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Relation Between Field Homogeneity and Multipole Coefficients for a Helical Dipole

T. Tominaka

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Collider Accelerator Department
Brookhaven National Laboratory

U.S. Department of Energy
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Alternating Gradient Synchrotron Department
Relativistic Heavy Ion Collider Project
BROOKHAVEN NATIONAL LABORATORY
Upton, New York 11973

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T. Tominaka (RIKEN, Japan)

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1. Introduction

The intrinsic limit of the field homogeneity for helical dipoles is calculated in this paper. In addition, the relation between field homogeneity and helical multipoles is investigated. This relation should be considered on the optimization of a helical dipole.

2. Intrinsic Limit of the Field Homogeneity for Helical Dipoles

The interior magnetic field of helical dipole coil with the infinite length is as follows, on the European definition, [1]

$$\begin{cases} B_r(r,\theta,z) = -\dfrac{\partial\psi_h}{\partial r} = B_{ref}(k)\, r_0 \displaystyle\sum_{n=1}^{\infty} n! \left[\frac{2}{n\,k\,r_0}\right]^n k\, I_n'(n\,k\,r) \left\{-a_n(k)\cos\left(n(\theta - k\,z)\right) + b_n(k)\sin\left(n(\theta - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = -\dfrac{1}{r}\dfrac{\partial\psi_h}{\partial\theta} = B_{ref}(k)\, r_0 \displaystyle\sum_{n=1}^{\infty} n! \left[\frac{2}{n\,k\,r_0}\right]^n \frac{I_n(n\,k\,r)}{r} \left\{a_n(k)\sin\left(n(\theta - k\,z)\right) + b_n(k)\cos\left(n(\theta - k\,z)\right)\right\} \\ B_z(r,\theta,z) = -\dfrac{\partial\psi_h}{\partial z} = B_{ref}(k)\, r_0 \displaystyle\sum_{n=1}^{\infty} (-k)\, n! \left[\frac{2}{n\,k\,r_0}\right]^n I_n(n\,k\,r) \left\{a_n(k)\sin\left(n(\theta - k\,z)\right) + b_n(k)\cos\left(n(\theta - k\,z)\right)\right\}, \end{cases} \tag{1}$$

where $k = 2\pi/L$, L is the helical pitch length, r_0 is the reference radius, and $I_n(nkr)$ is the modified Bessel function of the first kind of order n, and $K_n(nkr)$ is the modified Bessel function of the second kind of order n. Eq.(1) is equivalent with Eq.(4) - Eq.(7) of Fischer's [2], on the American definition. Therefore, the y component of field at z=0 $B_y(r,\theta,z=0)$ becomes,

$$B_y(r,\theta,z=0) = B_r(r,\theta,z=0)\sin\theta + B_\theta(r,\theta,z=0)\cos\theta. \tag{2}$$

On the case with $a_n(k)=b_{2n}(k)=0$ for $n=1, 2, 3, \ldots, \infty$, corresponding to the dipole symmetry,

$$\begin{aligned} B_y(r,\theta,z=0) &= B_{ref}(k) \sum_{n=1,3,5}^{\infty} b_n(k)\, r_0\, n! \left[\frac{2}{n\,k\,r_0}\right]^n \left\{k\, I_n'(n\,k\,r)\sin n\theta \sin\theta + \frac{I_n(n\,k\,r)}{r}\cos n\theta\cos\theta\right\} \\ &= B_{ref}(k) \sum_{n=1,3,5}^{\infty} b_n(k)\, M_n(k,r,\theta), \end{aligned} \tag{3}$$

where,

$$M_n(k,r,\theta) = r_0\, n! \left[\frac{2}{n\,k\,r_0}\right]^n \left\{k\, I_n'(n\,k\,r)\sin n\theta\sin\theta + \frac{I_n(n\,k\,r)}{r}\cos n\theta\cos\theta\right\}. \tag{4}$$

The limiting forms for small argument of $k\, I_n'(n\,k\,r)$ and $I_n(n\,k\,r)/r$ are as follows,[3]

$$\begin{cases} \lim_{k\to 0}\left[k\, I_n'(n\,k\,r)\right] = \lim_{k\to 0}\left[k\left(I_{n-1}(n\,k\,r) - \dfrac{1}{k\,r} I_n(n\,k\,r)\right)\right] \\ \quad = k\left(\dfrac{1}{(n-1)!}\left(\dfrac{n\,k\,r}{2}\right)^{n-1} - \dfrac{1}{k\,r}\dfrac{1}{n!}\left(\dfrac{n\,k\,r}{2}\right)^n\right) = \dfrac{1}{r}\dfrac{1}{n!}\left(\dfrac{n\,k\,r}{2}\right)^n \\ \lim_{k\to 0}\left[\dfrac{I_n(n\,k\,r)}{r}\right] = \dfrac{1}{r}\dfrac{1}{n!}\left(\dfrac{n\,k\,r}{2}\right)^n \end{cases} \tag{5}$$

Then, the asymptotic form for the y component of field at z=0 $B_y(r,\theta,z=0)$ as $k\to 0$ ($L\to\infty$) is,

$$\lim_{k\to 0}\left[B_{y,helix}(r,\theta,z=0)\right] = B_{y,2d}(r,\theta) = B_{ref} \sum_{n=1,3,5}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} b_n \cos\left[(n-1)\theta\right]. \tag{6}$$

Then, it results that the asymptotic form of By for helical dipoles is equal to the usual form for 2-dimensional dipoles.
With the assumption of $b_1=1$, $b_3=0$, $b_5=0$, ... , corresponding to the ideal dipole, the y component of field at $r=r_0$, $z=0$, $B_y(r=r_0,\theta,z=0)$ becomes,

$$
\begin{aligned}
B_y(r=r_0,\theta,z=0)\big|_{n=1} &= B_{ref}(k)\, M_1(k,r_0,\theta) = B_{ref}(k)\frac{2}{k}\left\{k\, I_1'(k\, r_0)\sin^2\theta + \frac{I_1(k\, r_0)}{r_0}\cos^2\theta\right\} \\
&= B_{ref}(k)\frac{2}{k}\left\{k\left(I_0(k\, r_0) - \frac{I_1(k\, r_0)}{kr_0}\right)\sin^2\theta + \frac{I_1(k\, r_0)}{r_0}\cos^2\theta\right\} \\
&= B_{ref}(k)\left\{I_0(k\, r_0) + \left(-I_0(k\, r_0) + \frac{2}{k}\frac{I_1(k\, r_0)}{r_0}\right)\cos 2\theta\right\} \\
&= B_{ref}(k)\left\{1.00165 - 8.24\times10^{-4}\cos 2\theta\right\},
\end{aligned}
\tag{7}
$$

where the helical pitch length $L=2.4$ m, $k = 2\pi/2.4 = 2.62$, $r_0=31$ mm are assumed. The 3D and contour plot of By for the ideal dipole are shown in Figs.1 and 2, with $B_{ref}(k)=B_y(r=0,\theta,z=0)=B_{y0}=4.0$ T. Similarly, the sextupole term of field at $r=r_0$, $z=0$, $B_y(r=r_0,\theta,z=0)$ becomes,

$$
\begin{aligned}
B_y(r=r_0,\theta,z=0)\big|_{n=3} &= B_{ref}(k)\, b_3(k)\, M_3(k,r_0,\theta) \\
&= B_{ref}(k)\, b_3(k)\, r_0\, 3!\left[\frac{2}{3\, k\, r_0}\right]^3\left\{k\, I_3'(3\, k\, r_0)\sin 3\theta\sin\theta + \frac{I_3(3\, k\, r_0)}{r_0}\cos 3\theta\cos\theta\right\} \\
&= B_{ref}(k)\, b_3(k)\, r_0\, 3!\left[\frac{2}{3\, k\, r_0}\right]^3\left\{\frac{k\, I_2(3\, k\, r_0)}{2}\cos 2\theta + \left(-\frac{k\, I_2(3\, k\, r_0)}{2} + \frac{I_3(3\, k\, r_0)}{r_0}\right)\cos 4\theta\right\} \\
&= B_{ref}(k)\, b_3(k)\left\{1.00495\cos 2\theta - 1.24\times10^{-3}\cos 4\theta\right\}.
\end{aligned}
\tag{8}
$$

Therefore, with the following value of the helical sextupole coefficient, $b_3(k)$,

$$
b_3(k) = -\frac{\left(-I_0(k\, r_0) + \frac{2}{k}\frac{I_1(k\, r_0)}{r_0}\right)}{r_0\, 3!\left[\frac{2}{3kr_0}\right]^3\frac{k\, I_2(3\, k\, r_0)}{2}} \approx 8.20\times10^{-4},
\tag{9}
$$

the $\cos 2\theta$ term of $B_y(r=r_0,\theta,z=0)$ vanishes. The similar results are obtained by other authors.[4,5] As a result, with the assumption of $b_1(k)=1$, $b_3(k)=8.2\times10^{-4}$, $b_5(k)=0$, ... , corresponding to the modified ideal dipole, the y component of field at $r=r_0$, $z=0$ $B_y(r=r_0,\theta,z=0)$ becomes,

$$
\begin{aligned}
B_y(r=r_0,\theta,z=0) &= B_{ref}(k)\left\{M_1(k,r_0,\theta) + M_3(k,r_0,\theta)\right\} \\
&= B_{ref}(k)\left\{I_0(k\, r_0) + b_3(k)\, r_0\, 3!\left[\frac{2}{3\, kr_0}\right]^3\left(-\frac{k\, I_2(3\, k\, r_0)}{2} + \frac{I_3(3\, k\, r_0)}{r_0}\right)\cos 4\theta\right\} \\
&= B_{ref}(k)\left\{1.00165 - 1.24\times10^{-3}\, b_3(k)\cos 4\theta\right\} = B_{ref}(k)\left\{1.00165 - 1.015\times10^{-6}\cos 4\theta\right\}.
\end{aligned}
\tag{10}
$$

The 3D and contour plot of By for the modified ideal dipole are shown in Figs.3 and 4. Therefore, the homogeneity of the dipole field By at the circular region of r=31 mm is limited. As a result, with the helical sextupole coefficient $b_3(k)=8.2\times10^{-4}$, the minimum value of $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ is about 0.165 % which is intrinsically determined from the value of the modified Bessel function of the first kind of order 0, $I_0(kr_0)$.

3. Relation between Field Homogeneity and Helical Multipoles

The relation between the field homogeneity at the circular region of r=31 mm and helical multipoles can be graphically investigated. The field homogeneity of the interior magnetic field for helical dipole coils with the infinite length is as follows with the definition of $b_1(k)=1$,

$$
\left|\frac{B_y(k,r,\theta,z=0) - B_{ref}(k)}{B_{ref}(k)}\right| = \left|(M_1(k,r,\theta) - 1) + b_3(k)\, M_3(k,r,\theta) + b_5(k)\, M_5(k,r,\theta) + b_7(k)\, M_7(k,r,\theta) + \ldots\right|.
\tag{11}
$$

Therefore, the requirement condition for the helical multipoles can be calculated from the prescribed homogeneity of the dipole field By at the boundary points of the circular region of r=31 mm shown in Fig.5, using Eq.(11). For example, for $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ of 0.2 % and 0.4 %, with $b_7(k)=0$, $b_9(k)=0,\cdots$, the satisfying region of $(b_3(k), b_5(k))$ for the prescribed field difference at two points, (r=31 mm, θ=0) and (r=31 mm, θ=π/2) are shown in Figs.6 and 7, respectively. The darker and narrower regions in Figs.6 and 7, corresponds to $|(By(r,\theta,z=0) - By_0)/By_0|$ of 0.2 %. The resultant satisfying regions of ($b_3(k)$, $b_5(k)$) for all boundary points for $|(By(r,\theta,z=0) - By_0)/By_0|$ of 0.18 % and 0.2 % are shown as the central white zones in Figs.8 and 9, respectively. It is also shown in Fig.10 that the satisfying region of ($b_3(k)$, $b_5(k)$) for all boundary points vanishes for $|(By(r,\theta,z=0) - By_0)/By_0|$ of 0.2 % with $b_7(k)=5 \times 10^{-4}$. Furthermore, for $|(By(r,\theta,z=0) - By_0)/By_0|$ of 0.3 %, the satisfying regions of ($b_3(k)$, $b_5(k)$) are shown in Figs.11, 12 and 13, with $b_7(k)=0$, $+5 \times 10^{-4}$, and -5×10^{-4}, respectively. In addition, with $b_7(k)=0$, $b_9(k)=0, \cdots$, the satisfying region of ($b_3(k)$, $b_5(k)$) for all boundary points vanishes for $|(By(r,\theta,z=0) - By_0)/By_0|$ of < 0.165 %. This result is equivalent with Eq.(10). For comparison, the relation between the field homogeneity and 2D multipoles is described in Appendix.

4. Conclusion

The intrinsic limit of field homogeneity and the relation between field homogeneity and helical multipoles of helical dipoles are obtained. It can be realized that the field homogeneity of helical dipoles is significantly different from that of the ordinary 2D dipoles. This relation will be useful to optimize the cross-sectional shape of helical dipole coils.

5. Acknowledgments

The author is indebted for helpful discussions and comments to Prof. T. Katayama of Institute of Nuclear Physics, University of Tokyo and RIKEN.

Appendix. Magnetic Field of 2-dimensional Dipole Coils

The interior magnetic field of 2-dimensional dipole coil with the infinite length is as follows, on the European definition, [6]

$$\begin{cases} B_r(r,\theta) = -\dfrac{\partial\psi}{\partial r} = \dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} = B_{ref}\sum_{n=1}^{\infty}\left(\dfrac{r}{r_0}\right)^{n-1}(-a_n\cos n\theta + b_n \sin n\theta) \\ B_\theta(r,\theta) = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = -\dfrac{\partial A_z}{\partial r} = B_{ref}\sum_{n=1}^{\infty}\left(\dfrac{r}{r_0}\right)^{n-1}(b_n\cos n\theta + a_n \sin n\theta) \end{cases} \quad \text{(A1)}$$

Then,

$$B_y(r,\theta) = B_{ref}\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}\{a_n \sin[(n-1)\theta] + b_n\cos[(n-1)\theta]\}. \quad \text{(A2)}$$

Therefore, on the case with the dipole symmetry, $a_n=b_{2n}=0$ for n=1, 2, 3, ... , ∞, the y component of field, $By(r,\theta)$ becomes with $b_1=1$,

$$B_y(r,\theta) = B_{ref}\sum_{n=1,3,5}^{\infty}\left(\frac{r}{r_0}\right)^{n-1} b_n\cos[(n-1)\theta] = B_{ref}\left(1 + b_3\left(\frac{r}{r_0}\right)^2\cos 2\theta + b_5\left(\frac{r}{r_0}\right)^4\cos 4\theta + b_7\left(\frac{r}{r_0}\right)^6\cos 6\theta + \ldots\right). \quad \text{(A3)}$$

Then,

$$\left|\frac{B_y(r,\theta) - B_{ref}}{B_{ref}}\right| = \left|b_3\left(\frac{r(\theta)}{r_0}\right)^2\cos 2\theta + b_5\left(\frac{r(\theta)}{r_0}\right)^4\cos 4\theta + b_7\left(\frac{r(\theta)}{r_0}\right)^6\cos 6\theta + \ldots\right|. \quad \text{(A4)}$$

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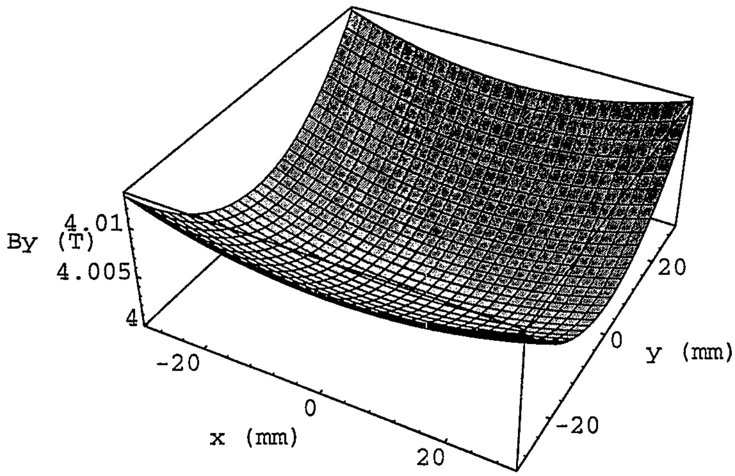

Fig.1 3D plot of By for the ideal helical dipole, with b1=1, and b3, b5,...=0.

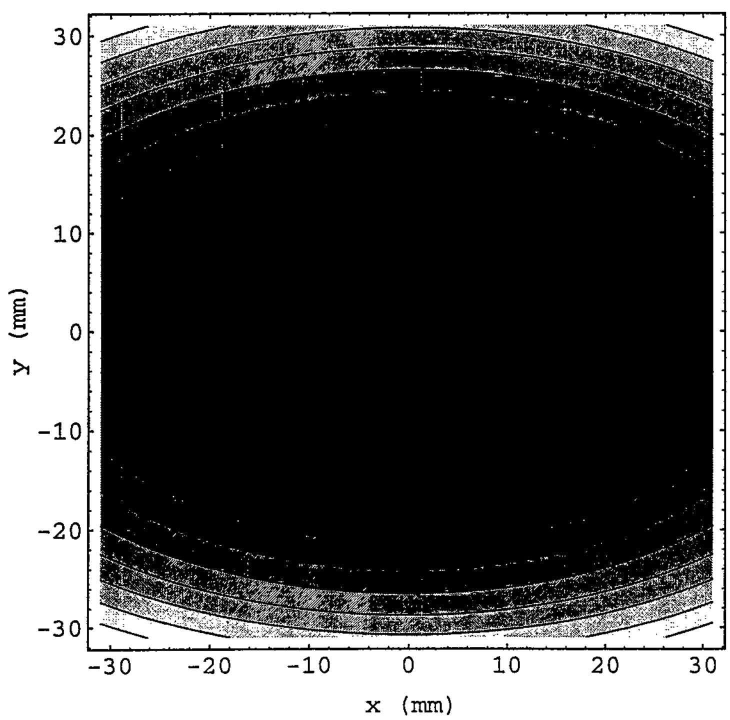

Fig.2 Contour plot of By for the ideal helical dipole, with b1=1, and b3, b5,...=0.

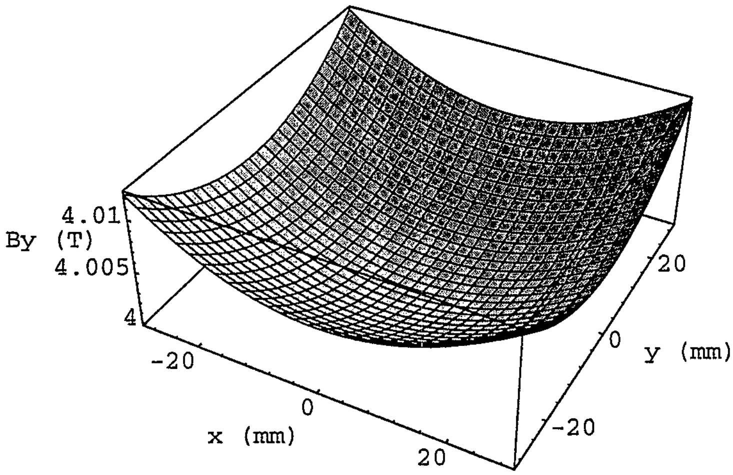

Fig.3 3D plot of By for the modified helical dipole, with b1=1, b3=0.00082, and b5, b7,...=0.

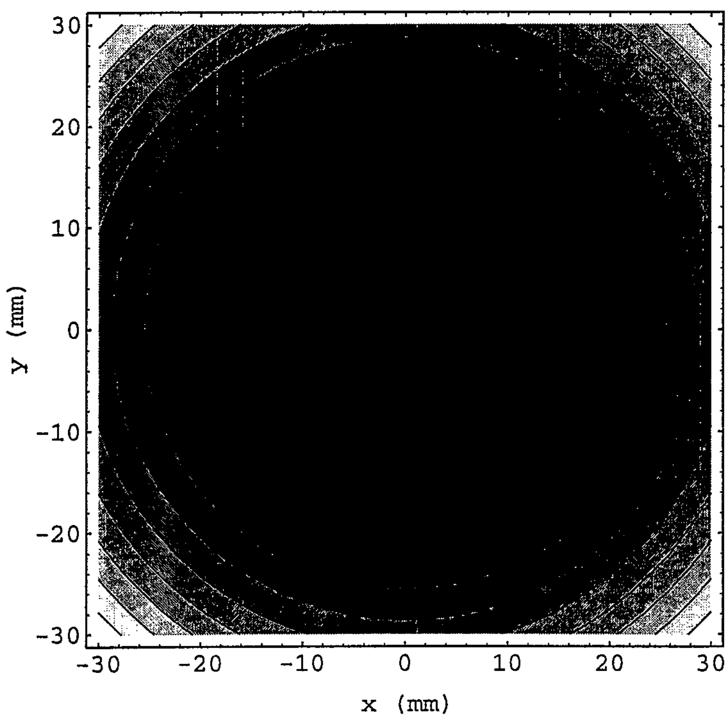

Fig.4 Contour plot of By for the modified helical dipole, with b1=1, b3=0.00082, and b5, b7,...=0.

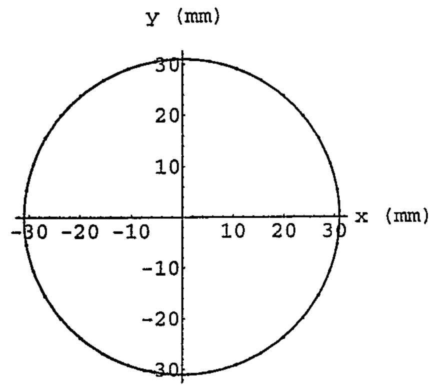

Fig.5 Homogeneous region of the magnetic field.

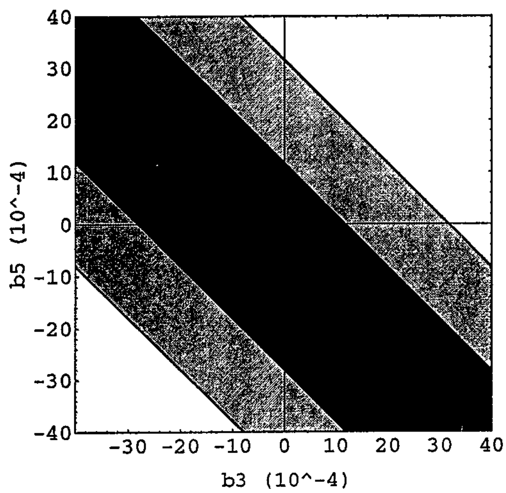

Fig.6 (b3, b5) relation of < 0.2 % and < 0.4 % of |By-By0|/By0 at (x=31mm,y=0mm) with b7=0, and b9, b11,...=0.

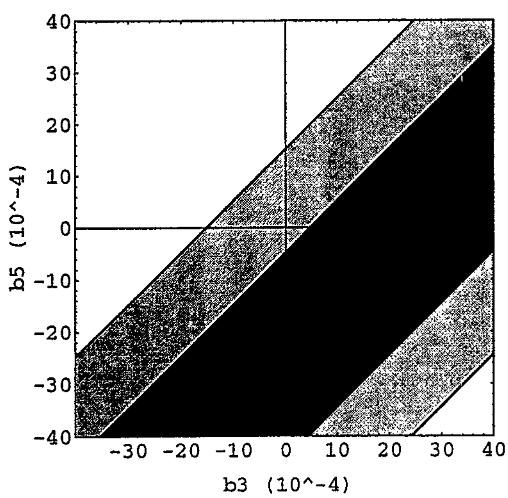

Fig.7 (b3, b5) relation of < 0.2 % and < 0.4 % of |By-By0|/By0 at (x=0mm,y=31mm) with b7=0, and b9, b11,...=0.

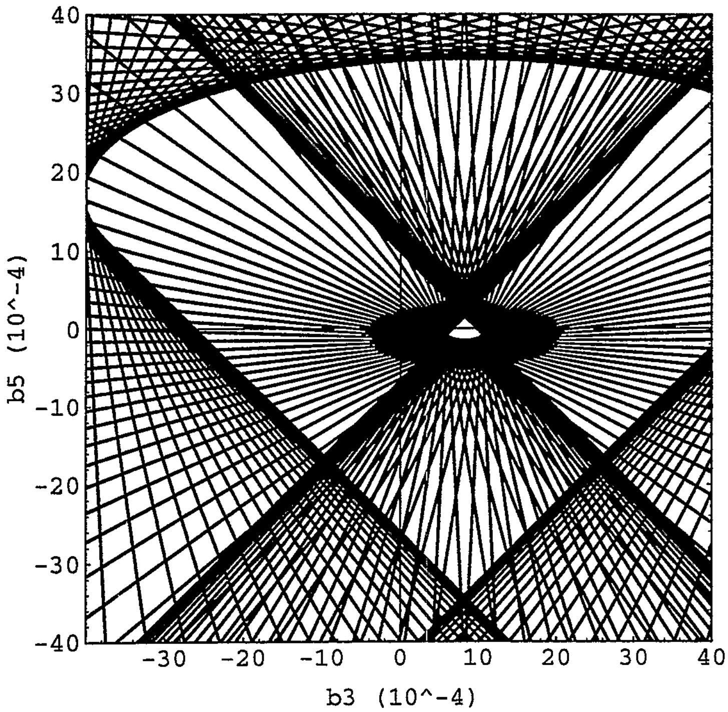

Fig.8 (b3, b5) region of < 0.18 % of |By-By0|/By0
with b7=0, and b9, b11,...=0.

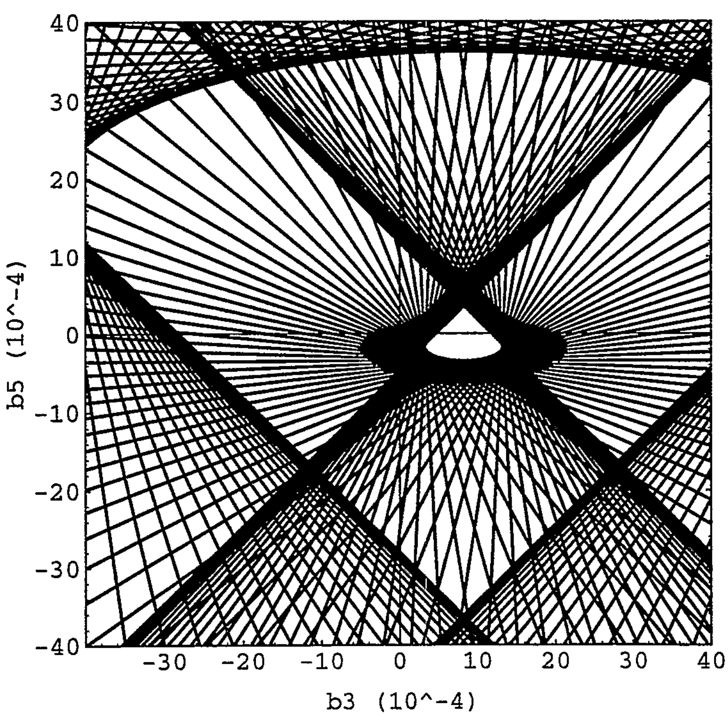

Fig.9 (b3, b5) region of < 0.2 % of |By-By0|/By0
with b7=0, and b9, b11,...=0.

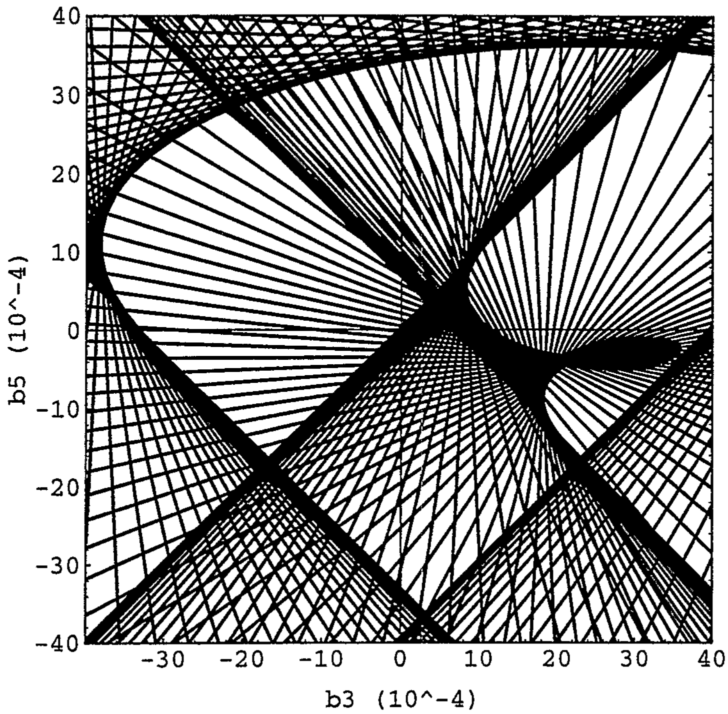

Fig.10 (b3, b5) region of < 0.2 % of |By-By0|/By0
with b7=0.0005, and b9, b11,...=0.

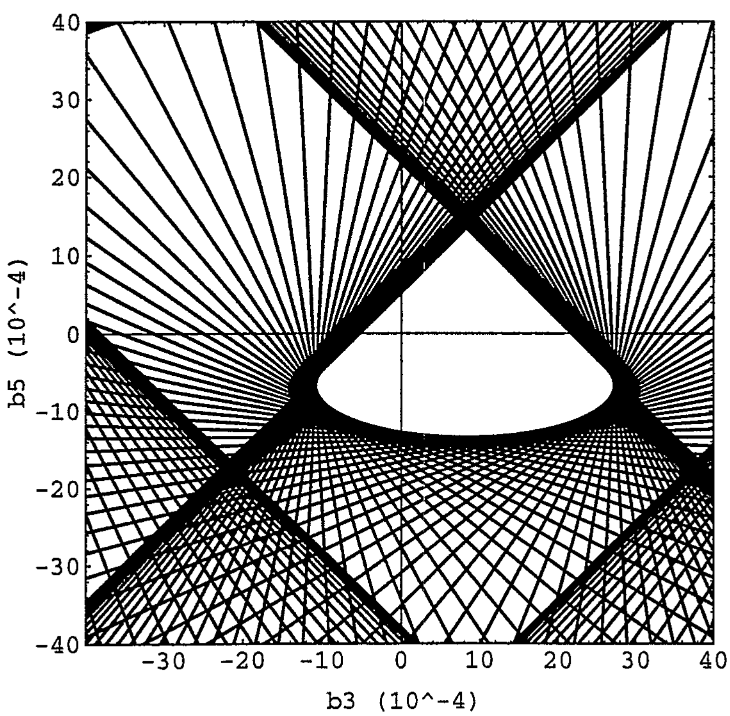

Fig.11 (b3, b5) region of < 0.3 % of |By-By0|/By0
with b7=0, and b9, b11,...=0.

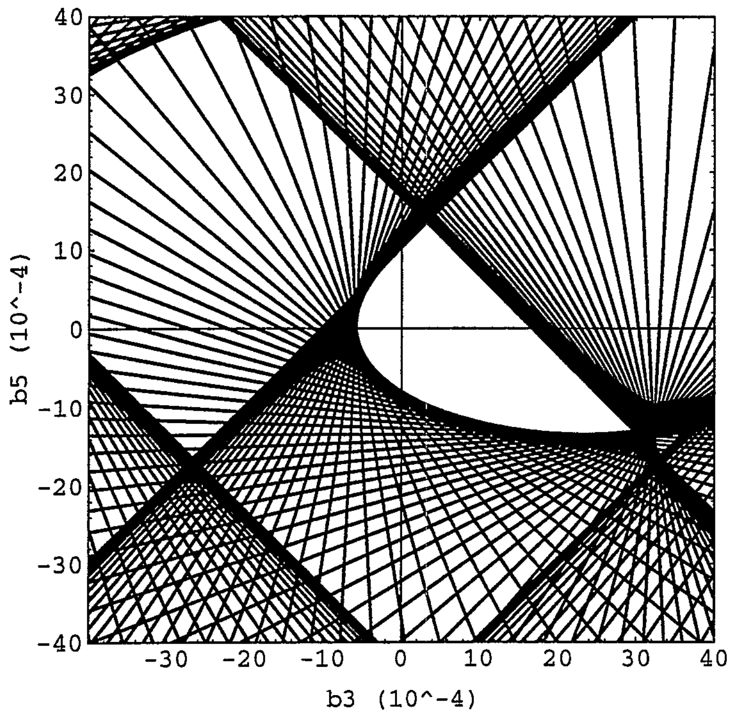

Fig.12 (b3, b5) region of < 0.3 % of |By-By0|/By0
with b7=0.0005, and b9, b11,...=0.

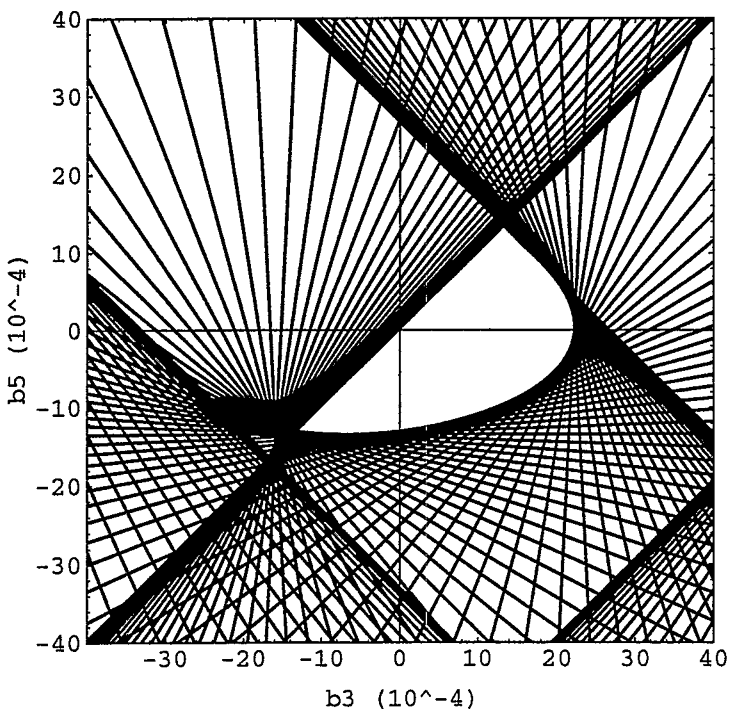

Fig.13 (b3, b5) region of < 0.3 % of |By-By0|/By0
with b7=-0.0005, and b9, b11,...=0.