# Comment on systematic resonances, space charge, and periodicity 

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## COMMENTS ON SYSTAMATIC RESONANCES, SPACE CHARGE and PERIODICITY

Booster Technical Note No. 48

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# ACCELERATOR DEVELOPMENT DEPARTMENT 

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## BOOSTER AT INJECTION

PROTONS

| Kinetic Energy | 200 MeV |
| :--- | ---: |
| Betatron Acceptance (Vert) | $50 \pi \mathrm{~mm} . \mathrm{mrad}$ |
| Intensity | $1.5 \times 1013$ |
|  | in 3 bunches |

CONCERNS:

Space Charge Limit
Eddy Currents
Chromatic Sextupoles
Systematic Resonances
"STANDARD" LATTICE:

Periodicity 6 (see later)
$Q_{H}=4.82, Q_{V}=4.83$

## BEAM DIMENSIONS

$$
\begin{aligned}
& \varepsilon_{\mathrm{H}}=\varepsilon_{\mathrm{V}}=50 \pi \mathrm{~mm} \cdot \mathrm{mrad} \quad \text { (full) } \\
& \Delta \mathrm{p} / \mathrm{p}= \pm 2.50 / 00 \quad \text { (full) }
\end{aligned}
$$

|  | QF | QD |
| :---: | :---: | :---: |
| $\beta_{H}$ | 13.865 m | 3.5754 m |
| $\beta_{\mathrm{V}}$ | 3.7033 | 13.644 |
| $\mathrm{X}_{\mathrm{p}}$ | 2.9515 | 0.54004 |

2d

a
a
b
d
$\sqrt{a^{2}+d^{2}}$
26.12 mm
13.61
7.38
27.14
13.44

$$
\frac{N_{\text {s.c. }}}{\varepsilon_{N}}=\left(\beta \gamma^{2}\right) \frac{4 B_{f}}{3 r_{o} F} \Delta Q \quad \varepsilon_{N}=(\beta \gamma) \varepsilon
$$

| Kinetic Enersy | 200 MeV |
| :---: | :---: |
| $\beta$ | 0.56616 |
| $\gamma$ | 1.2132 |
| ro $_{0}$ | $1.535 \times 10^{-18} \mathrm{~m}$ |
| F | 1 |
| Bf | 0.5 |


| $\Delta Q$ <br> $N_{S} \cdot c . / \varepsilon_{N}$ | 0.250 <br> $9 \times 10^{16} \mathrm{~m}^{-1}$ | 0.375 <br> $13.5 \times 10^{16} \mathrm{~m}^{-1}$ |
| :---: | :---: | :---: |
| $\varepsilon_{\mathrm{H}}=\varepsilon_{\mathrm{V}}$ |  |  |
| $\pi \mathrm{mm} . \mathrm{mrad}$ | 50 | 75 |
| $\mathrm{~N}_{\mathrm{S} . \mathrm{c} .}$ | $1.0 \times 10^{13}$ | $1.5 \times 10^{13}$ |

$$
\begin{array}{ll}
Q_{\mathrm{H}}=4.82 & \text { Periodicity }=6 \\
Q_{\mathrm{V}}=4.83 &
\end{array}
$$

P

$$
\frac{\perp n Q_{H}+m Q_{V}-p \perp}{n+m}
$$

1.180
0.820
0.827
0.387
1.180
1.170
0.320
0.330
0.325
0.005
0.177
0.397
0.180
0.173
0.387

```
Natural Chromaticity (H,V)
With Eddy Currents: }\mp@subsup{\varepsilon}{\textrm{H}}{
    \varepsilon
    SEXTUPOLES STRENGTH TO CANCEL CHROMATICITY WITH EDDY CURRENT
    SF~}~0.
    SD
    ~ 1.0
Very asymmetric
Very large average contribution (from E.C.) which enhances 2Q ( }
resonance.
SUGGESTION: Compensation of Eddy Currents with pole face windings in Dipole Magnets.
```


## SPACE CHARGE

$$
\begin{gathered}
p=\frac{\operatorname{Nf}(z)}{2 \pi \sigma^{2}} \exp \left(-\frac{r^{2}}{2 \sigma^{2}}\right) \\
\nabla 2 \phi=-4 \pi \rho \\
E_{r}=-\frac{\partial \phi}{\partial r}=-\frac{2 N}{\sigma^{2} r} f(z) \int_{0}^{n} u e-\frac{u^{2}}{2 \sigma^{2}} d u \\
\phi=\frac{2 N}{\sigma^{2}} f(z) \int_{0}^{r} \frac{d r^{1}}{r^{1}} \int_{0}^{r^{2}} u e-\frac{u^{2}}{2 \sigma^{2}} d u \\
\approx \text { constraint }\left\{\begin{array}{l}
x r^{2}-\frac{r^{4}}{8 \sigma^{2}}+\cdots \cdot \int_{1} \\
r^{4}= \\
\left(x^{2}+y^{2}\right)^{2}=\left(x^{4}+2 x^{2} y^{2}+y^{4}\right) \\
\text { ordinary octupole }-x^{4}-6 x^{2} y^{2}+y^{4}
\end{array}\right.
\end{gathered}
$$

$$
\left\{\begin{array}{c}
\text { RMS Beam Envelope Equations } \quad(\tilde{x}, \tilde{y}) \\
\tilde{x^{\prime}} / /+k_{x} \tilde{x}=\frac{E_{x}^{2}}{\tilde{x^{3}}}+\frac{8}{\tilde{x}+\tilde{y}} \\
\tilde{y} / /+k_{y} \tilde{y}=\frac{E_{y}^{2}}{\tilde{y} 3}+\frac{g}{\tilde{x}+\tilde{y}} \\
g=\frac{N r_{o}}{2 \pi R B f \beta^{2} \gamma^{3}}
\end{array}\right.
$$

Floquet tranformation:

$$
\begin{aligned}
\eta=\frac{\tilde{x}}{\sqrt{\beta_{\mathrm{x}}}} \quad \zeta=\frac{\tilde{y}}{\sqrt{\beta_{y}}} \quad & d s={\underset{v}{ } \beta_{H} d \phi}_{Q_{H}}=Q_{V}
\end{aligned}
$$

$$
\begin{aligned}
& \left\{\begin{array}{c}
\eta^{\prime \prime}+Q^{2} \eta=Q^{2} \beta_{x}^{3}{ }^{2} \frac{E_{x}^{2}}{\tilde{x^{3}}}+Q^{2} \beta_{x}^{3}{ }^{2} \frac{g}{\tilde{x}+\tilde{y}} \\
\zeta^{\prime \prime}+Q^{2} \zeta=Q^{2} \beta_{y^{3}}^{2} \frac{E_{y}^{2}}{\tilde{y^{3}}}+Q^{2} \beta_{y^{3}}^{2} \frac{g}{\tilde{x}+\tilde{y}}
\end{array}\right. \\
& \text { For } \mathrm{g}=0 \quad \mathrm{E}_{\mathrm{x}}=\frac{\tilde{x}^{2}}{\beta_{\mathrm{x}}}, \quad \mathrm{E}_{\mathrm{y}}=\frac{\tilde{y}^{2}}{\beta_{\mathrm{y}}} \\
& \text { For } g \neq 0 \text {, neglect } \eta^{\prime /}, \zeta^{\prime /} \text { take } \beta_{x} \sim \beta_{y} \sim \bar{\beta} \\
& \left\{\begin{array}{l}
\eta=\frac{E_{x}{ }^{2}}{\eta^{3}}+\frac{\bar{B} g}{n+\zeta} \\
\zeta=\frac{E_{y}{ }^{2}}{\zeta^{3}}+\frac{\bar{\beta} g}{\eta+\zeta}
\end{array}\right. \\
& \text { "Round" Beam } E_{X}=E_{y}=E \text { then } n=\zeta
\end{aligned}
$$

$$
\begin{aligned}
& \eta^{4}-\bar{\beta} g \eta^{2}-E^{2}=0 \\
& \frac{\eta^{2}}{E}=\sqrt{1+\alpha^{2}}+\alpha \\
& \alpha=\frac{\bar{\beta} g}{2 E}=\frac{\bar{\beta} N r_{0}}{4 \pi R E B f \beta^{2} \gamma^{3}}
\end{aligned}
$$

$$
\begin{array}{rlr}
N & =1.5 \times 10^{13} & \\
r_{o} & =1.535 \times 10^{18 \mathrm{~m}} & \\
\bar{B} & =8 \mathrm{~m} & \alpha=0.1 \\
2 \pi R & =201.78 \mathrm{~m} & \\
B_{f} & =0.5 & \\
B^{2} \gamma^{3} & =0.57232 & \frac{n^{2}}{E}=1.1 \\
E & =\frac{50}{3}(\pi) \mathrm{mm} . \mathrm{mrad} & \bar{E}
\end{array}
$$

$10 \%$ Dinition
"Nominal"6
B - functions, (quads) ..... 24 ..... 12
Dispersion ..... 6 ..... 12
Dipole, Eddy Currents ..... 8 ..... 12
Sextupole ArrangementQuite Acceptable12


FIG. 1 The Amplitude and Dispersion Functions of the Booster Lattice.


## —— DIRECTIUN DF BEAM

$$
\begin{aligned}
1 & =\text { FOCUSING QUADRUPOLE } \\
\dagger & =\text { DEFICUSING QUADRUPOLE } \\
\square & =\text { BENDING MAGNET GIPDLE) } \\
x & =\text { SEXTUPLE }
\end{aligned}
$$

FIG. 2 a) Schematic Diagram of the Booster and
b) Components of the Superperiod

COMBINED FUNCTION LATTICE
short dipole-. 5 m Ernax- 5936 T

(1) All ${ }^{\log }$ dipoles are identical
2.6 m ( 1.3 m focusing dipole, 113 m defocusig diple)
short apples $=0.5 \mathrm{~m}$ (for injection)
(2) all short space are $1 \mathrm{~m} \log$
$\operatorname{long}$ space $=4.015 \mathrm{~m} \quad \operatorname{ling}$
(3)

$$
\begin{aligned}
& \hat{B}=0.9936 \mathrm{~T} \text { (4) } B P=16.7 \mathrm{Tm} \\
& \hat{B}^{\prime}=4.0 \mathrm{~T} / \mathrm{m}
\end{aligned}
$$

(4)

$$
\begin{aligned}
& P=12 \\
& Q=4.83
\end{aligned}
$$

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Sumary .
Envelope equations for a continuous beam with uniprmarge density and elliptical cross-section were irst derived by Kapchinsky and Vladinirsky ${ }^{2}(\mathrm{~K}-\mathrm{V})$. In ifct, the $K-V$ equations are not restricted to uniformly parged beams, but are equally valid for any charge dispibution with elliptical symmetry, provided the beam pundary and emittance are defined by rms (root-meanquare) values. This results because (i) the second pments of any particle distribution depend only on the :inear part of the force (determined by least squares \#chod), while (ii) this linear part of the force in mrn depends only on the second moments of the distribu:ion. This is also true in practice for three-dimenional bunched beams with ellipsoidal symmetry, and Hlows the formulation of envelope equations that inIUde the effect of space charge on bunch length and sergy spread.

The urility of this ms approach was first demonirrated by Lapostolle ${ }^{3}$ for stationary distributions. jubsequently, Gluckstern ${ }^{4}$ proved that the rus version if the $K-V$ equations remain valid for all continuous zams with axial symmetry. In this report these re;ults are extended to continuous beams with elliptical ;ymmetry as well as to bunched beams with ellipsoidal iorm, and also to one-dimensional motion.

## Moment equations

Consider an ensemble of particles that obey the ingle-particle equations

$$
\begin{align*}
& \dot{x}=p \\
& \dot{p}=F(x, t), \tag{1.}
\end{align*}
$$

there $F(x, t)$ includes both the external force and the ielf-force, $F=F_{e}+F_{s}$. Averaging (1) over an arbiirary particle distribution $f(x, p, t)$, we obtain

$$
\begin{align*}
& \dot{\bar{x}}=\bar{p} \\
& \dot{\bar{p}}=\overline{\mathrm{F}}=\overline{\mathrm{F}}_{\mathrm{e}}, \tag{2}
\end{align*}
$$

there the last equation follows because $\bar{F}_{s}=0$ by
lewton's third law. (We neglect the small magnetic selftorces due to internal motion.) If $\mathrm{F}_{\mathrm{e}}(\mathrm{x}, \mathrm{t})$ is non-linear in $x$, the second equation of (2) involves the higher 30ments $\overline{x^{n}}$ of the distribution. However, for linear external forces, $F_{e}=-K(t) x$, equations (2) involve only the first moments $\bar{x}$ and $\bar{p}$, and therefore the centre-ofass motion depends only on the external force,

$$
\ddot{\bar{x}}+K(t) \bar{x}=0
$$

ind not on the detailed form of the distribution. In the remainder of this paper we consider only linear exiernal forces.

The second moments of $f(x, p, t)$ satisfy the equaticns

$$
\begin{align*}
& \dot{x^{2}}=2 \overline{x \dot{x}}=2 \overline{x p} \\
& \dot{\overline{x p}}=\overline{x p}+\overline{x p}=\overline{p^{2}}-K(t) \overline{x^{2}}+\overline{x F_{s}}  \tag{4}\\
& \overline{p^{2}}=2 \overline{p \dot{p}}=-2 k(t) \overline{x p}+2 \overline{p F_{s}},
\end{align*}
$$

Where the cerms $\overline{x F}_{S}$ and $\overline{p F}_{s}$ are usually functions of the iigher moments $x^{\sqrt{n}}$ and $\bar{x}^{n} p$. This is a general feature of
moment equations, namely tine equation for each moment involves the higher moments in an endless hierarchy. However, if the self-force is derived from the freespace Poisson equation, $\overline{x F}_{s}$ depends mainly on the second moments and very little, if at all, on the higher moments. This will be demonstrated in the following sections. The remaining term $\mathrm{PF}_{\mathrm{s}}$ is associated with emittance growth; we will avoid considering it by assuming that the rms emittance

$$
\begin{equation*}
E=\sqrt{\overline{x^{2}} \bar{p}^{2}-\overline{x p}^{2}} \tag{5}
\end{equation*}
$$

is either constant, or that its time dependence is known $a$ priori. Then $\bar{p}^{2}$ is given in terms of $\bar{x}^{2}, \overline{x p}$, and $E(t)$ by (5), and the first two equations of (4) form a closed set. They can be combined to give the $K-V$ type equation:

$$
\begin{equation*}
\ddot{\tilde{x}}+K(t) \tilde{x}-\frac{E^{2}}{\bar{x}^{3}}-\frac{\overline{x F}}{\bar{x}}=0, \tag{6}
\end{equation*}
$$

where $\tilde{x}$ is the rms value, $\tilde{x}=\sqrt{\overline{x^{2}}}$.
The space-charge term in this equation has an interesting interpretation. If we define the linear part of the force $F_{S}(x, t)$ as $\varepsilon(t) x$, where $\varepsilon(t)$ is determined by minimizing the difference

$$
\begin{equation*}
D=\int\left[\varepsilon(t) x-F_{s}(x, t)\right]^{2} n(x, t) d x \tag{7}
\end{equation*}
$$

for a fixed $t$, where $n(x, t)=\int f(x, p, t) d p$, then

$$
\begin{equation*}
\varepsilon(t) x=\frac{\overline{x F}_{s}}{\tilde{x}^{2}} x \tag{8}
\end{equation*}
$$

In other words, the rms envelope equation depends only on the linear part of the forces, determined by least squares method.

It is convenient to put equation (4) into matrix form. The assumption of constant rms emittance is equivalent to setting $\overline{\mathrm{pF}}_{\mathrm{s}}=\varepsilon(t) \overline{\mathrm{xp}}$. Then equation (4) has the form

$$
\begin{equation*}
\dot{\sigma}=F \sigma+\sigma F^{T} \tag{9}
\end{equation*}
$$

where $\sigma$ is the covariance matrix

$$
\sigma=\left[\begin{array}{ll}
\overline{x^{2}} & \overline{x p}  \tag{10}\\
\overline{x p} & \overline{p^{2}}
\end{array}\right]
$$

and $F$ is

$$
F=\left[\begin{array}{ll}
0 & 1  \tag{1I}\\
-K(t)+\varepsilon(t) & 0
\end{array}\right]
$$

Equation (9) is equivalent to $g(t+d t)=M \sigma(t) M^{T}$ where $M$ is the infinitesimal transfer matrix $M(t+d t, t)=$ $I+F(t) d t$.

This procedure is easily extended to two and three dimensions. For three dimensions, the $6 \times 6$ correlation matrix includes cross-correlation terms such as $\overline{x y}$, $\frac{x y}{x y}, \ldots$, while the $6 \times 6$ force matrix $F$ may include linear coupling terms from both space-charge and external forces. The three-dimensional equivalent of (9) has
been incorporated into program TRANSPORT ${ }^{5}$ to investigate both longitudinal and transverse space-ctarge effects in transfer lines ${ }^{6}$. In many cases the external forces will not involve coupling and the cross-correlation terms between the different directions will be zero or close to zero. In this case the envelope equations reduce to the $\mathrm{K}-\mathrm{V}$ form (6) for each direction.

## One-dimensional envelope equations

For a beam in free space that is very long in the $z$-direction and very wide in the $y$-direction, only the r-component of the self-force is important, and this is obtained from the Poisson equation

$$
\begin{equation*}
\frac{\partial \varepsilon}{\partial x}=4 \pi e n(x, t) \tag{12}
\end{equation*}
$$

The envelope equation is

$$
\begin{equation*}
\ddot{\tilde{x}}+K(t) \tilde{x}-\frac{E^{2}}{\tilde{x}^{3}}-\frac{e}{\tilde{m}} \frac{\overline{x \varepsilon}}{\tilde{x}}=0 \tag{13}
\end{equation*}
$$

where $N$ is the number of particles per unit area in $\Delta y$ 42 . This equation can be written as

$$
\begin{equation*}
\ddot{\tilde{x}}+K(t) \tilde{x}-\frac{E^{2}}{\tilde{x}^{3}}-\frac{2 \pi e^{2} N}{m} \lambda_{1}=0 \tag{14}
\end{equation*}
$$

where $\lambda_{1}$ is the dimensionless parameter

$$
\begin{equation*}
\lambda_{1}=\frac{2 \int_{-\infty}^{\infty} \operatorname{xh}(x) d x \int_{0}^{x} h\left(x^{\prime}\right) d x^{\prime}}{\left[\int_{-\infty}^{\infty} x^{2} h(x) d x\right]^{1 / 2}} \tag{15}
\end{equation*}
$$

and where $h(x)=(1 / N) n(x)$ specifies the distribution. For the four distributions
a) uniform,
$h(x)=\frac{1}{2}$
for $x \leq 1$
$=0 \quad$ for $x>1$
b) parabolic

$$
h(x)=\frac{3}{4}\left(1-x^{2}\right) \quad \text { for } x \leq 1
$$

$=0$
for $x>1$
c) gaussian,
$h(x)=\frac{1}{2} e^{-x^{2} / 2}$
d) hollow,
$h(x)=\frac{1}{2} x^{2} e^{-x^{2} / 2}$,
the values of $\lambda_{1}$ are given in Table 1 .

## Table 1

| Distribution | $\sqrt{3} \lambda_{1}$ | $\frac{10 \sqrt{5}}{3} \lambda_{2}$ | $5 \sqrt{5} \lambda_{3}$ |
| :--- | :---: | :---: | :---: |
| uniform | 1 | 1.08 | 1 |
| parabolic | 0.996 | 1 | 1.01 |
| gaussian | 0.977 | 1.05 | 1.05 |
| hollow | 0.987 | 1.37 | 1.02 |

Thus, for the range of distributions likely to be encountered in practice, the variation in $\lambda_{1}$ is negligible and the rms envelope motion will be accurately described by Eq. (14) with constant $\lambda_{1}$, for example $\lambda_{1}=1 / \sqrt{3}$.

A second type of one-dimensional envelope equation arises in the study of longitudinal oscillations of a bunched beam inside a conducting pipe ${ }^{7}$. The longitudinal self-field is determined by

$$
\begin{equation*}
E(z, t)=-e g \frac{\partial n(z, t)}{\partial z}, \tag{16}
\end{equation*}
$$

where $g=1+2 \ln$ (pipe radius/beam radius), and the corresponding envelope equation is

$$
\begin{equation*}
\ddot{\tilde{z}}+K(t) \tilde{z}-\frac{E^{2}}{\tilde{z}^{3}}-\frac{g e^{2} N}{m} \frac{\lambda_{2}}{\tilde{z}^{2}}=0 \tag{17}
\end{equation*}
$$

where $N$ is the number of particles per bunch and

$$
\begin{equation*}
\lambda_{2}=\frac{1}{2}\left[\int_{-\infty}^{\infty} z^{2} h(z) d z\right] \int_{-\infty}^{1 / 2} h^{2}(z) d z \tag{18}
\end{equation*}
$$

with values of $\lambda_{2}$ listed in Table 1 . For this case of a shielded electric field, the envelope equation does depend on the type of distribution. However, if the form of the distribution varies only slightly during its evolution, for example remains within the range unifor-parabolic-Gaussian, then the envelope equation (17) can be used with confidence.

## Envelope equations for continuous beams

In the absence of cross-correlations and coupling terras, the envelope equations have the form (13) where the space-charge terms involve the average $\overline{\mathrm{XE}} \mathrm{x}$ and $\overline{\mathrm{V}}, \mathrm{y}$. These averages will depend only on the second moments $\bar{x}$ and $\tilde{y}$ and not on the higher moments provided the charge distribution has the elliptical symmetry

$$
\begin{equation*}
n(x, y, t)=n\left(\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}, t\right) \tag{19}
\end{equation*}
$$

In this case the solution to Poisson's equation is

$$
\begin{equation*}
\varepsilon_{x}=2 \pi e a b x \int_{0}^{\infty} \frac{n(T) d s}{\left(a^{2}+s\right)^{3 / 2}\left(b^{2}+s\right)^{1 / 2}}, \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
T=\frac{x^{2}}{a^{2}+s}+\frac{y^{2}}{b^{2}+s} \tag{21}
\end{equation*}
$$

with a similar expression for $\varepsilon_{y}$. The term $\overline{x \varepsilon} x$ is there
fore
$\overline{x \varepsilon}_{x}=2 \pi e a b \int_{0}^{\infty} d s \int_{-\infty}^{\infty} \frac{x^{2} d x}{\left(a^{2}+s\right)^{3 / 2}} \int_{-\infty}^{\infty} \frac{d y}{\left(b^{2}+s\right)^{1 / 2}} n(T) n\left(\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}\right)$
which suggests the change of variables

$$
\begin{equation*}
r \cos \theta=\frac{x}{\sqrt{a^{2}+s}}, \quad r \sin \theta=\frac{y}{\sqrt{b^{2}+s}} \tag{23}
\end{equation*}
$$

With the new variables, the integration over $\theta$ can be performed giving

$$
\begin{equation*}
\bar{x} \bar{\varepsilon}_{x}=\frac{4 \pi e a^{3} b^{2}}{a+b} \int_{0}^{\infty} n\left(r^{2}\right) 2 \pi r d r \int_{r}^{\infty} n\left(r^{\prime 2}\right) 2 \pi r^{\prime} d r^{\prime} \tag{24}
\end{equation*}
$$

The remaining integrals can be evaluated with the help of the definition

$$
\begin{equation*}
N=\int_{-\infty}^{\infty} \int_{-\infty} n(x, y) d x d y=a b \int_{0}^{\infty} n\left(r^{2}\right) 2 \pi r d r \tag{25}
\end{equation*}
$$

where $N$ is the number of particle per unit length. Then

$$
\begin{equation*}
Q(r)=a b \int_{r}^{r} n\left(r^{\prime 2}\right) 2 \pi r^{\prime} d r^{\prime} \tag{26}
\end{equation*}
$$

is the number of particles within radius $r$, and Eq. (24) yocomes

$$
\begin{equation*}
\overline{x E}_{x}=\frac{2 e a}{a+b} \int_{0}^{\infty} \frac{d Q}{d r^{\prime}}\left[N-Q\left(r^{\prime}\right)\right] d r^{\prime}, \tag{27}
\end{equation*}
$$

dich is easily integrated,

$$
\begin{equation*}
\overline{x \varepsilon}_{x}=\frac{e N^{2} a}{a+b}=\frac{e y^{2} \bar{x}}{\bar{x}+\tilde{y}} \tag{28}
\end{equation*}
$$

fing this and the expression for $\overline{y \varepsilon}_{y}$, we obtain the avelope equations

$$
\begin{align*}
& \ddot{\tilde{x}}+K_{x}(t) \tilde{x}-\frac{E_{x}^{2}}{\tilde{x}^{3}}-\frac{e^{2} N}{m} \frac{1}{\dot{x}+\tilde{y}}=0 \\
& \ddot{\tilde{y}}+K_{y}(t) \tilde{y}-\frac{E_{y}^{2}}{\dot{y}^{3}}-\frac{e^{2} N}{m} \frac{1}{\ddot{x}+\tilde{y}}=0 . \tag{29}
\end{align*}
$$

aese equations are identical to the $\mathrm{K}-\mathrm{V}$ equarions if be rms ralues $\tilde{x}, E_{x}, \tilde{y}$, $E_{y}$ are replaced by the physial boundary for a uniform distribution, namely
$I=2 \bar{x}, \ldots$. However, they are not restricted to the $t-V$ distribution but are valid for any distribution with :te elliptical symetry (19).

## Envelope equations for bunched beams

The procedure in two-dimensions can be repeated :or bunched beams with the ellipsoidal symmetry

$$
\begin{equation*}
n(x, y, z, t)=n\left(\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}+\frac{z^{2}}{c^{2}}, t\right) \tag{30}
\end{equation*}
$$

The electric field is ${ }^{9}$

$$
\begin{equation*}
\varepsilon_{x}=2 \pi e a b c x \int_{0}^{\infty} \frac{n(T) d s}{\left(a^{2}+s\right)^{3 / 2}\left(b^{2}+s\right)^{1 / 2}\left(c^{2}+s\right)^{1 / 2}}, \tag{31}
\end{equation*}
$$

inere

$$
\begin{equation*}
I=\frac{x^{2}}{a^{2}+s}+\frac{y^{2}}{b^{2}+s}+\frac{z^{2}}{c^{2}+s} \tag{32}
\end{equation*}
$$

ind with analogous expressions for $\varepsilon_{y}$ and $\varepsilon_{z}$. The term

$$
\begin{equation*}
\overline{x \varepsilon}_{x}=\frac{e \tilde{x}^{2} \lambda_{3}}{\tilde{x}} g_{x}\left(\frac{b}{a}, \frac{c}{a}\right) \tag{33}
\end{equation*}
$$

mere $N$ is the number of particles per bunch and

$$
\begin{equation*}
g_{x}=\frac{3}{2} \int_{0}^{\infty} \frac{d s}{(1+s)^{3 / 2}\left(\frac{b^{2}}{a^{2}}+s\right)^{1 / 2}\left(\frac{c^{2}}{e l^{2}}+s\right)^{1 / 2}} . \tag{34}
\end{equation*}
$$

Te integral in (34) can be expressed in terms of ellip:ic integrals of the second kind, but direct numerical :raluation with the Gaussian integration method is isier and also quick and accurate. The complete enelope equation for $\tilde{x}$ is

$$
\ddot{\tilde{x}}+K_{x}(t) \tilde{x}-\frac{E_{x}^{2}}{\tilde{x}^{3}}-\frac{e^{2} J \lambda_{3}}{m_{x^{2}}^{2}} g_{x}\left(\begin{array}{l}
\left.\frac{\tilde{y}}{\tilde{x}}, \frac{\ddot{z}}{\ddot{x}}\right)=0, ~ \text {, } \tag{35}
\end{array}\right)=0,
$$

"ere
$:=\frac{1}{3 \sqrt{3}}\left[\int_{0}^{\infty} h\left(r^{2}\right) r^{4} d r\right]^{1 / 2} \int_{0}^{\infty} h\left(r^{2}\right) r^{2} d r \int_{r}^{\infty} h\left(\rho^{2}\right) \rho d \rho$
with the normalization

$$
\begin{equation*}
\int_{0}^{\infty} h\left(r^{2}\right) r^{2} d r=1 \tag{37}
\end{equation*}
$$

The parameter $\lambda_{3}$ depends only weakly on the type of distribution as shown in Table 1. Thus for practical distributions, the dependence of the envelope equations on the cype of distribution can be neglected. The same statement also applies if cross-correlations or linear excernal coupling forces are present; in this case the more general matrix form (9) of the rms equations can be used.

## Conclusion

A rather surprising and useful result has been found for beams in free space, namely that the linear part of the self-field depends mainly on the rms size of the distriburion and only very weakly on its exact form. Using this result, envelope equations for the rms beam size have been derived that are exact for continuous beams of elliptical symmetry, and in practice also valid for bunched beams of ellipsoidal form. The main restriction in applying these equations is that the time dependence of the rms emittance must be known a priori.

Possible uses of the equations include the specification of stationary or matched states in the presence of space charge. For example, the periodic solution of Eq. (35) for alternating-gradient structures, including radio frequency cavities, specifies the matched beam size (both longitudinal and transverse) as a function of rms emittances and intensity. The largest matched size attainable without exceeding aperture limits or bucket size determines a space-charge limit. For a beam matched in this way, envelope oscillations about the periodic solution are suppressed, although higher modes of oscillations (sextupole, octupole, etc.) may occur. Suppression of the higher modes will require constraints, as yet undetermined, on the higher moments of the distribution. Another use is the design of lowenergy bean transfer lines.

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