



BNL-103680-2014-TECH

AGS/RHIC/SN 054;BNL-103680-2014-IR

Relation Between Field Homogeneity and Multipole Coefficients for a Helical Dipole

T. Tominaka

April 1997

Collider Accelerator Department
Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No. DE-AC02-76CH00016 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Alternating Gradient Synchrotron Department
Relativistic Heavy Ion Collider Project
BROOKHAVEN NATIONAL LABORATORY
Upton, New York 11973

Spin Note

AGS/RHIC/SN No. 054

**Relation Between Field Homogeneity and Multipole
Coefficients for a Helical Dipole**

T. Tominaka

April 25, 1997

For Internal Distribution Only

Relation between Field Homogeneity and Multipole Coefficients for a Helical Dipole

T. Tominaka (RIKEN, Japan)

April 18, 1997

1. Introduction

The intrinsic limit of the field homogeneity for helical dipoles is calculated in this paper. In addition, the relation between field homogeneity and helical multipoles is investigated. This relation should be considered on the optimization of a helical dipole.

2. Intrinsic Limit of the Field Homogeneity for Helical Dipoles

The interior magnetic field of helical dipole coil with the infinite length is as follows, on the European definition, [1]

$$\left\{ \begin{array}{l} B_r(r,\theta,z) = -\frac{\partial \Psi_h}{\partial r} = B_{\text{ref}}(k) r_0 \sum_{n=1}^{\infty} n! \left[\frac{2}{n k r_0} \right]^n k I_n'(n k r) \{ -a_n(k) \cos(n(\theta - k z)) + b_n(k) \sin(n(\theta - k z)) \} \\ B_\theta(r,\theta,z) = -\frac{1}{r} \frac{\partial \Psi_h}{\partial \theta} = B_{\text{ref}}(k) r_0 \sum_{n=1}^{\infty} n! \left[\frac{2}{n k r_0} \right]^n \frac{I_n(n k r)}{r} \{ a_n(k) \sin(n(\theta - k z)) + b_n(k) \cos(n(\theta - k z)) \} \\ B_z(r,\theta,z) = -\frac{\partial \Psi_h}{\partial z} = B_{\text{ref}}(k) r_0 \sum_{n=1}^{\infty} (-k) n! \left[\frac{2}{n k r_0} \right]^n I_n(n k r) \{ a_n(k) \sin(n(\theta - k z)) + b_n(k) \cos(n(\theta - k z)) \} \end{array} \right. \quad (1)$$

where $k = 2\pi/L$, L is the helical pitch length, r_0 is the reference radius, and $I_n(nkr)$ is the modified Bessel function of the first kind of order n , and $K_n(nkr)$ is the modified Bessel function of the second kind of order n . Eq.(1) is equivalent with Eq.(4) - Eq.(7) of Fischer's [2], on the American definition. Therefore, the y component of field at $z=0$ $B_y(r,\theta,z=0)$ becomes,

$$B_y(r,\theta,z=0) = B_r(r,\theta,z=0) \sin \theta + B_\theta(r,\theta,z=0) \cos \theta. \quad (2)$$

On the case with $a_n(k)=b_{2n}(k)=0$ for $n=1, 2, 3, \dots, \infty$, corresponding to the dipole symmetry,

$$\begin{aligned} B_y(r,\theta,z=0) &= B_{\text{ref}}(k) \sum_{n=1,3,5}^{\infty} b_n(k) r_0 n! \left[\frac{2}{n k r_0} \right]^n \left\{ k I_n'(n k r) \sin n\theta \sin \theta + \frac{I_n(n k r)}{r} \cos n\theta \cos \theta \right\} \\ &= B_{\text{ref}}(k) \sum_{n=1,3,5}^{\infty} b_n(k) M_n(k,r,\theta), \end{aligned} \quad (3)$$

where,

$$M_n(k,r,\theta) = r_0 n! \left[\frac{2}{n k r_0} \right]^n \left\{ k I_n'(n k r) \sin n\theta \sin \theta + \frac{I_n(n k r)}{r} \cos n\theta \cos \theta \right\}. \quad (4)$$

The limiting forms for small argument of $k I_n'(n k r)$ and $I_n(n k r)/r$ are as follows,[3]

$$\left\{ \begin{array}{l} \lim_{k \rightarrow 0} [k I_n'(n k r)] = \lim_{k \rightarrow 0} \left[k \left(I_{n-1}(n k r) - \frac{1}{k r} I_n(n k r) \right) \right] \\ = k \left(\frac{1}{(n-1)!} \left(\frac{n k r}{2} \right)^{n-1} - \frac{1}{k r} \frac{1}{n!} \left(\frac{n k r}{2} \right)^n \right) = \frac{1}{r} \frac{1}{n!} \left(\frac{n k r}{2} \right)^n \\ \lim_{k \rightarrow 0} \left[\frac{I_n(n k r)}{r} \right] = \frac{1}{r} \frac{1}{n!} \left(\frac{n k r}{2} \right)^n \end{array} \right. \quad (5)$$

Then, the asymptotic form for the y component of field at $z=0$ $B_y(r,\theta,z=0)$ as $k \rightarrow 0$ ($L \rightarrow \infty$) is,

$$\lim_{k \rightarrow 0} [B_{y,\text{helix}}(r,\theta,z=0)] = B_{y,2d}(r,\theta) = B_{\text{ref}} \sum_{n=1,3,5}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} b_n \cos [(n-1)\theta], \quad (6)$$

Then, it results that the asymptotic form of B_y for helical dipoles is equal to the usual form for 2-dimensional dipoles.

With the assumption of $b_1=1, b_3=0, b_5=0, \dots$, corresponding to the ideal dipole, the y component of field at $r=r_0, z=0, B_y(r=r_0, \theta, z=0)$ becomes,

$$\begin{aligned} B_y(r=r_0, \theta, z=0)|_{n=1} &= B_{\text{ref}}(k) M_1(k, r_0, \theta) = B_{\text{ref}}(k) \frac{2}{k} \left\{ k I_1(k r_0) \sin^2 \theta + \frac{I_1(k r_0)}{r_0} \cos^2 \theta \right\} \\ &= B_{\text{ref}}(k) \frac{2}{k} \left\{ k \left(I_0(k r_0) - \frac{I_1(k r_0)}{k r_0} \right) \sin^2 \theta + \frac{I_1(k r_0)}{r_0} \cos^2 \theta \right\} \quad (7) \\ &= B_{\text{ref}}(k) \left\{ I_0(k r_0) + \left(-I_0(k r_0) + \frac{2}{k} \frac{I_1(k r_0)}{r_0} \right) \cos 2\theta \right\} \\ &= B_{\text{ref}}(k) \left\{ 1.00165 - 8.24 \times 10^{-4} \cos 2\theta \right\}, \end{aligned}$$

where the helical pitch length $L=2.4$ m, $k = 2\pi/2.4 = 2.62$, $r_0=31$ mm are assumed. The 3D and contour plot of B_y for the ideal dipole are shown in Figs.1 and 2, with $B_{\text{ref}}(k)=B_y(r=0, \theta, z=0)=B_{y0}=4.0$ T. Similarly, the sextupole term of field at $r=r_0, z=0, B_y(r=r_0, \theta, z=0)$ becomes,

$$\begin{aligned} B_y(r=r_0, \theta, z=0)|_{n=3} &= B_{\text{ref}}(k) b_3(k) M_3(k, r_0, \theta) \\ &= B_{\text{ref}}(k) b_3(k) r_0 3! \left[\frac{2}{3 k r_0} \right]^3 \left\{ k I_3(3 k r_0) \sin 3\theta \sin \theta + \frac{I_3(3 k r_0)}{r_0} \cos 3\theta \cos \theta \right\} \quad (8) \\ &= B_{\text{ref}}(k) b_3(k) r_0 3! \left[\frac{2}{3 k r_0} \right]^3 \left\{ \frac{k I_2(3 k r_0)}{2} \cos 2\theta + \left(-\frac{k I_2(3 k r_0)}{2} + \frac{I_3(3 k r_0)}{r_0} \right) \cos 4\theta \right\} \\ &= B_{\text{ref}}(k) b_3(k) \left\{ 1.00495 \cos 2\theta - 1.24 \times 10^{-3} \cos 4\theta \right\}. \end{aligned}$$

Therefore, with the following value of the helical sextupole coefficient, $b_3(k)$,

$$b_3(k) = - \frac{\left(-I_0(k r_0) + \frac{2}{k} \frac{I_1(k r_0)}{r_0} \right)}{r_0 3! \left[\frac{2}{3 k r_0} \right]^3 \frac{k I_2(3 k r_0)}{2}} \approx 8.20 \times 10^{-4}, \quad (9)$$

the $\cos 2\theta$ term of $B_y(r=r_0, \theta, z=0)$ vanishes. The similar results are obtained by other authors.[4,5] As a result, with the assumption of $b_1(k)=1, b_3(k)=8.2 \times 10^{-4}, b_5(k)=0, \dots$, corresponding to the modified ideal dipole, the y component of field at $r=r_0, z=0, B_y(r=r_0, \theta, z=0)$ becomes,

$$\begin{aligned} B_y(r=r_0, \theta, z=0) &= B_{\text{ref}}(k) \left\{ M_1(k, r_0, \theta) + M_3(k, r_0, \theta) \right\} \\ &= B_{\text{ref}}(k) \left\{ I_0(k r_0) + b_3(k) r_0 3! \left[\frac{2}{3 k r_0} \right]^3 \left(-\frac{k I_2(3 k r_0)}{2} + \frac{I_3(3 k r_0)}{r_0} \right) \cos 4\theta \right\} \quad (10) \\ &= B_{\text{ref}}(k) \left\{ 1.00165 - 1.24 \times 10^{-3} b_3(k) \cos 4\theta \right\} = B_{\text{ref}}(k) \left\{ 1.00165 - 1.015 \times 10^{-6} \cos 4\theta \right\}. \end{aligned}$$

The 3D and contour plot of B_y for the modified ideal dipole are shown in Figs.3 and 4. Therefore, the homogeneity of the dipole field B_y at the circular region of $r=31$ mm is limited. As a result, with the helical sextupole coefficient $b_3(k)=8.2 \times 10^{-4}$, the minimum value of $|(B_y(r, \theta, z=0) - B_{y0})/B_{y0}|$ is about 0.165 % which is intrinsically determined from the value of the modified Bessel function of the first kind of order 0, $I_0(kr_0)$.

3. Relation between Field Homogeneity and Helical Multipoles

The relation between the field homogeneity at the circular region of $r=31$ mm and helical multipoles can be graphically investigated. The field homogeneity of the interior magnetic field for helical dipole coils with the infinite length is as follows with the definition of $b_1(k)=1$,

$$\left| \frac{B_y(k, r, \theta, z=0) - B_{\text{ref}}(k)}{B_{\text{ref}}(k)} \right| = \left| (M_1(k, r, \theta) - 1) + b_3(k) M_3(k, r, \theta) + b_5(k) M_5(k, r, \theta) + b_7(k) M_7(k, r, \theta) + \dots \right|. \quad (11)$$

Therefore, the requirement condition for the helical multipoles can be calculated from the prescribed homogeneity of the dipole field B_y at the boundary points of the circular region of $r=31$ mm shown in Fig.5, using Eq.(11). For example, for $|(B_y(r, \theta, z=0) - B_{y0})/B_{y0}|$ of 0.2 % and 0.4 %, with $b_7(k)=0, b_9(k)=0, \dots$, the satisfying region of $(b_3(k), b_5(k))$ for the prescribed field difference at two points, ($r=31$

mm, $\theta=0$) and ($r=31$ mm, $\theta=\pi/2$) are shown in Figs.6 and 7, respectively. The darker and narrower regions in Figs.6 and 7, corresponds to $|(\text{By}(r,\theta,z=0) - \text{By}_0)/\text{By}_0|$ of 0.2 %. The resultant satisfying regions of ($b_3(k)$, $b_5(k)$) for all boundary points for $|(\text{By}(r,\theta,z=0) - \text{By}_0)/\text{By}_0|$ of 0.18 % and 0.2 % are shown as the central white zones in Figs.8 and 9, respectively. It is also shown in Fig.10 that the satisfying region of ($b_3(k)$, $b_5(k)$) for all boundary points vanishes for $|(\text{By}(r,\theta,z=0) - \text{By}_0)/\text{By}_0|$ of 0.2 % with $b_7(k)=5 \times 10^{-4}$. Furthermore, for $|(\text{By}(r,\theta,z=0) - \text{By}_0)/\text{By}_0|$ of 0.3 %, the satisfying regions of ($b_3(k)$, $b_5(k)$) are shown in Figs.11, 12 and 13, with $b_7(k)=0$, $+5 \times 10^{-4}$, and -5×10^{-4} , respectively. In addition, with $b_7(k)=0$, $b_9(k)=0, \dots$, the satisfying region of ($b_3(k)$, $b_5(k)$) for all boundary points vanishes for $|(\text{By}(r,\theta,z=0) - \text{By}_0)/\text{By}_0|$ of < 0.165 %. This result is equivalent with Eq.(10). For comparison, the relation between the field homogeneity and 2D multipoles is described in Appendix.

4. Conclusion

The intrinsic limit of field homogeneity and the relation between field homogeneity and helical multipoles of helical dipoles are obtained. It can be realized that the field homogeneity of helical dipoles is significantly different from that of the ordinary 2D dipoles. This relation will be useful to optimize the cross-sectional shape of helical dipole coils.

5. Acknowledgments

The author is indebted for helpful discussions and comments to Prof. T. Katayama of Institute of Nuclear Physics, University of Tokyo and RIKEN.

Appendix. Magnetic Field of 2-dimensional Dipole Coils

The interior magnetic field of 2-dimensional dipole coil with the infinite length is as follows, on the European definition, [6]

$$\begin{cases} B_r(r,\theta) = -\frac{\partial \Psi}{\partial r} = \frac{1}{r} \frac{\partial A_z}{\partial \theta} = B_{\text{ref}} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (-a_n \cos n\theta + b_n \sin n\theta) \\ B_\theta(r,\theta) = -\frac{1}{r} \frac{\partial \Psi}{\partial \theta} = -\frac{\partial A_z}{\partial r} = B_{\text{ref}} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (b_n \cos n\theta + a_n \sin n\theta) \end{cases} \quad (A1)$$

Then,

$$B_y(r,\theta) = B_{\text{ref}} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \{a_n \sin [(n-1)\theta] + b_n \cos [(n-1)\theta]\} \quad (A2)$$

Therefore, on the case with the dipole symmetry, $a_n=b_{2n}=0$ for $n=1, 2, 3, \dots, \infty$, the y component of field, $B_y(r,\theta)$ becomes with $b_1=1$,

$$B_y(r,\theta) = B_{\text{ref}} \sum_{n=1,3,5}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} b_n \cos [(n-1)\theta] = B_{\text{ref}} \left(1 + b_3 \left(\frac{r}{r_0}\right)^2 \cos 2\theta + b_5 \left(\frac{r}{r_0}\right)^4 \cos 4\theta + b_7 \left(\frac{r}{r_0}\right)^6 \cos 6\theta + \dots\right) \quad (A3)$$

Then,

$$\left| \frac{B_y(r,\theta) - B_{\text{ref}}}{B_{\text{ref}}} \right| = \left| b_3 \left(\frac{r(\theta)}{r_0}\right)^2 \cos 2\theta + b_5 \left(\frac{r(\theta)}{r_0}\right)^4 \cos 4\theta + b_7 \left(\frac{r(\theta)}{r_0}\right)^6 \cos 6\theta + \dots \right| \quad (A4)$$

References

- 1) T. Tominaka, "Magnetic Field Calculation of Helical Dipole Coils", AGS/RHIC/SN No.24, April 29, (1996).
- 2) W. Fischer, "On Measurements of Helical Magnetic Fields Using Devices for Straight Magnets", AGS/RHIC/SN No.32, July 30, (1996).
- 3) M. Abramowitz and I. A. Stegun (eds.), "Handbook of Mathematical Functions", Dover, pp.374-376 (1970).
- 4) T. Katayama, "Multipole Field Expansion of Helical Magnet", AGS/RHIC/SN No.38, September 27, (1996).
- 5) M. Okamura, "Multipole Components in the BNL Type Helical Dipole Magnet", AGS/RHIC/SN No.46, November 29, (1996).
- 6) K. -H. Mess, P. Schmüser and S. Wolff, "Superconducting Accelerator Magnets", World Scientific, pp.44-53 (1996).

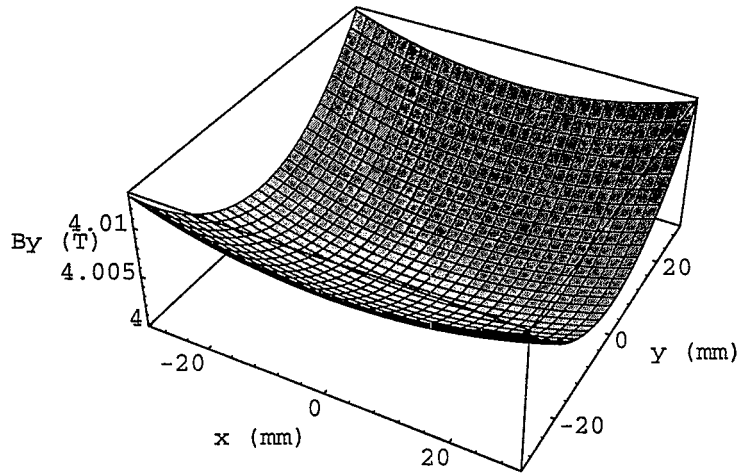


Fig.1 3D plot of B_y for the ideal helical dipole, with $b_1=1$, and $b_3, b_5, \dots=0$.

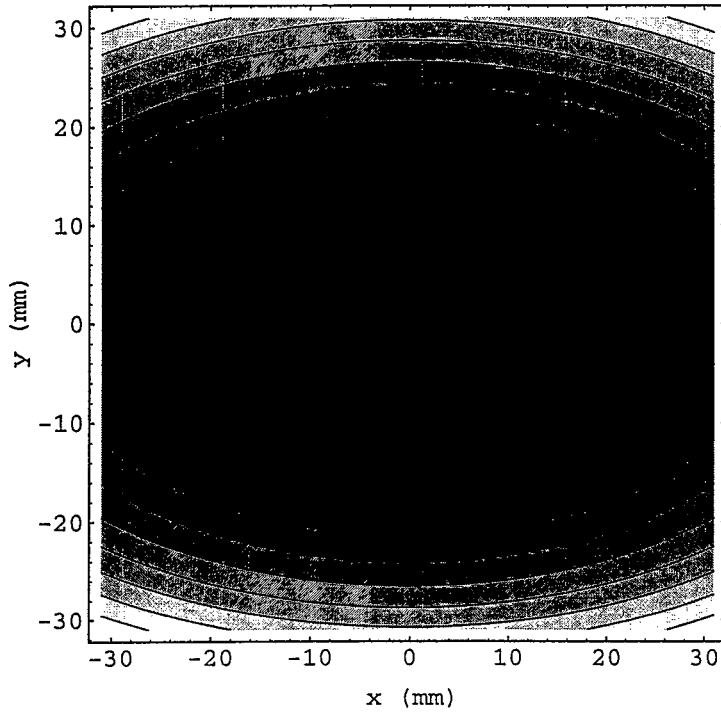


Fig.2 Contour plot of B_y for the ideal helical dipole, with $b_1=1$, and $b_3, b_5, \dots=0$.

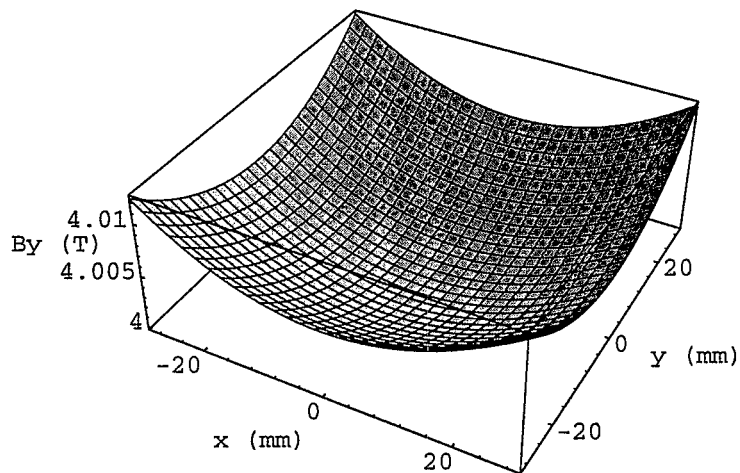


Fig.3 3D plot of B_y for the modified helical dipole, with $b_1=1$, $b_3=0.00082$, and $b_5, b_7, \dots=0$.

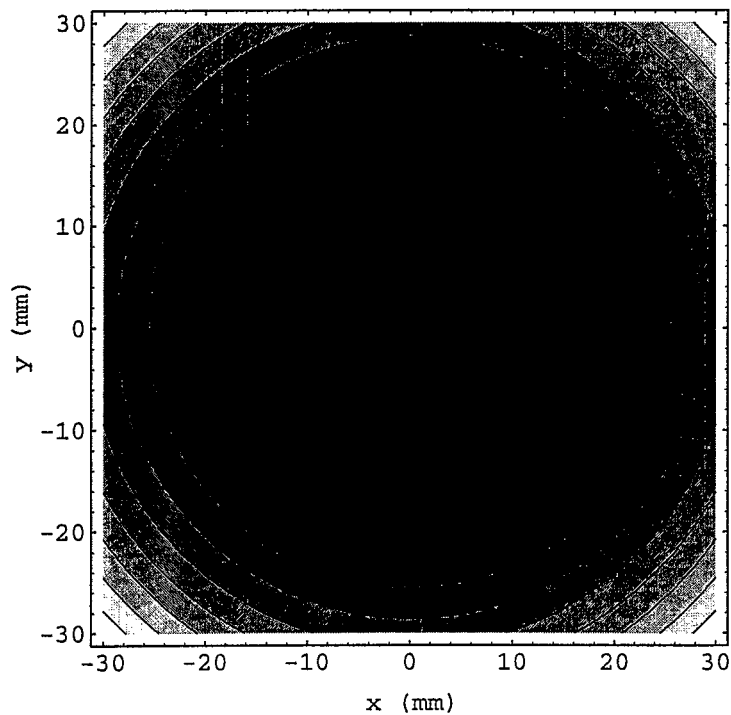


Fig.4 Contour plot of B_y for the modified helical dipole, with $b_1=1$, $b_3=0.00082$, and $b_5, b_7, \dots=0$.

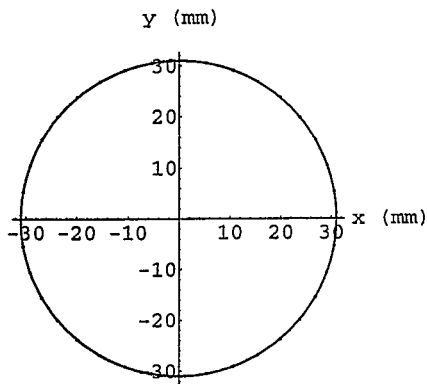


Fig.5 Homogeneous region of the magnetic field.

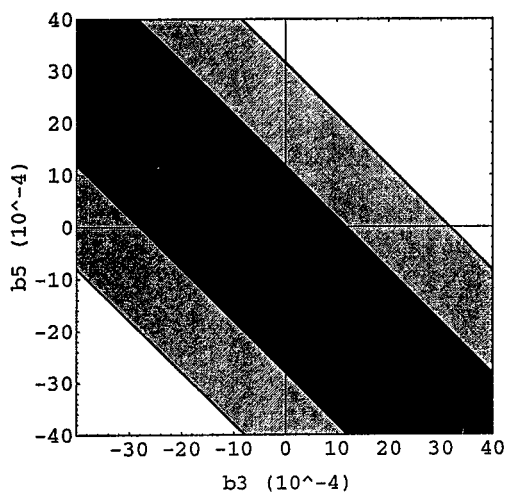


Fig.6 (b3, b5) relation of $< 0.2\%$ and $< 0.4\%$ of $|B_y - B_{y0}|/B_{y0}$ at $(x=31\text{mm}, y=0\text{mm})$ with $b_7=0$, and $b_9, b_{11}, \dots=0$.

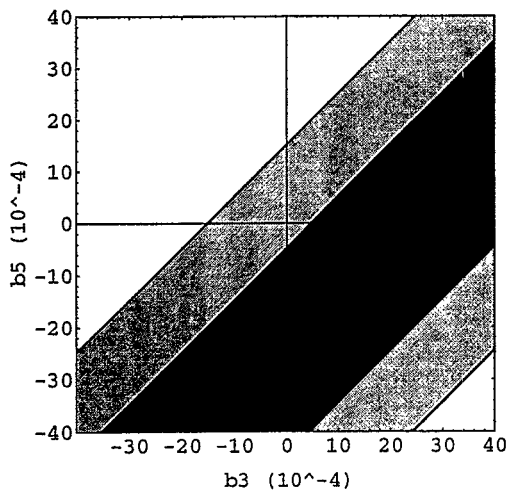


Fig.7 (b3, b5) relation of $< 0.2\%$ and $< 0.4\%$ of $|B_y - B_{y0}|/B_{y0}$ at $(x=0\text{mm}, y=31\text{mm})$ with $b_7=0$, and $b_9, b_{11}, \dots=0$.

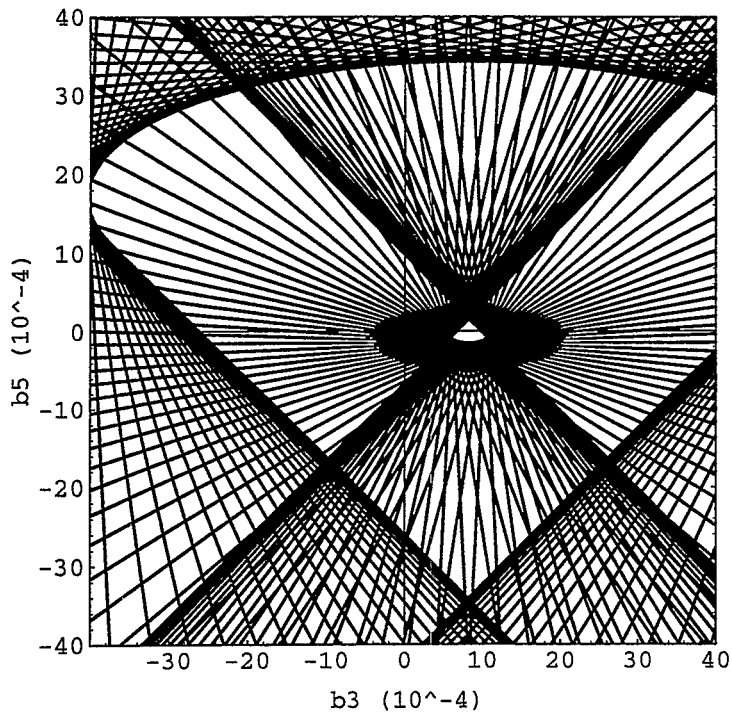


Fig.8 (b_3, b_5) region of $< 0.18\%$ of $|By-By_0|/By_0$ with $b_7=0$, and $b_9, b_{11}, \dots=0$.

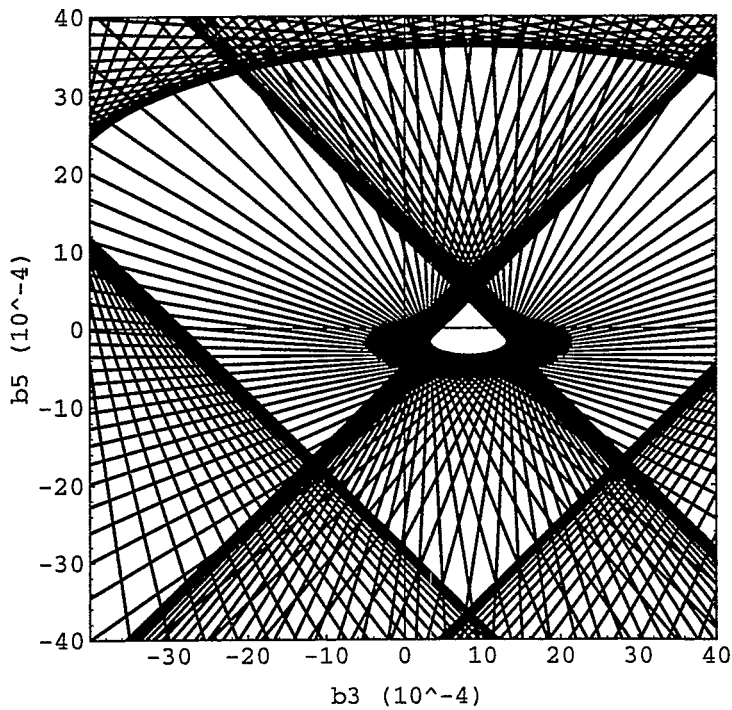


Fig.9 (b_3, b_5) region of $< 0.2\%$ of $|By-By_0|/By_0$ with $b_7=0$, and $b_9, b_{11}, \dots=0$.

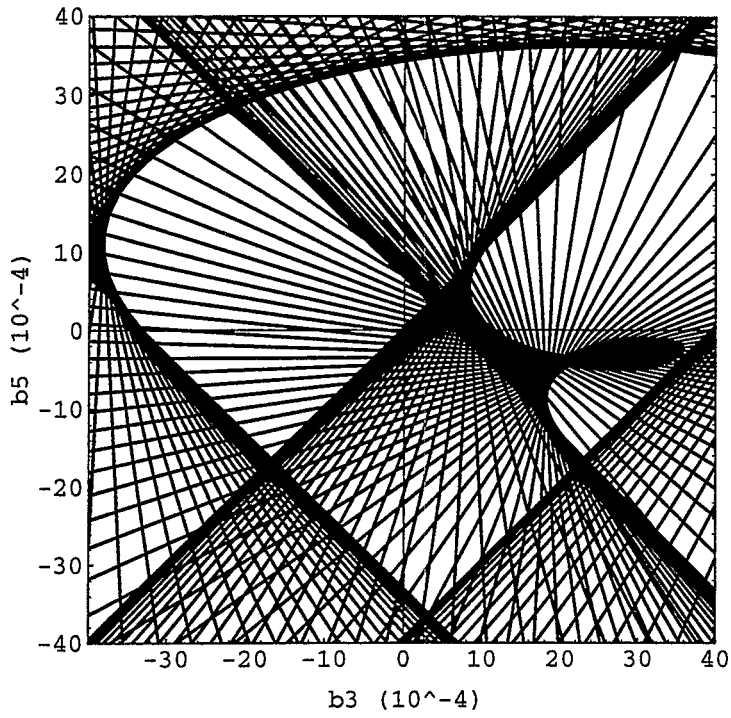


Fig.10 (b_3, b_5) region of $< 0.2\%$ of $|By-By_0|/By_0$ with $b_7=0.0005$, and $b_9, b_{11}, \dots=0$.

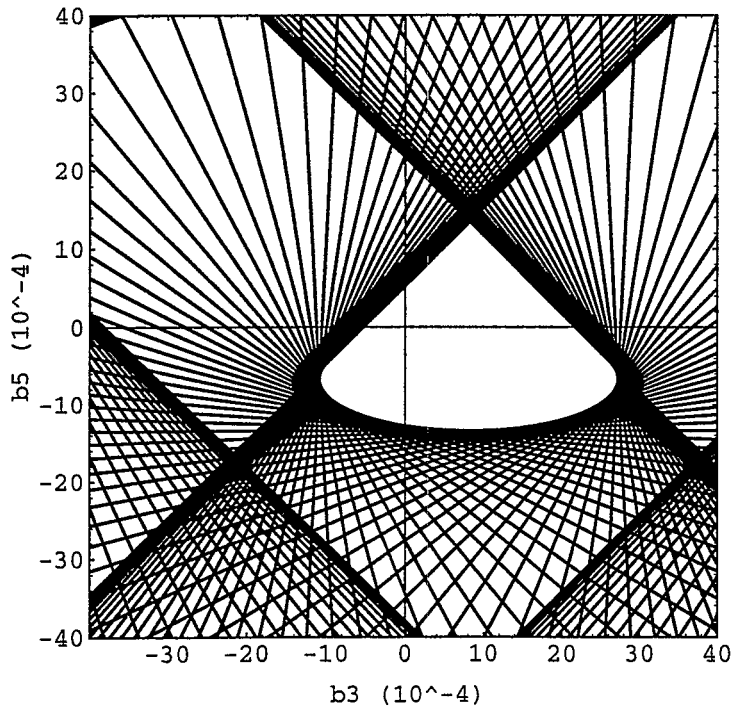


Fig.11 (b_3, b_5) region of $< 0.3\%$ of $|By-By_0|/By_0$ with $b_7=0$, and $b_9, b_{11}, \dots=0$.

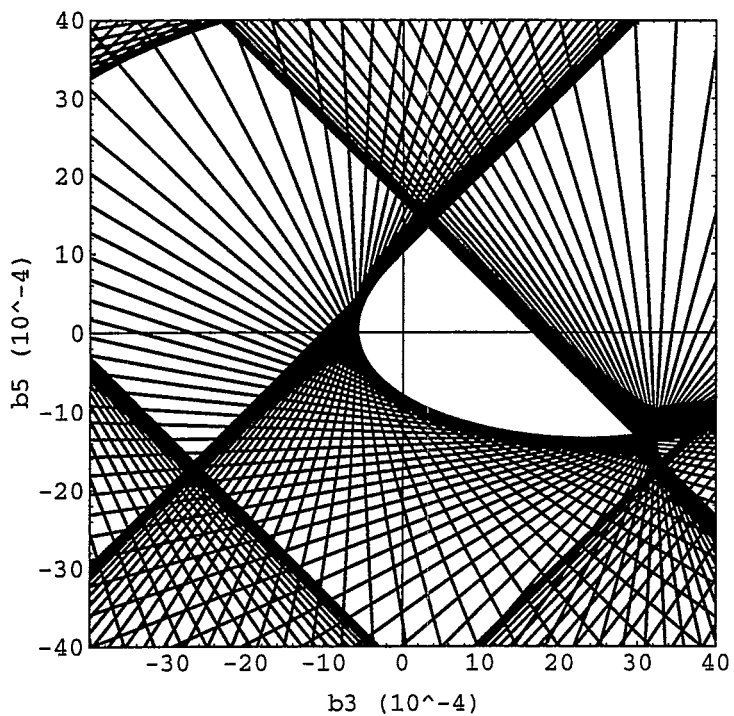


Fig.12 (b_3, b_5) region of $< 0.3\%$ of $|By-By_0|/By_0$ with $b_7=0.0005$, and $b_9, b_{11}, \dots=0$.

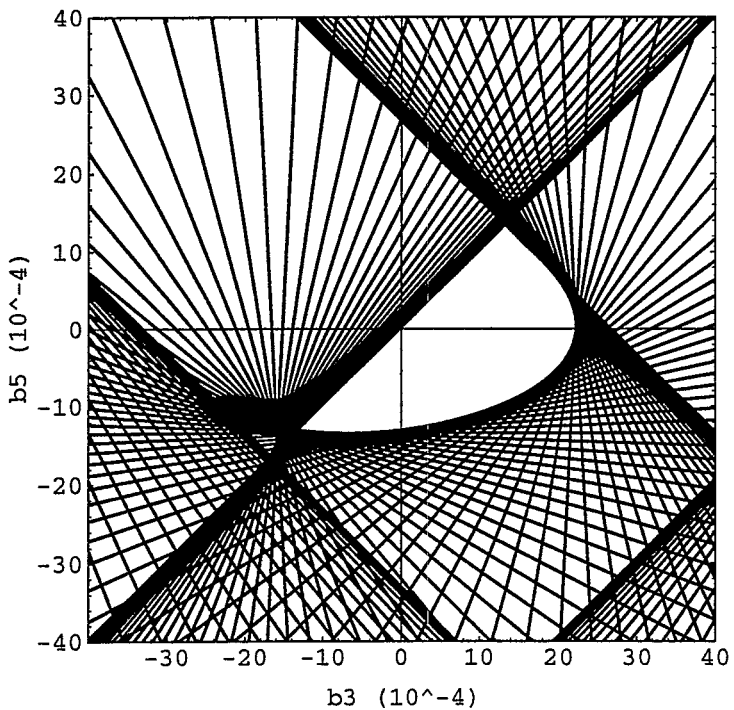


Fig.13 (b_3, b_5) region of $< 0.3\%$ of $|By-By_0|/By_0$ with $b_7=-0.0005$, and $b_9, b_{11}, \dots=0$.