ,  $E_y = \frac{\tilde{y}^2}{\beta_y}$ 

For  $g \neq 0$ , neglect  $\eta''$ ,  $\zeta''$  take  $\beta_X \sim \beta_y \sim \overline{\beta}$ 

$$\int \eta = \frac{E_{x}^{2}}{\eta^{3}} + \frac{\overline{\beta} g}{\eta + \zeta}$$
$$\zeta = \frac{E_{y}^{2}}{\zeta^{3}} + \frac{\overline{\beta} g}{\eta + \zeta}$$

•

"Round" Beam  $E_X = E_y = E$  then  $\eta = \zeta$ 

$$n^{4} - \bar{\beta}gn^{2} - E^{2} = 0$$

$$\frac{n^{2}}{E} = \sqrt{1 + \alpha^{2}} + \alpha$$

$$\alpha = \frac{\overline{\beta}g}{2E} = \frac{\overline{\beta}Nr_0}{4\pi REBf\beta^2\gamma^3}$$

 $N = 1.5 \times 10^{13}$  $r_0 \approx 1.535 \times 10^{18} m$  $\bar{B} = 8m$  $\alpha = 0.1$  $2\pi R = 201.78m$  $B_{f} = 0.5$  $\beta^2 \gamma^3 = 0.57232$ n<sup>2</sup> 50 Е

$$= \frac{50}{-1} = 1.1$$
  
3 ( $\pi$ ) mm.mrad E

10% Dihition

### PERIODICITY

, -

	STANDARD	COMBINED
"Nominal"	6	12
$\beta$ - functions, (quads)	24	12
Dispersion	6	12
Dipole, Eddy Currents	8	12
Sextupole Arrangement	6 - 24	12

Quite Acceptable

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FIG. 1 The Amplitude and Dispersion Functions of the Booster Lattice.







- FOCUSING QUADRUPOLE
   DEFOCUSING QUADRUPOLE
   BENDING MAGNET (DIPOLE)
- FIG. 2 a) Schematic Diagram of the Booster and
  - b) Components of the Superperiod

X = SEXTUPULE



Frank J. Sacherer CERN, Geneva, Switzerland

#### Summary

Envelope equations for a continuous beam with uniun charge density and elliptical cross-section were irst derived by Kapchinsky and Vladimirsky<sup>2</sup>(K-V). In act, the K-V equations are not restricted to uniformly barged beams, but are equally valid for any charge disribution with elliptical symmetry, provided the beam wundary and emittance are defined by rms (root-meanguare) values. This results because (i) the second ments of any particle distribution depend only on the inear part of the force (determined by least squares sthod), while (ii) this linear part of the force in urn depends only on the second moments of the distribuion. This is also true in practice for three-dimenional bunched beams with ellipsoidal symmetry, and glows the formulation of envelope equations that in-Jude the effect of space charge on bunch length and mergy spread.

The utility of this rms approach was first demonstrated by Lapostolle<sup>3</sup> for stationary distributions. subsequently, Gluckstern' proved that the rms version if the K-V equations remain valid for all continuous seams with axial symmetry. In this report these rejults are extended to continuous beams with elliptical symmetry as well as to bunched beams with ellipsoidal form, and also to one-dimensional motion.

#### Moment equations

Consider an ensemble of particles that obey the single-particle equations

$$\dot{\mathbf{x}} = \mathbf{p}$$
  
 $\dot{\mathbf{p}} = \mathbf{F}(\mathbf{x}, \mathbf{t})$ , (1)

where F(x,t) includes both the external force and the self-force,  $F = F_e + F_s$ . Averaging (1) over an arbitrary particle distribution f(x,p,t), we obtain

$$\overline{\mathbf{x}} = \overline{\mathbf{p}}$$

$$\frac{\mathbf{p}}{\mathbf{p}} = \overline{\mathbf{F}} = \overline{\mathbf{F}}_{\mathbf{p}} , \qquad (2)$$

where the last equation follows because  $\overline{F}_s = 0$  by 'ewton's third law. (We neglect the small magnetic self- where  $\sigma$  is the covariance matrix forces due to internal motion.) If  $F_e(x,t)$  is non-linear in x, the second equation of (2) involves the higher ments  $\overline{x^n}$  of the distribution. However, for linear external forces,  $F_e = -K(t)x$ , equations (2) involve only the first moments x and p, and therefore the centre-oflass motion depends only on the external force,

$$\overline{\mathbf{x}} + \mathbf{K}(\mathbf{t})\overline{\mathbf{x}} = 0 , \qquad (3)$$

and not on the detailed form of the distribution. In the remainder of this paper we consider only linear external forces.

$$\overline{x^2} = 2 \overline{xx} = 2 \overline{xp}$$

$$\overline{xp} = \overline{xp} + \overline{xp} = \overline{p^2} - K(t)\overline{x^2} + \overline{xF_s}$$

$$\overline{p^2} = 2 \overline{pp} = -2K(t)\overline{xp} + 2 \overline{pF_s},$$
(4)

where the terms  $\overline{xF}_s$  and  $\overline{pF}_s$  are usually functions of the higher moments  $\overline{x^n}$  and  $\overline{x^np}$ . This is a general feature of

moment equations, namely the equation for each moment involves the higher moments in an endless hierarchy. However, if the self-force is derived from the freespace Poisson equation, xFs depends mainly on the second moments and very little, if at all, on the higher moments. This will be demonstrated in the following sections. The remaining term  $\overline{pF}_s$  is associated with emittance growth; we will avoid considering it by assuming that the rms emittance

$$E = \sqrt{\frac{1}{x^2} p^2 - \frac{1}{xp^2}}$$
(5)

is either constant, or that its time dependence is known a priori. Then  $p^2$  is given in terms of  $x^2$ , xp, and E(t) by (5), and the first two equations of (4) form a closed set. They can be combined to give the K-V type equation:

$$\dot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{\tilde{x}\tilde{F}_s}{\tilde{x}} = 0 , \qquad (6)$$

where  $\tilde{x}$  is the rms value,  $\tilde{x} = \sqrt{\overline{x^2}}$ .

The space-charge term in this equation has an interesting interpretation. If we define the linear part of the force  $F_{s}(x,t)$  as  $\varepsilon(t)x$ , where  $\varepsilon(t)$  is determined by minimizing the difference

$$D = \int \left[ \varepsilon(t) x - F_{s}(x,t) \right]^{2} n(x,t) dx$$
 (7)

for a fixed t, where  $n(x,t) = \int f(x,p,t) dp$ , then

$$\varepsilon(t)x = \frac{\overline{xF}_{s}}{\overline{x}^{2}} x . \qquad (8)$$

In other words, the rms envelope equation depends only on the linear part of the forces, determined by least squares method.

It is convenient to put equation (4) into matrix form. The assumption of constant rms emittance is equivalent to setting  $\overline{pF}_S = \varepsilon(t)\overline{xp}$ . Then equation (4) has the form

$$\dot{\sigma} = F\sigma + \sigma F^{\mathrm{T}} \tag{9}$$

$$\sigma = \begin{bmatrix} \overline{x^2} & \overline{xp} \\ \\ \overline{xp} & \overline{p^2} \end{bmatrix}$$
(10)

and F is

$$F = \begin{bmatrix} 0 & 1 \\ -K(t) + \varepsilon(t) & 0 \end{bmatrix} .$$
 (11)

The second moments of f(x,p,t) satisfy the equations Equation (9) is equivalent to  $\sigma(t + dt) = M\sigma(t)M^{T}$  where M is the infinitesimal transfer matrix M(t + dt, t) =I + F(t) dt.

> This procedure is easily extended to two and three dimensions. For three dimensions, the 6 × 6 correlation matrix includes cross-correlation terms such as  $\overline{xy}$ ,  $\overline{xy}'$ , ..., while the 6 × 6 force matrix F may include linear coupling terms from both space-charge and external forces. The three-dimensional equivalent of (9) has

been incorporated into program TRANSPORT<sup>5</sup> to investigate both longitudinal and transverse space-charge effects in transfer lines<sup>6</sup>. In many cases the external forces will not involve coupling and the cross-correlation terms between the different directions will be zero or close to zero. In this case the envelope equations reduce to the K-V form (6) for each direction.

#### One-dimensional envelope equations

For a beam in free space that is very long in the z-direction and very wide in the y-direction, only the x-component of the self-force is important, and this is obtained from the Poisson equation

$$\frac{\partial \varepsilon}{\partial \mathbf{x}} = 4\pi \mathrm{en}(\mathbf{x}, \mathbf{t})$$
 (12)

The envelope equation is

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{e}{mN}\frac{\overline{x}\overline{c}}{\tilde{x}} = 0 , \qquad (13)$$

where N is the number of particles per unit area in  $\Delta y$ Az. This equation can be written as

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{2\pi e^2 N}{m} \lambda_1 = 0$$
, (14)

where  $\lambda_1$  is the dimensionless parameter

$$\lambda_{1} = \frac{2\int_{-\infty}^{\infty} xh(x) dx \int_{0}^{x} h(x') dx'}{\left[\int_{-\infty}^{\infty} x^{2}h(x) dx\right]^{\frac{1}{2}}}$$
(15)

and where h(x) = (1/N)n(x) specifies the distribution. For the four distributions

a) uniform, 
$$h(x) = \frac{1}{2}$$
 for  $x \le 1$   
 $= 0$  for  $x > 1$   
b) parabolic  $h(x) = \frac{3}{4}(1 - x^2)$  for  $x \le 1$   
 $= 0$  for  $x > 1$   
c) gaussian,  $h(x) = \frac{1}{2} e^{-x^2/2}$ 

d) hollow, 
$$h(x) = \frac{1}{2} x^2 e^{-x^2/2}$$
,

the values of  $\lambda_1$  are given in Table 1.

Table 1

, Distribution	$\sqrt{3} \lambda_1$	$\frac{10\sqrt{5}}{3} \lambda_2$	5√5 λ₃
uniform	1	1.08	1
parabolic	0.996	1	1.01
gaussian	0.977	1.05	1.05
hollow	0.987	1.37	1.02

Thus, for the range of distributions likely to be encountered in practice, the variation in  $\lambda_1$  is negligible and the rms envelope motion will be accurately described by Eq. (14) with constant  $\lambda_1,$  for example  $\lambda_1$  =  $1/\sqrt{3}$  .

A second type of one-dimensional envelope equation arises in the study of longitudinal oscillations of a bunched beam inside a conducting pipe7. The longitudinal self-field is determined by

$$\varepsilon(z,t) = -eg \frac{\partial n(z,t)}{\partial z}$$
, (16)

where g = 1 + 2 ln (pipe radius/beam radius), and the corresponding envelope equation is

$$\ddot{\tilde{z}} + K(t)\tilde{z} - \frac{E^2}{\tilde{z}^3} - \frac{ge^2N}{m}\frac{\lambda_2}{\tilde{z}^2} = 0$$
, (17)

where N is the number of particles per bunch and

$$\lambda_2 = \frac{1}{2} \left[ \int_{-\infty}^{\infty} z^2 h(z) dz \right] \int_{-\infty}^{\gamma_2} \int_{-\infty}^{\infty} h^2(z) dz \qquad (18)$$

. .

with values of  $\lambda_2$  listed in Table 1. For this case of a shielded electric field, the envelope equation  $\underline{does}$ depend on the type of distribution. However, if the form of the distribution varies only slightly during its evolution, for example remains within the range uniformparabolic-Gaussian, then the envelope equation (17) can be used with confidence.

#### Envelope equations for continuous beams

In the absence of cross-correlations and coupling terms, the envelope equations have the form (13) where the space-charge terms involve the average  $\overline{x\epsilon}_x$  and  $\overline{y\epsilon}_y$ . These averages will depend only on the second moments'  $\tilde{x}$  and  $\tilde{y}$  and not on the higher moments provided the charge distribution has the elliptical symmetry

$$n(x,y,t) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}, t\right)$$
 (19)

In this case the solution to Poisson's equation is

**v**<sup>2</sup>

$$\varepsilon_{\rm x} = 2\pi {\rm eabx} \int_{0}^{\infty} \frac{n({\rm T}) \, {\rm ds}}{({\rm a}^2 + {\rm s})^{\frac{3}{2}} ({\rm b}^2 + {\rm s})^{\frac{1}{2}}}, \qquad (20)$$

where

$$T = \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s},$$
 (21)

with a similar expression for  $\varepsilon_v$ . The term  $\overline{x\varepsilon}_x$  is there fore

$$\overline{x\varepsilon}_{x} = 2\pi eab \int_{0}^{\infty} ds \int_{-\infty}^{\infty} \frac{x^{2} dx}{(a^{2} + s)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{dy}{(b^{2} + s)^{\frac{3}{2}}} n(T) n\left(\frac{x^{2}}{a^{2}} + \frac{y^{2}}{b^{2}}\right)$$
(22)

which suggests the change of variables

æ

$$r\cos\theta = \frac{x}{\sqrt{a^2 + s}}$$
,  $r\sin\theta = \frac{y}{\sqrt{b^2 + s}}$ . (23)

With the new variables, the integration over  $\theta$  can be performed giving

$$\overline{x}\overline{e}_{x} = \frac{4\pi e a^{3}b^{2}}{a+b} \int_{0}^{\pi} n(r^{2})2\pi r \, dr \int_{r}^{\pi} n(r'^{2})2\pi r' \, dr' . \quad (24)$$

The remaining integrals can be evaluated with the help of the definition

$$N = \int_{-\infty}^{\infty} \int n(x,y) dx dy = ab \int_{0}^{\infty} n(r^{2}) 2\pi r dr , \quad (25)$$

where N is the number of particle per unit length. Then

$$Q(r) = ab \int_{r}^{r} n(r'^{2}) 2\pi r' dr'$$
 (26)

is the number of particles within radius r, and Eq. (24) with the normalization xcomes

$$\overline{x\varepsilon}_{x} = \frac{2ea}{a+b} \int_{a}^{\infty} \frac{dQ}{dr'} \left[ N - Q(r') \right] dr' , \qquad (27)$$

mich is easily integrated,

$$\overline{x\varepsilon}_{x} = \frac{eN^{2}a}{a+b} = \frac{eN^{2}\tilde{x}}{\tilde{x}+\tilde{y}}.$$
 (28)

ging this and the expression for  $\overline{y\varepsilon_v}$ , we obtain the avelope equations

$$\ddot{\tilde{x}} + K_{x}(t)\tilde{x} - \frac{E_{x}^{2}}{\tilde{x}^{3}} - \frac{e^{2}N}{m}\frac{1}{\tilde{x} + \tilde{y}} = 0$$

$$\ddot{\tilde{y}} + K_{y}(t)\tilde{y} - \frac{E_{y}^{2}}{\tilde{y}^{3}} - \frac{e^{2}N}{m}\frac{1}{\tilde{x} + \tilde{y}} = 0.$$
(29)

gese equations are identical to the K-V equations if be rms values  $\tilde{x}$ ,  $E_{\chi}$ ,  $\tilde{y}$ ,  $E_{\gamma}$  are replaced by the physial boundary for a uniform distribution, namely  $I = 2 \tilde{x}, \ldots$  However, they are not restricted to the t-V distribution but are valid for any distribution with the elliptical symmetry (19).

#### Envelope equations for bunched beams

The procedure in two-dimensions can be repeated for bunched beams with the ellipsoidal symmetry

$$n(x,y,z,t) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, t\right) .$$
 (30)

he electric field is8

$$\varepsilon_{\rm x} = 2\pi \text{eabcx} \int_{0}^{\infty} \frac{n(T) \, ds}{(a^2 + s)^{\frac{3}{2}} (b^2 + s)^{\frac{1}{2}} (c^2 + s)^{\frac{1}{2}}}, \quad (31)$$

where

$$T = \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{c^2 + s},$$
 (32)

ind with analogous expressions for  $\varepsilon_v$  and  $\varepsilon_z$ . The term  $\varepsilon_x$  can be reduced to the form

$$\overline{x\varepsilon}_{x} = \frac{eN^{2}\lambda_{3}}{\tilde{x}} g_{x}\left(\frac{b}{a}, \frac{c}{a}\right) , \qquad (33)$$

Mere N is the number of particles per bunch and

$$g_{\chi} = \frac{3}{2} \int_{0}^{1} \frac{ds}{(1+s)^{\frac{3}{2}} \left(\frac{b^{2}}{a^{2}} + s\right)^{\frac{1}{2}} \left(\frac{c^{2}}{a^{2}} + s\right)^{\frac{1}{2}}} (34)$$

he integral in (34) can be expressed in terms of ellipic integrals of the second kind, but direct numerical "aluation with the Gaussian integration method is asier and also quick and accurate. The complete enelope equation for  $\tilde{x}$  is

$$\ddot{\tilde{x}} + K_{x}(t)\tilde{x} - \frac{E_{x}^{2}}{\tilde{x}^{3}} - \frac{e^{2}N\lambda_{3}}{m\tilde{x}^{2}} g_{x}\left(\frac{\tilde{y}}{\tilde{x}}, \frac{\tilde{z}}{\tilde{x}}\right) = 0 , \qquad (35)$$

Tere

$$h = \frac{1}{3\sqrt{3}} \left[ \int_{0}^{\infty} h(r^{2})r^{4} dr \right]^{\frac{1}{2}} \int_{0}^{\infty} h(r^{2})r^{2} dr \int_{r}^{\infty} h(\rho^{2})\rho d\rho$$
(35)

$$\int_{0}^{0} h(r^{2})r^{2} dr = 1 .$$
 (37)

The parameter  $\lambda_3$  depends only weakly on the type of distribution as shown in Table 1. Thus for practical distributions, the dependence of the envelope equations on the type of distribution can be neglected. The same statement also applies if cross-correlations or linear external coupling forces are present; in this case the more general matrix form (9) of the rms equations can be used.

#### Conclusion

A rather surprising and useful result has been found for beams in free space, namely that the linear part of the self-field depends mainly on the rms size of the distribution and only very weakly on its exact form. Using this result, envelope equations for the rms beam size have been derived that are exact for continuous beams of elliptical symmetry, and in practice also valid for bunched beams of ellipsoidal form. The main restriction in applying these equations is that the time dependence of the rms emittance must be known a priori.

Possible uses of the equations include the specification of stationary or matched states in the presence of space charge. For example, the periodic solution of Eq. (35) for alternating-gradient structures, including radio frequency cavities, specifies the matched beam size (both longitudinal and transverse) as a function of rms emittances and intensity. The largest matched size attainable without exceeding aperture limits or bucket size determines a space-charge limit. For a beam matched in this way, envelope oscillations about the periodic solution are suppressed, although higher modes of oscillations (sextupole, octupole, etc.) may occur. Suppression of the higher modes will require constraints, as yet undetermined, on the higher moments of the distribution. Another use is the design of lowenergy beam transfer lines.

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