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DA (Differential Algebraic) Method and Symplectification for Field Map Generated Matrices of Siberian Snake

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Spin Note

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1 Introduction

During the acceleration of polarized proton beam in RHIC, it will be necessary to preserve the spin orientation to a high degree. For this reason, we must include spin tracking in simulations and the effect of the proposed Siberian snakes.

Spin precession of the proton depends on the ambient magnetic field and thus it is coupled with the orbital motion. The spin matrix, a rotation in spin space, provides the required rotation of the spin and the correct angle of the precession axis. The Siberian Snake was designed to turn the direction of spin by 180^o with minimal effect on orbital motion.

The laws of physics require that the spin matrix be unitary and the orbit matrix be symplectic. In fact, the entire nonlinear orbit map must be symplectic. In general it is not clear that an approximate solution for the spin matrix and the orbital map will produce unitary matrices for the spin and symplectic maps for the orbital part. The violation of the unitary/symplectic condition becomes unacceptable whenever the simulation code is used in an iterative mode, i.e., whenever several turns of a machine are tracked. A violation of the unitary/symplectic condition then leads to a growth or shrinkage of the phase space; occasionally it leads to more complex but still unphysical behavior. It is this issue that motivates the present paper.

The study of spin and orbital motion in Siberian Snakes for a realistic description of the magnetic field was done using a numerical approach [1].The magnetic field of the full snake was constructed with symmetrical conditions based on the field map given numerically on the grids

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of 1/4 helical magnet. The spline function method was used to interpolate the magnetic field in three dimensions and the classical fourth-order Runge-Kutta method was taken to integrate the equations of orbital motion and spin precession. From the entrance of the snake, rays with randomly generated initial conditions in an ellipse defined by the emittance, are tracked through the snake, and a polynomial fit of the dependence of the final results on the initial conditions was performed. Then, the spin matrix and the first order orbital matrix were obtained.

However, there are several problems in the above calculation :

1. The variables (x, x', y, y') used to integrate the equations of orbital motion are not canonical variables. Therefore, we cannot easily evaluate the symplectic condition using the resulting matrix.

2. The matrix defined with randomly generated rays is, for a nonlinear system, a kind of average matrix on which it makes no sense to impose the symplectic condition.

3. The area of the fringing magnetic field along the entrance and the exit of the snake have to be considered as long as possible, and the fringing magnetic field between two helical magnets needs to be taken into account too;

4. The field strengths of both the first and second helical magnet have to be adjusted so that the spin precesses from $(0, 1, 0)$ to $(0, -1, 0)$ as close as possible.

The issue of symplectic integration has been addressed by several authors [2, 6]. Grossly speaking symplectic integrators can be divided into the explicit and implicit types. Most accelerator physics codes, of the "kick-code" variety, are explicit integrators. These codes are fast and conceptually simple as a variety of effects can be added to them in a more or less self-consistent way. Unfortunately, it seems very hard to produce explicit symplectic integrators for a non-ideal magnetic field; the complexity is compounded by the absence of an analytical representation for the magnetic field.

Therefore we have opted here to keep using an ordinary integrator from which we extract a map. This has a small advantage: because the integrator is not symplectic, any violation of the symplectic condition will be a reflection of the integration accuracy or the non-Maxwellian aspects of the fields. In other words it gives us an idea of the absolute error before any symplectic "fudging" algorithm masks it.

In this paper, to insure that the matrix is correctly computed, we abandon techniques based on numerical differentiation in favor of the more accurate numerical method based on the DA(Differential Algebra) approach [7, 9] pioneered by Berz in beam physics. This method permits the calculation of a truncated Taylor map of an arbitrary element to any order. The Taylor coefficients of the resulting truncated map will be acurate to machine precision. However since the magnetic field interpolated by spline function is smooth in the first derivatives and continuous in the second derivatives, it will be possible to introduce DA methods to calculate the map of the full snake to second order in the phase space variables without refitting the fields.

In fact, the matrices calculated from a field map are not symplectic in general. This is a problem resulting not only from the method we used, but also from the deviation of the magnetic field from the Maxwellian property. The symplectification for the maps obtained by the DA approach will be introduced and symplectified maps which satisfy exactly the symplectic conditions will be presented in this report.

2 Canonical variables $(x, p_x)(y, p_y)(l, \delta)$ and equations of orbital motion

In reference [1], the following equations of orbital motion in natural coordinates were used for the snake from the entrance to the exit.

$$\frac{ds}{ds} = 1, \tag{1}$$
$$\frac{dx}{ds} = x' = u, \tag{2}$$
$$\frac{dy}{ds} = y' = v, \tag{3}$$
$$\frac{du}{ds} = w \cdot [uvB_x - (1+u^2)B_y + vB_s], \tag{4}$$
$$\frac{dv}{ds} = w \cdot [(1+v^2)B_x - uvB_y - vB_s]. \tag{5}$$

where $w = \frac{e \cdot \sqrt{1+u^2+v^2}}{m_0 \gamma V}$, V is the velocity of a charged particle. B_x, B_y and B_s are the components of magnetic field in x, y and s directions, respectively. These are simply the rectangular coordinates along the axis of the snake. (The variable s is just the usual Cartesian z-axis)

In the field free region, the general Hamiltonian with time t as independent variable is

$$H = \sqrt{m^2 c^4 + c^2 P^2}. \tag{6}$$

If we take s as the independent variable, and if we normalized the momenta by P_0, a Halmiltonian K can be derived for the s-dependent motion:

$$K = -P_s = -\sqrt{\frac{(H^2 - m^2c^4)}{c^2} - P_\perp^2}$$
$$= -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \quad (7)$$

where

$$\delta = \frac{\Delta P}{P_0}, p_x = \frac{P_x}{P_0}, p_y = \frac{P_y}{P_0},$$

then

$$x' = \frac{dx}{ds} = [x, K] = \frac{\partial K}{\partial p_x} = \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} \quad (8)$$
$$y' = \frac{dy}{ds} = [y, K] = \frac{\partial K}{\partial p_y} = \frac{p_y}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} \quad (9)$$

and

$$p_x = \frac{(1+\delta)\cdot x'}{\sqrt{1 + x'^2 + y'^2}} \quad (10)$$
$$p_y = \frac{(1+\delta)\cdot y'}{\sqrt{1 + x'^2 + y'^2}} \quad (11)$$

The canonical variables were defined in this way. Suppose, the magnetic field in the region before the entrance and after the exit of the snake is zero, then we can find a gauge for which $A_x = A_y = 0$.[1] Then we can convert back and forth between $(x, p_x)(y, p_y)$ and $(x, x')(y, y')$ using eq.(10) - eq.(11) or eq.(8) - (9).With these tranformations, we can still use eq.(2) - (5)

Another pair of canonical variables[2] is

$$l' = \frac{dl}{ds} = [l, K] = -\frac{\partial K}{\partial \delta} = \frac{1+\delta}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}. \quad (12)$$

On the other hand,

$$\frac{dl}{ds} = \frac{v_s}{v} = \frac{1}{\sqrt{1 + x'^2 + y'^2}}, \quad (13)$$

so that path-length L was calculated by

$$L = \int_0^l \sqrt{1 + x'^2 + y'^2}\, ds \quad (14)$$

[1] $A_s = 0$ is not necessary.

[2] The reader will notice that path length l and momentum δ form a complete set if and only if the motion is ultrarelativistic. For normal motion we must substitute time and energy. This can be done afterwards on the map of the snake.

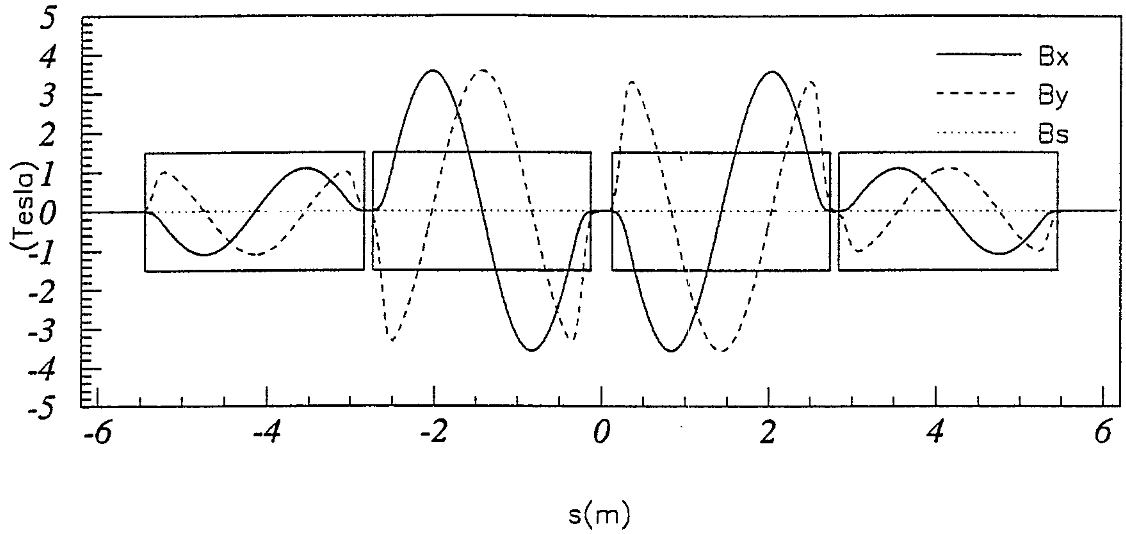

Figure 1: schematic view of Siberian snake and its magnetic field distribution on the axis ($x = 0, y = 0$)

and it will be used in the program for getting 6-dimensional matrices. The design path-length can be removed and non-relativistic corrections (using time and energy) can be exactly added to the matrix if necessary.

3 The magnetic field distribution of the full snake and the central ray

Fig. 1 gives the schematic view of the Siberian snake and its magnetic field distribution on the axis ($x = 0, y = 0$). Based on the field map on the grids of 1/4 helical magnet, the magnetic field distribution of the full snake was constructed. For the region of $y < 0$, the following symmetrical conditions were used,

$$\begin{cases} \bar{B}_x(x,-y,s) &= B_x(x,y,s) \\ \bar{B}_y(x,-y,s) &= B_y(-x,y,s) \\ \bar{B}_s(x,-y,s) &= -B_s(-x,y,s) \end{cases} \tag{15}$$

For another half length ($s = -\frac{L}{2} - 0$) inside one helical magnet, the following symmetrical conditions were used,

$$\begin{cases} \bar{B}_x(-x,-y,s) &= -B_x(x,y,s) \\ \bar{B}_y(-x,-y,s) &= B_y(-x,y,s) \\ \bar{B}_s(-x,-y,s) &= -B_s(-x,y,s) \end{cases} \tag{16}$$

Magnetic field		Orbital		Spin	
Field map	Map827.grid	$x_e(mm)$	-0.12385485	S_{xe}	0.00094950
$B_{1m}(T)$	1.2433899	$p_{xe}(1)$	0.00000405	S_{ye}	-0.99999902
$B_{2m}(T)$	3.8813956	$y_e(mm)$	-0.41663179	S_{se}	-0.00103215
		$p_{ye}(1)$	-0.00006766	$\phi(^o)$	45.00361191
				$\theta(^o)$	0.00167157

Table 1: The results of the central ray at the exit of the Snake.

For another half ($s > 0$) of the full snake, the symmetrical conditions were

$$\begin{cases} \bar{B}_x(x,y,s) &= \quad B_x(-x,y,-s) \\ \bar{B}_y(x,y,s) &= -B_y(-x,y,-s) \\ \bar{B}_s(x,y,s) &= \quad B_s(-x,y,-s) \end{cases} \tag{17}$$

The magnetic field between two helical magnets is simply the overlap of the fringing fields of the two helical magnets. The fringing field effect in the region of the entrance and the exit of the snake has been taken into account as much as possible.

The field strengths of four helical magnets are set so as to turn the direction of the spin of the proton by 180^o without any effect on the orbital motion. The more important point is to make the spin precess from (0 1 0) to (0 -1 0) as close as possible. The orbital deviation from $(0,0,0,0)$ for the central ray could be corrected by other way if both condition could not be satisfied simultaneously. Based on this point and considering the symmetry $B_3 = -B_2, B_4 = -B_1$, the peak fields B_{1m}, B_{2m}, which represent the maximum field strength of the first and the second helical magnet respectively, were adjusted at injection ($\gamma = 27$) to be as follows,

$$B_{1m} = 1.2433899\ Tesla.$$

$$B_{2m} = 3.8813956\ Tesla.$$

Table 1 lists the results of the central ray, Fig.2 gives its trajectory and the corresponding spin precession.

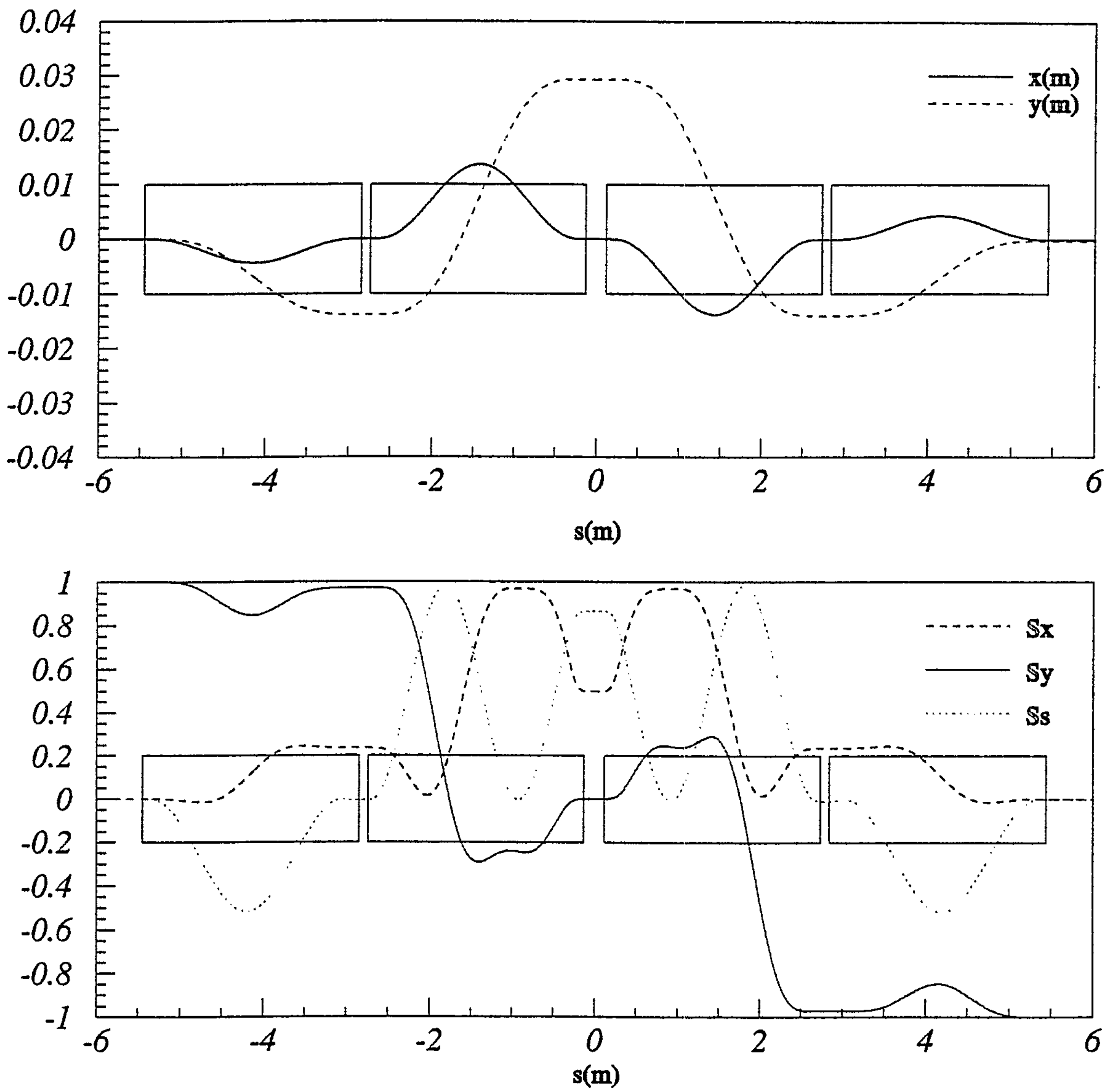

Figure 2: the trajectory of the central ray and its corresponding spin procession.

4 Spin matrix

The equations of spin precession are

$$\frac{dS_x}{ds} = S_sP_y - S_yP_s \quad (18)$$

$$\frac{dS_y}{ds} = S_xP_s - S_sP_x \quad (19)$$

$$\frac{dS_s}{ds} = S_yP_x - S_xP_y \quad (20)$$

where

$$\begin{aligned} P_x &= \Gamma_1 \cdot [-uB_s + (1+v^2)B_x - uvB_y] + u \cdot \Gamma_2 \\ P_y &= \Gamma_1 \cdot [-vB_s + (1+u^2)B_x - uvB_y] + v \cdot \Gamma_2 \\ P_s &= \Gamma_1 \cdot [-uB_x - vB_y + (u^2+v^2)B_s] + \Gamma_2 \end{aligned}$$

and

$$\begin{aligned} \Gamma_1 &= \frac{h}{B\rho} \cdot (1+G\gamma), \\ \Gamma_2 &= \frac{h}{B\rho} \cdot (1+G)(uB_x + vB_y + B_s), \end{aligned}$$

$$h = \frac{1}{\sqrt{u^2+v^2+1}}, B\rho = m_0\gamma V, G = 1.7928456.$$

The axis of spin precession was calculated from the results of the integration of two groups of spin precession $(S_x, S_y, S_s), (S_x^A, S_y^A, S_s^A)$ as follows,

$$\begin{cases} \tan\phi &= \frac{\sigma_x}{\sigma_z} \\ \tan\theta &= \frac{\sigma_y}{\sqrt{\sigma_x^2+\sigma_z^2}} \end{cases} \quad (21)$$

where

$$\begin{aligned} \vec{\sigma} &= \delta\vec{S} \times \delta\vec{S}^A \\ \vec{\delta S} &= \vec{S}_f - \vec{S}_i \\ \delta\vec{S}^A &= \vec{S}_f^A - \vec{S}_i^A \end{aligned}$$

and $\vec{S}_f$, $\vec{S}_f^A$, $\vec{S}_i$, $\vec{S}_i^A$ are the final and initial value of two groups of the spin precession respectively.

The spin matrix was obtained by sending three representative central rays with inital spin $(1,0,0),(010),(001)$ respectively, and taking the final results of them at exit. The result is

$$M_s = \begin{bmatrix} 0.00012657 & -0.00103203 & 0.99999946 \\ 0.00094950 & -0.99999902 & -0.00103215 \\ 0.99999954 & 0.00094963 & -0.00012559 \end{bmatrix},$$

The deviation of this spin matrix from unitary is 2.7806033×10^{-9}. It is very close to unitary as required. The angle ϕ of the spin precession axis is 45.00361191^o, and θ is 0.00167157^o.

5 Orbital matrix calculation by the DA approach

5.1 DA approach

Differential algebra is a technique for systematically propagating the derivatives of a function $f(x_i)$ through mathematical transformations on f by simply applying the familiar sum, product and chain rule of differentiation. The derivatives of any complicated function which may be obtained by successive mapping can be calculated by extending any function f to a vector $\mathbf{f}$ which contains the value of the function as the first element and the values of the derivatives with respect to all the variables up to the desired order in the subsequent elements. These vectors are called "DA-vectors"[10].

$$\begin{aligned} f(x_i) \rightarrow \mathbf{f}(\mathbf{x_i}) &= \{f, .., \frac{\partial f}{\partial x_i} .., .. \frac{\partial^2 f}{\partial x_i \partial x_j} .., ...\} \\ &= \{f, ..f_{x_i} .., ..f_{x_i x_j} .., ...\} \end{aligned} \tag{22}$$

For map tracking, all phase-space coordinates z_i becomes the DA vectors $\mathbf{z_i}$. The first element of $\mathbf{z_i}$ contains the current value of the coordinates z_i, and the subsequent elements contain the derivatives with respect to the initial values of z_i. The vectors $\mathbf{z_i}$ are initialized by setting the first element to the initial value of z_i, the element which contains the first derivative with respect to the $i-th$ initial coordinate is set to one and all other elements are zero. Each mathematical operation which involves phase-space variables is replaced by a vector operation.

In our code we used the old DA package of Berz later modified by Bengtson and Forest at the Lawrence Berkeley Laboratory. This package contains FORTRAN routines which must replace the usual FORTRAN calls and operations whenever necessary. It should be noted that this task is greatly simplified if the DA-package (this one or any other) is linked to a code written in a language that allows operator overloading such as C++.

5.2 The expression of the magnetic field for DA

The force terms in the equations of orbital motion is the magnetic field, and they are given numerically. In order to make use of DA, the magnetic field interpolated by spline function was reconstructed again to second order as follows,

$$B_w(x,y,s) = b_w(x,y,s) + \frac{\partial b_w(x,y,s)}{\partial x} \cdot x + \frac{\partial b_w(x,y,s)}{\partial y} \cdot y$$
$$+\frac{1}{2} \cdot \frac{\partial^2 b_w(x,y,s)}{\partial x^2} \cdot x^2 + \frac{\partial^2 b_w(x,y,s)}{\partial x \partial y} \cdot xy + \frac{1}{2} \cdot \frac{\partial^2 b_w(x,y,s)}{\partial y^2} \cdot y^2$$
$$+ \dots \tag{23}$$
$$w = x, y, \text{ or } s$$

The magnetic field $b_w(x,y,s)$ as well as its first and second derivatives are expressed by spline interpolation function in two steps.

1) Fitted on the plane $(s = s_k)$

Bicubic spline function was taken to fit $b_w(x,y,s_k), (w = x, y \text{ or } s), (k = 1, 2, \dots, L_k)$, as well as their derivatives at any sub-region as follows,

$$b_w(x,y,s_k) = \sum_{K,L=1}^{4} A_{wijKL}(x,y)(x-x_i)^{K-1}(y-y_j)^{L-1} \tag{24}$$
$$\frac{\partial b_w(x,y,s_k)}{\partial x} = \sum_{K,L=1}^{4} (K-1)A_{wijKL}(x,y)(x-x_i)^{K-2}(y-y_j)^{L-1} \tag{25}$$
$$\frac{\partial b_w(x,y,s_k)}{\partial y} = \sum_{K,L=1}^{4} (L-1)A_{wijKL}(x,y)(x-x_i)^{K-1}(y-y_j)^{L-2} \tag{26}$$
$$\frac{\partial^2 b_w(x,y,s_k)}{\partial x^2} = \sum_{K,L=1}^{4} (K-1)(K-2)A_{wijKL}(x,y)(x-x_i)^{K-3}(y-y_j)^{L-1} \tag{27}$$
$$\frac{\partial^2 b_w(x,y,s_k)}{\partial x \partial y} = \sum_{K,L=1}^{4} (K-1)(L-1)A_{wijKL}(x,y)(x-x_i)^{K-2}(y-y_j)^{L-2} \tag{28}$$
$$\frac{\partial^2 b_w(x,y,s_k)}{\partial y^2} = \sum_{K,L=1}^{4} (L-1)(L-2)A_{wijKL}(x,y)(x-x_i)^{K-1}(y-y_j)^{L-3} \tag{29}$$
$$x_i \le x \le x_{i+1}, y_j \le y \le y_{j+1} (i = 1, 2, \dots, m-1; j = 1, 2, \dots, n-1)$$

The coefficients A_{wijKL} are determined by one-dimensional cubic spline function,and a two - dimensional 3-points Lagrangian function

$$b_w(x,y,s_k) = \sum_{i,j=1}^{3} [\prod_{\substack{m,n=1,\\ m\ne i, n\ne j}}^{3} (\frac{x-x_m}{x_i-x_m})(\frac{x-x_m}{x_i-x_m})] \cdot b_w(x_i,y_j,s_k) \tag{30}$$

was constructed to calculate the first and second derivatives at the boundaries.

$$\frac{\partial b_w(x, y_j, s_k)}{\partial x}(x = x_1, x_m; j = 1, 2, \ldots, n)$$
$$\frac{\partial b_w(x_i, y, s_k)}{\partial y}(y = y_1, y_n; i = 1, 2, \ldots, m)$$
$$\frac{\partial^2 b_w(x, y, s_k)}{\partial x \partial y}(x = x_1, x_m; y = y_1, y_n)$$

2) Fitted in s-direction

By using the coefficients A_{wijKL}, the magnetic field $b_w(x, y, s_k)$ and its first and second derivatives at every plane of $s = s_k$ were obtained first. Then, $b_w(x, y, s)$ and its first and second derivatives were fitted again by one dimensional spline function in s-direction.

$$b_w(x, y, s) = \sum_{N=1}^{4} A_{wkN}(s)(s - s_k)^{N-1}, \tag{31}$$
$$\frac{\partial b_w(x, y, s)}{\partial x} = \sum_{N=1}^{4} A10_{wkN}(s)(s - s_k)^{N-1}, \tag{32}$$
$$\frac{\partial b_w(x, y, s)}{\partial y} = \sum_{N=1}^{4} A01_{wkN}(s)(s - s_k)^{N-1}, \tag{33}$$
$$\frac{\partial^2 b_w(x, y, s)}{\partial x^2} = \sum_{N=1}^{4} A20_{wkN}(s)(s - s_k)^{N-1}, \tag{34}$$
$$\frac{\partial^2 b_w(x, y, s)}{\partial x \partial y} = \sum_{N=1}^{4} A11_{wkN}(s)(s - s_k)^{N-1}, \tag{35}$$
$$\frac{\partial^2 b_w(x, y, s)}{\partial y^2} = \sum_{N=1}^{4} A02_{wkN}(s)(s - s_k)^{N-1} \tag{36}$$
$$s_k \leq s \leq s_{s+1}, (k = 1, 2, \ldots, L_k)$$

As we can see from the formulae above, the first derivatives of the magnetic field are smooth, and the second derivatives are continuous.

5.3 Orbital matrice to the second order by DA

The LBNL version of the DA package written by Berz was used and the program SSSTRA was modified into a DA-version, Named DA-SSSTRA. The variables (x, x'), (y, y') and (δ, L), as well as the three components of the magnetic field were converted into DA-vector. The Runge-Kutta integrator was also converted into DA integrator since it obviously handles phase space variables, Then, the truncated Taylor map to second order can be obtained around any ray started from the entrance of the snake.

Setting $\delta = 0$, the four-dimensional map to the second order around the central ray is calculated, and the results for 0th and first order are given as follows,

$$\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} = \begin{bmatrix} -0.12385485 \\ 0.00404683 \\ -0.41663179 \\ -0.0676637 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.99742496 & 12.27518760 & -0.00102426 & -0.02715057 \\ -0.00041083 & 0.99751220 & -0.00004195 & -0.00387867 \\ -0.00309773 & 0.02360347 & 0.93446509 & 11.92662949 \\ -0.00010177 & 0.00608138 & -0.01058291 & 0.93505074 \end{bmatrix} \begin{bmatrix} x_o \\ p_{x_o} \\ y_o \\ p_{y_o} \end{bmatrix}$$

The second order coeficients are given as follows,

x_o^2	$x_o \cdot p_{x_o}$	$p_{x_o}^2$	$x_o \cdot y_o$	$p_{x_o} \cdot y_o$
0.00020947	0.00316546	0.00990645	−0.00215393	−0.01120040
0.00004617	0.00071181	0.00276977	−0.00036805	−0.00236419
0.00056057	0.00478956	0.01465320	0.00146453	0.00255622
0.00008808	0.00105557	0.00408687	0.00008888	−0.00034413

$x_o \cdot p_{y_o}$	$p_{x_o} \cdot p_{y_o}$	y_o^2	$y_o \cdot p_{y_o}$	$p_{y_o}^2$
−0.01128183	−0.07119621	−0.00061085	−0.00185626	0.00461867
−0.00239398	−0.01809435	−0.00002344	0.00067186	0.00520059
0.00252052	−0.00964764	−0.00053097	−0.00741909	−0.02429417
−0.00035647	−0.00683366	−0.00009814	−0.00135330	−0.00464027

The unit is [mm.] for x_o, y_o, and $[10^{-3}]$ for p_{x_o}, p_{y_o}.

6 Symplectification

6.1 Symplectic condition checking for the first order matrix

A $2n \times 2n$ matrix M is said to be symplectic if the matrix M

$$M^T \cdot J \cdot M = J \tag{37}$$

where M^T is the transpose of M, and J is the matrix

$$J = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}.$$

It follows from (37) that a symplectic matrix has a unit determinant. For 2×2 case, the necessary and sufficient condition for a matrix to be symplectic is that it has a determinant of +1.

The 4×4 linear matrix of the snake obtained by DA method is

$$M_T = \begin{bmatrix} 0.99742496 & 12.27518760 & -0.00102426 & -0.02715057 \\ -0.00041083 & 0.99751220 & -0.00004195 & -0.00387867 \\ -0.00309773 & 0.02360347 & 0.93446509 & 11.92662949 \\ -0.00010177 & 0.00608138 & -0.01058291 & 0.93505074 \end{bmatrix}$$

then,

$$M_T^T \cdot J \cdot M_T - J$$

$$= \begin{bmatrix} 0.00000000E+00 & -0.29804412E-04 & 0.85622196E-04 & -0.55626436E-02 \\ 0.29804412E-04 & 0.00000000E+00 & -0.54258057E-02 & -0.70988243E-01 \\ -0.85622196E-04 & 0.54258057E-02 & 0.00000000E+00 & -0.64557730E-05 \\ 0.55626436E-02 & 0.70988243E-01 & 0.64557730E-05 & 0.00000000E+00 \end{bmatrix}$$

The determinant of the matrix is $1 + 1.4432899 \times 10^{-14}$.

The results show that although the deviation of the determinant of the orbital matrix from 1 is $1.4432899 \times 10^{-14}$, it does not satisfy the symplectic conditions completely, the maximun error among the matrix elements is 7.0988243×10^{-2}. The Maxwellian property of the magnetic field around the central ray was checked and it was found that the maximum error is around 10^{-2}, the same order as the symplectic property of the orbital matrix.

6.2 A method for symplectifying the map

In the appendix we clarify the meaning of "symplectification." For our purpose here, let's evaluate the map of Siberian Sanke first. The map M (including second order) from the entrance to the exit of the snake is nearly the map for a drift of length equal to that of the full snake. Therefore, construct a new map M_K, so that

$$M = D(\frac{L}{2}) \cdot M_K \cdot D(\frac{L}{2}) \tag{38}$$

D(L/2) is a drift of half length of the full snake, and

$$M_K = D^{-1}(\frac{L}{2}) \cdot M \cdot D^{-1}(\frac{L}{2}), \tag{39}$$

The linear part of the M_K is a matrix near the identity.

Rewrite M_K in the form as follows,

$$M_K = \exp(\vec{F} \cdot \nabla) Id \tag{40}$$

where Id=identity map

Now we know that $\vec{F} \cdot \nabla$ is a Poisson Bracket operator if and only if M_K is symplectic,

$$\vec{F} \cdot \nabla \Rightarrow : f :$$

Suppose $\vec{g}$ is a vector function of phase space then,

$$: f : \vec{g} = \nabla f^T \cdot J \vec{g} = -J \nabla f \cdot \nabla \vec{g} \tag{41}$$

Since $\vec{g}$ is an arbitary vector function, then

$$\begin{aligned} : f : &= \vec{F} \cdot \nabla = -J \nabla f \cdot \nabla \\ \vec{F} &= -J \nabla f \qquad (42) \\ f &= \int_0^x J \cdot \vec{F} \cdot dx' \qquad (43) \end{aligned}$$

If $\vec{F}$ is symplectic, then this computation involves an integral of a curl free function. Thus the function f is unique. If $\vec{F}$ is slightly non-symplectic, then the function f is one possible symplectification of the vector field $\vec{F}$, and it will depend on the path of the integration.

In the program DA-SSSTRA, a possible symplectic matrix $(M_K)_{symp}$ for M_K was computed by an iterative process as follows. First we compute the vector $\vec{F}$ such that $M_K = \exp\left(\vec{F} \cdot \nabla\right) Id$. Later we will find a Poisson bracket vector field approximating this general vector field.

Like all iterative procedure we assume that at a given step during the iteration we have obtained a vector field $\vec{F_j}$ which is not yet the generator of M_K. We then compute

$$vectorm = exp(-\vec{F_j} \cdot \nabla) \cdot M_K - Id \tag{44}$$

as a correction to $\vec{F}$, and check whether *vectorm* is close enough to zero. If not, then we set

$$\vec{F}_{j+1} \Leftarrow \vec{F}_j + vectorm \tag{45}$$

and repeat the iteration.

Notice that this computation does not separate the map to be symplectified into the first and second order, so it can be used for arbitrary order symplectification. Once $: f :$ is obtained from $\vec{F}$, the resulting symplectified map for M_k is

$$(M_k)_{symp.} = \exp(: f :) Id \tag{46}$$

and

$$(M)_{symp.} = D(L/2) \cdot (M_K)_{symp.} \cdot D(L/2) \tag{47}$$

6.3 Symplectification of the map for the Siberian Snake

The map for the Siberian Snake generated by numerical magnetic field including second order was symplectified, and the result is given as follows,

Symplectified linear matrix

$$\begin{bmatrix} 0.99742235 & 12.27524147 & -0.00425569 & -0.02841177 \\ -0.00041110 & 0.99752264 & -0.00008353 & -0.00108963 \\ 0.00001192 & 0.02441573 & 0.93446289 & 11.92668196 \\ -0.00009117 & 0.00316904 & -0.01058277 & 0.93506167 \end{bmatrix}$$

The result of symplecticity checking for the linear matrix is

$$\begin{bmatrix} 0.00000000E+00 & 0.00000000E+00 & 0.00000000E+00 & 0.21684043E-18 \\ 0.00000000E+00 & 0.00000000E+00 & 0.86736174E-18 & 0.00000000E+00 \\ -0.22499313E-20 & -0.92157185E-18 & 0.00000000E+00 & 0.00000000E+00 \\ -0.24563955E-18 & 0.34694470E-17 & -0.22204460E-15 & 0.00000000E+00 \end{bmatrix}$$

The deviation of the matrix determinant from 1 is $2.2204460 \times 10^{-16}$.

The 4-dimensional symplectified second order matrix is

x_o^2	$x_o \cdot p_{x_o}$	$p_{x_o}^2$	$x_o \cdot y_o$	$p_{x_o} \cdot y_o$
0.00020872	0.00315862	0.00994728	−0.00106071	−0.00589758
0.00004584	0.00070896	0.00277490	−0.00018663	−0.00123443
−0.00052941	−0.00592733	−0.01900004	0.00058169	0.00112769
−0.00093622	−0.00125001	−0.00473489	0.00004734	0.00000969

$x_o \cdot p_{y_o}$	$p_{x_o} \cdot p_{y_o}$	y_o^2	$y_o \cdot p_{y_o}$	$p_{y_o}^2$
−0.00593241	−0.03783892	0.00030068	0.00121178	−0.00132230
−0.00124618	−0.00939109	0.00002554	0.00002510	−0.00045167
0.00111315	−0.00311945	−0.00053122	−0.00744960	−0.02442827
0.00000883	−0.00097815	−0.00009812	−0.00135791	−0.00466336

Symplectification results show that there is a little adjustment for the linear map of the snake calculated by DA to be symplectic, but for second order map, the elements corresponding to cross terms, such as $x_0 \cdot p_{x_o}$ and so on, change a lot.

Further work is being done to put the symplectified map into the full lattice of RHIC for spin tracking. The accuracy will have to be gauged in an actual run.

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[11] It was pointed out by Berz that one can partially invert a Taylor series map in terms of a subset of variables (momenta for example) and then compute a mixed generating function by integration. Berz, in his packages, provides this functionality. In fact his COSY Infinity program (see his WEB page) provides generating function tracking. It must be said that they were originally in the code Marylie of Dragt et al. using an akward connection between Poisson bracket operators and generating functions.

[12] D.Abell, *Analytic Properties and Cremona Approximation of Transfer Maps for Hamiltonian Systems* University of Maryland Ph.D. thesis, 1995.

Appendix

There are two distinct concepts in the area of "symplectification" which are often confused. In order to distinguish them we must introduce the concept of symplectic truncation. Consider a symplectic map of phase space $\vec{m}_\varepsilon(\vec{x},\vec{p})$ that depends on a parameter ε. If by assumption this map is symplectic for all values of ε, then there cannot be any issue of symplectification associated to the map $\vec{m}_\varepsilon$. However let us assume that we expand this map in powers of ε to a specified order k and denote this map by $\vec{m}_\varepsilon^k$:

$$\vec{m}_\varepsilon^k \;=\; \vec{T}^0 + \vec{T}^1\varepsilon + \cdots + \vec{T}^k\varepsilon^k. \tag{A.1}$$

Now we ask the following question: are the Taylor series coefficients $\vec{T}^j$, which are themselves functions of phase space, truly arbitrary? Or do there exist special relationships between these functions imposed by the symplecticity of the original map $\vec{m}_\varepsilon(\vec{x},\vec{p})$?

Indeed we find that there exist special conditions between the coefficients of a Taylor series. Therefore, if an approximate map is extracted as a Taylor series ordered in a parameter or ordered in the phase space degree, it is already clear that the Taylor series coefficients will themselves reveal the lack of symplecticity of the underlying tracking code. If the underlying tracking code is not symplectic, the coefficients need to be adjusted. It is this process which is described in the body of this article. Let us call this "Symplectic Completion of the Taylor Coefficients" and let us summarize it mathematically. We start with an almost map $\vec{n}_\varepsilon(\vec{x},\vec{p})$ and expand it to order k:

$$\vec{n}_\varepsilon^k \;=\; \vec{W}^0 + \vec{W}^1\varepsilon + \cdots + \vec{W}^k\varepsilon^k. \tag{A.2}$$

Then, "Symplectic Completion of the Taylor Coefficients" consists in adding to each term $\vec{W}^j$ a correction $\Delta\vec{W}^j$ such that the new Taylor expansion $\vec{n} + \Delta\vec{n}$ to order k

$$\vec{n}_\varepsilon^k + \Delta\vec{n}_\varepsilon^k \;=\; \sum_{j=0}^{k}\left(\Delta\vec{W}^j + \vec{W}^j\right)\varepsilon^j \tag{A.3}$$

is consistent with the expansion of a symplectic map. In this paper we will describe a simple way to produce a symplectic completion of Taylor coefficients.

Now, there is another problem more akin to symplectic integration and it is called "Symplectic completion of a map." To understand this issue we assume that we already have a Taylor series to order k consistant with the symplectic condition. Secondly we would like to use this in a tracking code and iterate the resulting map. Will the map be symplectic? The answer is NO. It will be symplectic to order ε^k but it will violate the symplectic condition at higher order in ε. Generally one must add a very large or, depending on the method, an infinite number of powers in ε so as to make sure that the map is symplectic to all orders in ε. There is one notable exception to this rule and it concerns linear maps in the phase space variables. Therefore if we extract a matrix from a nonsymplectic code, then it suffices to adjust the coefficients of this matrix (coefficient completion) to create a true *bona fide* symplectic map. For higher order maps, symplectic completion of the map is necessary but it will not be discussed here. Suffices to say that it is possible to do either through mixed variables generating functions techniques [11] or through kick/jolt factorization (see reference [12] for a marvellous piece of work on this topic).

Symplectic Coefficients Completion

To understand how symplectic completion of the coefficients works let us introduce the Dragt-Finn representation for the expanded symplectic map $\vec{m}_\varepsilon$. First we notice that the map $\vec{T}^0$ for $\varepsilon = 0$ is symplectic. So we might as well factor it out of the problem as follow:

$$\vec{r}_\varepsilon^{\,k} = \vec{m}_\varepsilon^{\,k} \circ \left(\vec{T}^0\right)^{-1} \quad = \quad Id + \vec{t}^1\varepsilon + \cdots + \vec{t}^k\varepsilon^k \;\; \text{where } \vec{t}^j = \vec{T}^j \circ \left(\vec{T}^0\right)^{-1}. \tag{A.4}$$

The Dragt-Finn factorization states that the residual map $\vec{r}_\varepsilon^{\,k}$ can be factorized as follows

$$\vec{r}_\varepsilon^{\,k} \quad = \quad \exp\left(\varepsilon : g_1 :\right) \cdots \exp\left(\varepsilon^k : g_k :\right) Id \tag{A.5}$$

where Id is the identity map in phase space and the operators $: g_j :$ are defined in terms of the well-known Poisson bracket, that is to say, the action of $: g_j :$ on an arbitrary function f is given by

$$: g_j : f \quad = \quad [g_j, f]\,. \tag{A.6}$$

The important point here is that the Dragt-Finn factorization is valid if and only if the expansion $\vec{r}_\varepsilon^{\,k}$ is consistant with the expansion of a symplectic map.

Equivalently we can rewrite this map as a single exponent

$$\begin{aligned} \vec{r}_\varepsilon^{\,k} \quad &= \quad \exp\left(\varepsilon : g_1 :\right) \cdots \exp\left(\varepsilon^k : g_k :\right) Id \\ &= \quad \exp\left(\varepsilon : f_1 + \cdots + \varepsilon^k f_k :\right) Id. \end{aligned} \tag{A.7}$$

In the case of the residual map it is always possible to do so because the residual is connected to the identity through the parameter ε. In general this is not possible unless the map is close to the identity. (See Dragt-Lecture notes[6])

If the map is not the expansion of a symplectic map, it is nevertheless possible to factorize it in terms of general vector field operators rather than Poisson bracket operators:

$$\vec{r}_\varepsilon^{\,k} \quad = \quad \exp\left(\varepsilon \vec{G}_1 \cdot \nabla\right) \cdots \exp\left(\varepsilon^k \vec{G}_k \cdot \nabla\right) Id$$

$$= \exp\left(\left\{\varepsilon\vec{F}_1 + \cdots + \varepsilon^k\vec{F}_k\right\}\cdot\nabla\right) Id. \quad (A.8)$$

We note that the Poisson bracket operator is a special case of the general vector field operator

$$\begin{aligned} :f:g &= [f,g] = \nabla f^T \cdot J \cdot \nabla g \\ &= \underbrace{-J\nabla f}_{\vec{F}} \cdot \nabla g\ . \end{aligned} \quad (A.9)$$

Therefore the method described in the main part of the paper consists in first finding a general vector field expansion for a properly defined residual map and then compute a Poisson bracket exponent for it. If the residual map is consistant with the symplectic condition, this process should be a mathematical tautology. By that we mean that any correct computation of the Poisson bracket function, followed by a recomputation of the vector fields should lead to the original vector fields. However if the map is slightly nonsymplectic, then a sensible algorithm should produce a small correction in the vector field, i.e., the vector field associated to the Poisson bracket should be a bit different from the original ones.

In our case we do not have an expansion in a parameter ε, therefore the major difference with the above discussion will be to first transform our Taylor map so as to bring its linear part near the identity. On that map we will apply an algorithm which assumes the existence of a single vector field representation. Then we will apply an iterative procedure to find this vector field representation. Of course convergence will provide a check on the method. Then we will extract potential candidates for the Poisson bracket operator. At that point recomputation of the Taylor series from these new operators will produced the symplectified coefficients. (This can be checked by yet applying the algorithm once more as it should display the tautology mentioned above). Finally, because this algorithm is implemented strickly with the DA tools, it is indeed order independent. The readers who are versed in these matters will also noticed that there are an infinite number of variations possible for the algorithm: this might help if the iterative procedure has a hard time to converge.