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Multipole Expansion for a Single Helical Current Conductor

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1. Introduction

The purpose of this paper is to give the expression of the multipole expansion for a single helical current conductor. This analytical expression will be useful for various helical coils such as helical dipole coils, multifilamentary superconductors and superconducting strand. In addition, the comparison between the analytical and numerical calculations is presented for a single helical current conductor.

2. Multipole Expansion for a Single Helical Current Conductor

The magnetic field of helical coils has been examined by several authors [1,2,3,4,5,6,7]. The present treatment of the multipole expansion for a single helical current conductor is derived as the extension of the case for a single straight current conductor. [8] 3-dimensional Laplace's equation in circular cylindrical coordinates is as follows,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (1)$$

Since the winding is periodic in z with a pitch length L, the general solution is,

$$\phi_h(r,\theta,z) = \sum_{n=1}^{\infty}\left(c_n' I_n(nkr) + d_n' K_n(nkr)\right)\left\{a_n' \cos\left(n(\theta - kz)\right) + b_n' \sin\left(n(\theta - kz)\right)\right\} + e' z + f' \theta \qquad (2)$$

where $k = 2\pi/L$, $I_n(nkr)$ is the modified Bessel function of the first kind of order n, and $K_n(nkr)$ is the modified Bessel function of the second kind of order n.

The form of the ascending series of $I_n(nkr)$ is as follows,

$$I_n(nkr) = \sum_{j=0}^{\infty}\frac{1}{j!\,(n+j)!}\left(\frac{n\,k\,r}{2}\right)^{2j+n} \qquad (3)$$

For the interior scalar potential nearer the axis than conductors of helical coil, we can define the following form for r < a,

$$\phi_{h,in}(r,\theta,z) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}(n-1)!\left[\frac{2}{n\,k\,a}\right]^n I_n(n\,k\,r)\left\{-a_n(k)\cos\left(n(\theta - k\,z)\right) + b_n(k)\sin\left(n(\theta - k\,z)\right)\right\} - \frac{\mu_0 I}{2\pi}k\,z \qquad (4)$$

Then, the asymptotic form for this scalar potential as $k\to 0$ ($L\to\infty$) is,

$$\mathrm{Lim}_{k\to 0}\left[\phi_{h,in}(r,\theta,z)\right] = \phi_{2d,in}(r,\theta) \qquad (5)$$

where

$$\phi_{2d,in}(r,\theta) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n\left(-a_n\cos(n\theta) + b_n\sin(n\theta)\right) \qquad (6)$$

where $\phi_{2d,in}(r,\theta)$ is the scalar potential nearer the axis than conductors of 2-dimensional non-spiral coil. If we assume that the current is located at a point (a, ϕ), the scalar potential is for r < a, [8]

$$\phi_{2d.in}(r,\theta) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n \sin(n(\theta-\phi)) \tag{7}$$

Then, the following relation is required, as proved later.

$$\begin{cases} \mathrm{Lim}_{k\to 0}\left[a_n(k)\right] = a_n = \sin n\phi \\ \mathrm{Lim}_{k\to 0}\left[b_n(k)\right] = b_n = \cos n\phi \end{cases} \tag{8}$$

From this scalar potential, the interior magnetic field of helical coil is for r < a,

$$\begin{cases} B_r(r,\theta,z) = -\frac{\partial\phi_h}{\partial r} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n k\, I_n'(n k r)\left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} \\ B_\theta(r,\theta,z) = -\frac{1}{r}\frac{\partial\phi_h}{\partial\theta} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty} n!\left[\frac{2}{n k a}\right]^n \frac{I_n(n k r)}{r}\left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} \\ B_z(r,\theta,z) = -\frac{\partial\phi_h}{\partial z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}(-k)\, n!\left[\frac{2}{n k a}\right]^n I_n(n k r)\left\{a_n(k)\sin\left(n(\theta - k z)\right) + b_n(k)\cos\left(n(\theta - k z)\right)\right\} + \frac{\mu_0 I}{2\pi}k \end{cases} \tag{9}$$

Then,

$$B_z(r=0,\theta,z) = \frac{\mu_0 I}{2\pi}k = \frac{\mu_0 I}{2\pi}\frac{2\pi}{L} = \frac{\mu_0 I}{L} = \mu_0 n I \tag{10}$$

This field B_z at the z axis coincides with the result due to the Biot and Savart's Law. [9]
On the other hand, for the exterior scalar potential of helical coil, we can define the following form for r > a,

$$\phi_{h.ex}(r,\theta,z) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}(n-1)!\left[\frac{2}{n k a}\right]^n \frac{I_n'(n k a)}{K_n'(n k a)} K_n(n k r)\left\{-a_n(k)\cos\left(n(\theta - k z)\right) + b_n(k)\sin\left(n(\theta - k z)\right)\right\} - \frac{\mu_0 I}{2\pi}\theta \tag{11}$$

Then, the asymptotic form for this scalar potential as $k\to 0$ ($L\to\infty$) is,

$$\mathrm{Lim}_{k\to 0}\left[\phi_{h,ex}(r,\theta,z)\right] = \phi_{2d,ex}(r,\theta) \tag{12}$$

$$\phi_{2d,ex}(r,\theta) = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n(-a_n\cos(n\theta) + b_n\sin(n\theta)) - \frac{\mu_0 I}{2\pi}\theta \tag{13}$$

where $\phi_{2d,ex}(r,\theta)$ is the exterior scalar potential of 2-dimensional non-spiral coil. If we assume that the current is located at a point (a, ϕ) similarly, the scalar potential is for r > a, [8]

$$\phi_{2d.ex}(r,\theta) = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n \sin(n(\theta-\phi)) - \frac{\mu_0 I}{2\pi}\theta \tag{14}$$

Then, the exterior magnetic field of helical coil is for r > a,

$$\begin{cases} B_r(r,\theta,z) = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty} n!\left[\frac{2}{n\,k\,a}\right]^n k\,\frac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)}\,K_n'(n\,k\,r)\left\{-a_n(k)\cos\left(n(\theta - k\,z)\right) + b_n(k)\sin\left(n(\theta - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty} n!\left[\frac{2}{n\,k\,a}\right]^n \frac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)}\,\frac{K_n(n\,k\,r)}{r}\left\{a_n(k)\sin\left(n(\theta - k\,z)\right) + b_n(k)\cos\left(n(\theta - k\,z)\right)\right\} + \frac{\mu_0 I}{2\pi}\frac{1}{r} \\ B_z(r,\theta,z) = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty} (-k)\,n!\left[\frac{2}{n\,k\,a}\right]^n \frac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)}\,K_n(n\,k\,r)\left\{a_n(k)\sin\left(n(\theta - k\,z)\right) + b_n(k)\cos\left(n(\theta - k\,z)\right)\right\} \end{cases} \quad (15)$$

On the situation that the currents are confined to lie on the surface of a circular cylinder of radius a, the surface currents will give rise to a discontinuity of the components B_z, B_θ, at the interface of radius a, but the radial component B_r will pass continuously through this interface. The values of $a_n(k)$, $b_n(k)$ can be determined for the current element. Appling Ampere's law for a closed path in z=constant plane enclosing the current element at radius a, we can obtain the following equation,

$$\left(B_{\theta,out} - B_{\theta,in}\right)\big|_{r=a} = \mu_0\, J_z\, \Delta a \quad (16)$$

Then, the coefficients $a_n(k)$ and $b_n(k)$ are obtained with the Wronskian relation,

$$I_n(n\,k\,a)\,K_n'(n\,k\,a) - I_n'(n\,k\,a)\,K_n(n\,k\,a) = -\frac{1}{n\,k\,a} \quad (17)$$

as follows,

$$\begin{cases} a_n(k) = -\frac{2}{I}\frac{1}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a \int j_z \sin n\theta\, d\theta \\ b_n(k) = -\frac{2}{I}\frac{1}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a \int j_z \cos n\theta\, d\theta \end{cases} \quad (18)$$

Then, for a helical line currents : current +I , radius a, angle ϕ ,
$a_n(k)$, $b_n(k)$ are calculated as follows,

$$\begin{cases} a_n(k) = -\frac{2}{I}\frac{1}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a\,\frac{I}{a\,\Delta\theta\,\Delta a}\sin n\varphi\,\Delta\theta \\ b_n(k) = -\frac{2}{I}\frac{1}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n k\,a^2\,K_n'(n\,k\,a)\,\Delta a\,\frac{I}{a\,\Delta\theta\,\Delta a}\cos n\varphi\,\Delta\theta \end{cases} \quad (19)$$

With the following relation,

$$K_n'(n\,k\,a) = -\left(K_{n-1}(n\,k\,a) + \frac{1}{k\,a}K_n(n\,k\,a)\right) \quad (20)$$

Then, we can obtain the following expression,

$$\begin{cases} a_n(k) = \frac{2}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n \left(k\,a\,K_{n-1}(n\,k\,a) + K_n(n\,k\,a)\right)\sin n\varphi \\ b_n(k) = \frac{2}{(n-1)!}\left(\frac{n\,k\,a}{2}\right)^n \left(k\,a\,K_{n-1}(n\,k\,a) + K_n(n\,k\,a)\right)\cos n\varphi \end{cases} \quad (21)$$

When n is fixed and $z \to 0$, the limiting forms for small arguments of the modified Bessel function of the second kind of order n, $K_n(nkr)$ are as follows,

$$\begin{cases} K_0(n\,k\,a) \approx -\ln(n\,k\,a) \\ K_n(n\,k\,a) \approx -\frac{1}{2}\Gamma(n)\left(\frac{2}{n\,k\,a}\right)^n, \quad n\geq 1 \end{cases} \tag{22}$$

Then,

$$\begin{cases} \mathrm{Lim}_{k\to 0}\left[a_n(k)\right] = a_n = \sin n\phi \\ \mathrm{Lim}_{k\to 0}\left[b_n(k)\right] = b_n = \cos n\phi \end{cases} \tag{8}$$

As a result, the interior magnetic field of a single helical conductor with the current +I , located at radius a and angle ϕ is for r < a,

$$\begin{cases} B_r(r,\theta,z) = \frac{\mu_0 I}{\pi} k^2 a \sum_{n=1}^{\infty} n\, K_n'(n\,k\,a)\, I_n'(n\,k\,r) \sin\left(n(\theta - \phi - k\,z)\right) \\ B_\theta(r,\theta,z) = \frac{\mu_0 I}{\pi} k\, a \sum_{n=1}^{\infty} n\, K_n'(n\,k\,a) \frac{I_n(n\,k\,r)}{r} \cos\left(n(\theta - \phi - k\,z)\right) \\ B_z(r,\theta,z) = -\frac{\mu_0 I}{\pi} k^2 a \sum_{n=1}^{\infty} n\, K_n'(n\,k\,a)\, I_n(n\,k\,r) \cos\left(n(\theta - \phi - k\,z)\right) + \frac{\mu_0 I}{2\pi} k \end{cases} \tag{23}$$

Similarly, the exterior magnetic field of a single helical conductor is for r > a,

$$\begin{cases} B_r(r,\theta,z) = \frac{\mu_0 I}{\pi} k^2 a \sum_{n=1}^{\infty} n\, I_n'(n\,k\,a)\, K_n'(n\,k\,r) \sin\left(n(\theta - \phi - k\,z)\right) \\ B_\theta(r,\theta,z) = \frac{\mu_0 I}{\pi} k\, a \sum_{n=1}^{\infty} n\, I_n'(n\,k\,a) \frac{K_n(n\,k\,r)}{r} \cos\left(n(\theta - \phi - k\,z)\right) + \frac{\mu_0 I}{2\pi}\frac{1}{r} \\ B_z(r,\theta,z) = -\frac{\mu_0 I}{\pi} k^2 a \sum_{n=1}^{\infty} n\, I_n'(n\,k\,a)\, K_n(n\,k\,r) \cos\left(n(\theta - \phi - k\,z)\right) \end{cases} \tag{24}$$

The above expression for the magnetic field of helical coil is the function of r and θ - kz, and is helically symmetric. Therefore, for many helical line currents : current $+I_i$, radius a_i, angle ϕ_i (i=1,2,3, ...) , or helical current blocks : current density $+j_i$, radii a_{1i}, a_{2i}, limiting angles ϕ_{1i}, ϕ_{2i}, the magnetic field can be calculated.

3. Comparison between the analytical and numerical calculations

For a single helical conductot with current +I , radius a, angle ϕ , with

Radius of helical line current a = 0.33 mm,
Angle of helical line current $\phi = 0$,
Current I = 100 A,
Pitch length L = 9.51 mm,
$k = 2\pi/L = 2\pi/9.51\ \mathrm{mm}^{-1}$,

as shown in Figs.1 and 2, the comparison between the analytical and numerical calculations is made. The numerical calculation is made for a single helical conductot with the infinite length. The agreement between the analytical and numerical calculations is quite good, except the region near the radius r=a. The sums of Eqs.(23) and (24) do not approach the same value in the limit $r \to a$, as shown in Fig.3. In Fig.3, the comparison among the numerical calculation with the Biot and Savart's Law, the analytical calculation with Eqs.(23) and (24) to n=51, and the analytical calculation with the following Eqs.(25) and (26) to N=51 is made. Then, the following expressions due to Cesaro's method of summation are adopted. As a result, the interior magnetic field of a single helical conductor with the current +I , located at radius a and angle ϕ is for r < a,

$$\begin{cases} B_r(r,\theta,z) = \frac{\mu_0 I}{\pi} k^2 a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n K_n'(n k a) I_n'(n k r) \sin\left(n(\theta - \phi - k z)\right) \right\} \\ B_\theta(r,\theta,z) = \frac{\mu_0 I}{\pi} k a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n K_n'(n k a) \frac{I_n(n k r)}{r} \cos\left(n(\theta - \phi - k z)\right) \right\} \\ B_z(r,\theta,z) = -\frac{\mu_0 I}{\pi} k^2 a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n K_n'(n k a) I_n(n k r) \cos\left(n(\theta - \phi - k z)\right) \right\} + \frac{\mu_0 I}{2\pi} k \end{cases} \quad (25)$$

Similarly, the exterior magnetic field of a single helical conductor is for $r > a$,

$$\begin{cases} B_r(r,\theta,z) = \frac{\mu_0 I}{\pi} k^2 a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n I_n'(n k a) K_n'(n k r) \sin\left(n(\theta - \phi - k z)\right) \right\} \\ B_\theta(r,\theta,z) = \frac{\mu_0 I}{\pi} k a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n I_n'(n k a) \frac{K_n(n k r)}{r} \cos\left(n(\theta - \phi - k z)\right) \right\} + \frac{\mu_0 I}{2\pi} \frac{1}{r} \\ B_z(r,\theta,z) = -\frac{\mu_0 I}{\pi} k^2 a \frac{1}{N} \sum_{m=1}^{m=N} \left\{ \sum_{n=1}^{n=m} n I_n'(n k a) K_n(n k r) \cos\left(n(\theta - \phi - k z)\right) \right\} \end{cases} \quad (26)$$

The calculated results of B_Z are shown in Figs.4 to 7. Similarly, the calculated results of B_X and B_Y are shown in Figs.8 to 9 and Figs.10 to 11, respectively. The agreement between the analytical and numerical calculations is quite good.

4. Conclusion

An analytical expression for the magnetic field of a single helical coil is obtained. This expression will be useful to estimate the various electromagnetic characteristics of helical coils.

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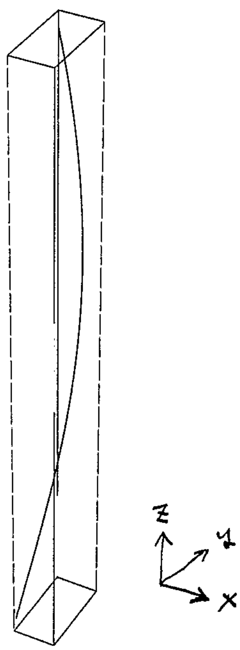

Fig.1 Schematic view of a single helical current conductor.

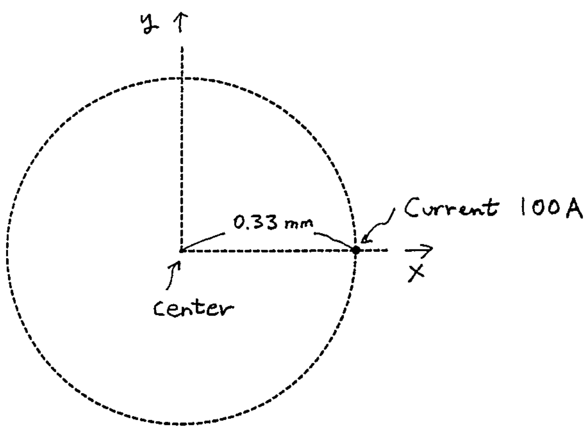

Fig.2 Cross section of a single helical current conductor at z=0.

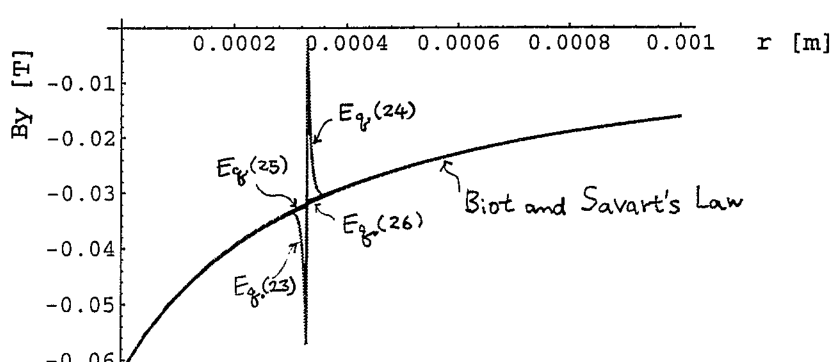

Fig.3 Comparison of By(r,theta=π) along the x axis (n=51, N=51).

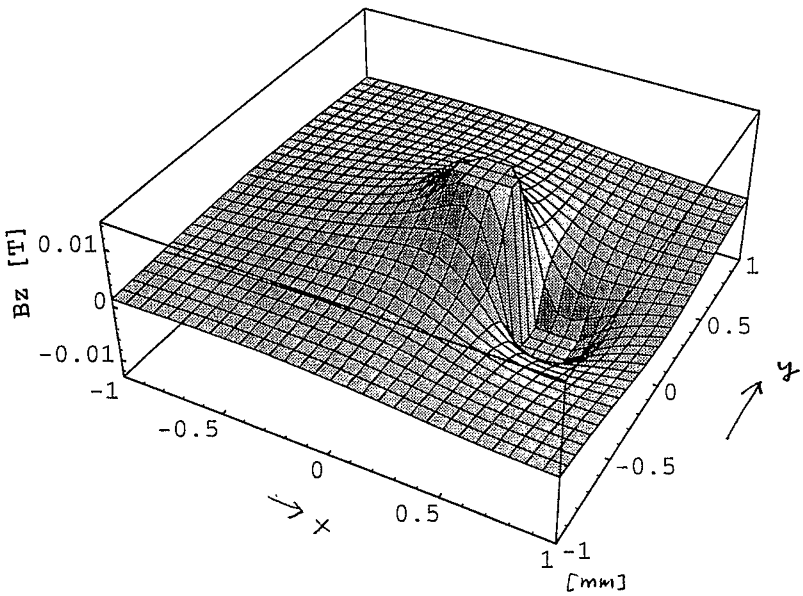

Fig.4 The field distribution of Bz at z=0 numerically calculated with the Biot and Savart's Law.

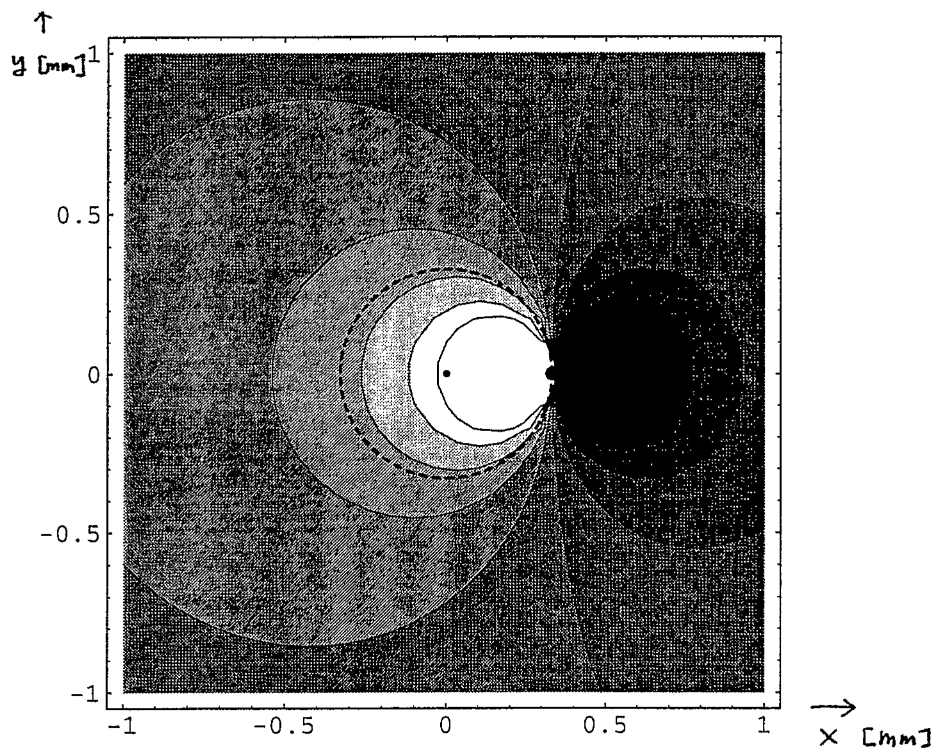

Fig.5 Contour plot of Bz at z=0 numerically calculated with the Biot and Savart's Law.

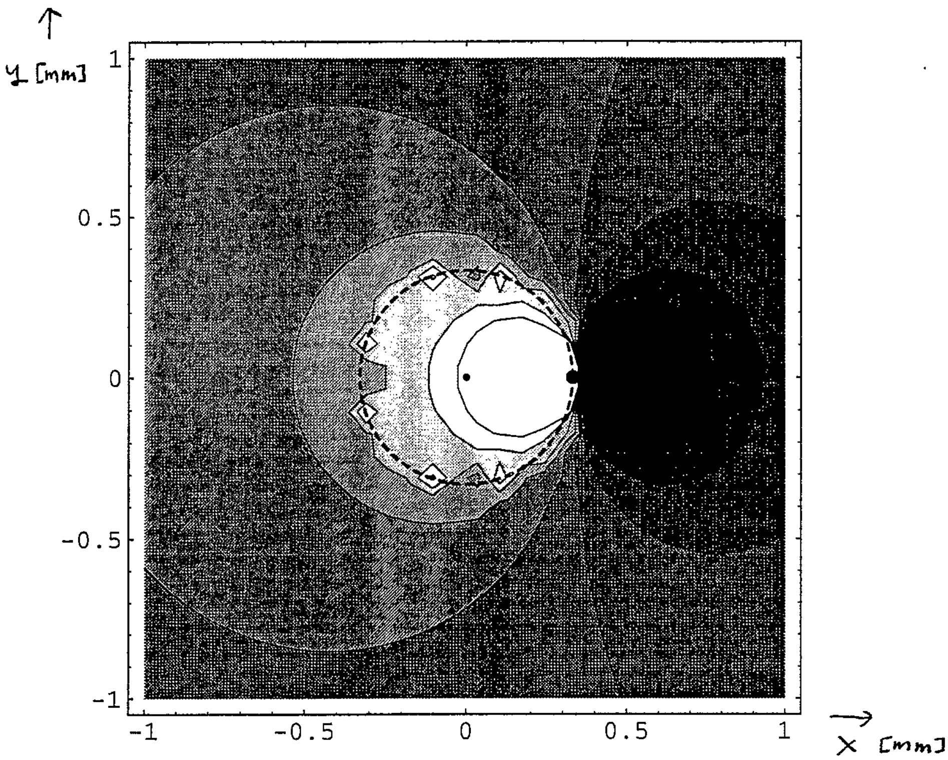

Fig.6 Contour plot of Bz at z=0 analytically calculated with Eqs.(23) and (24) to n=20.

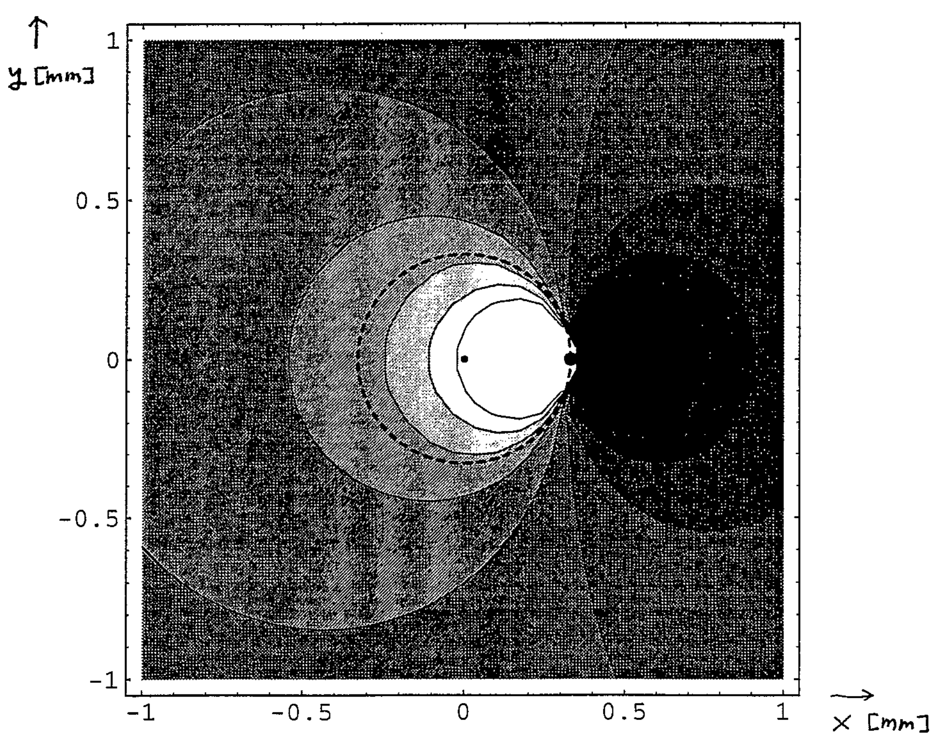

Fig.7 Contour plot of Bz at z=0 analytically calculated with Eqs.(25) and (26) to N=20.

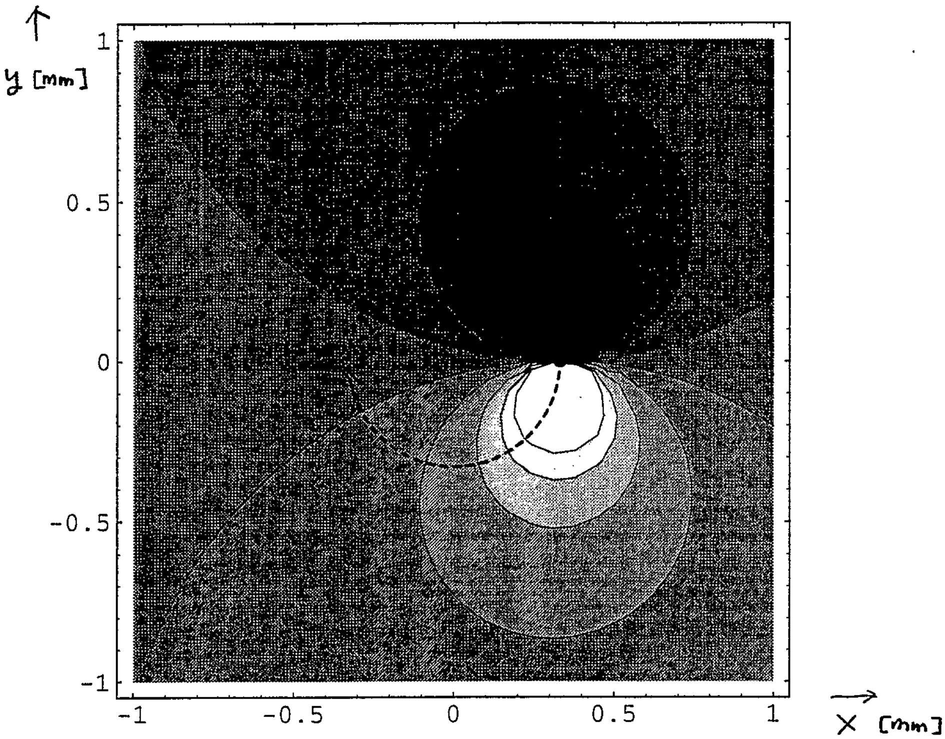

Fig.8 Contour plot of Bx at z=0 numerically calculated with the Biot and Savart's Law.

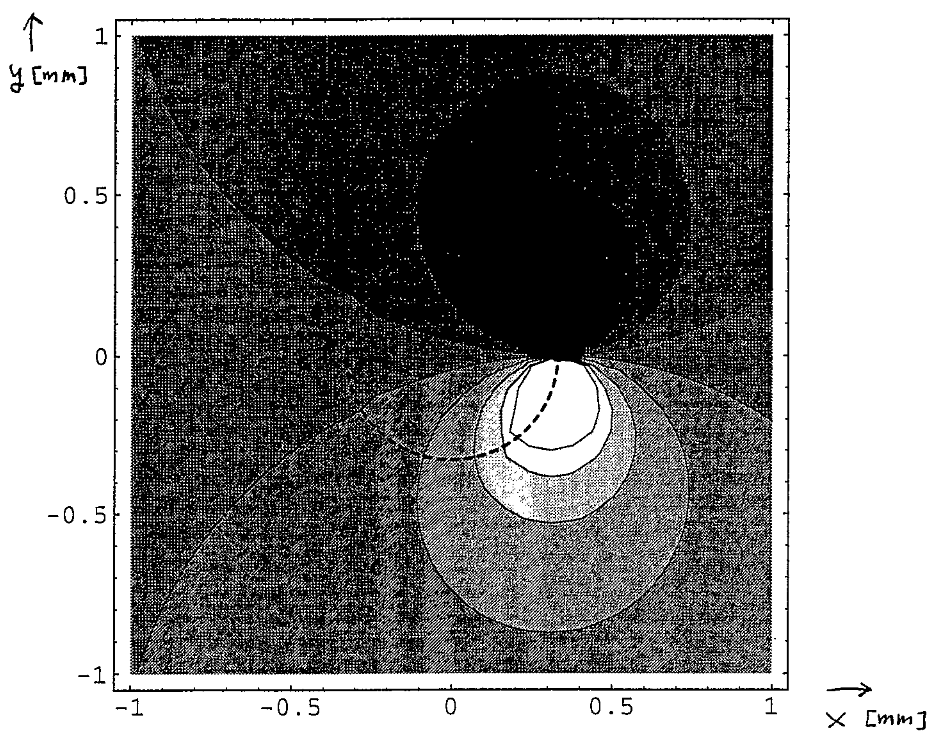

Fig.9 Contour plot of Bx at z=0 analytically calculated with Eqs.(25) and (26) to N=20.

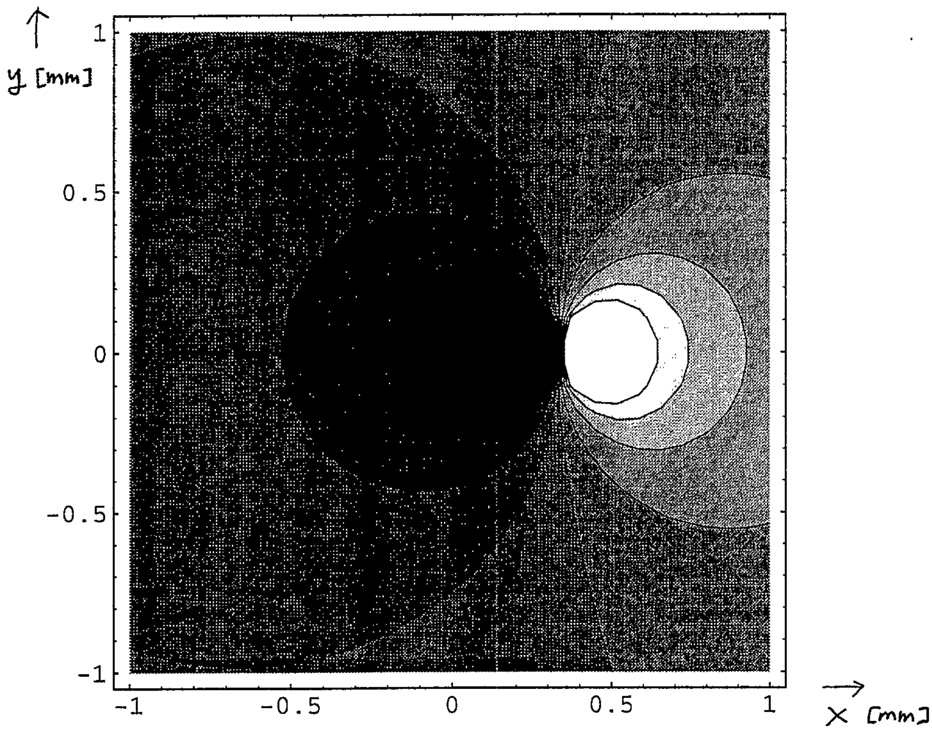

Fig.10 Contour plot of By at z=0 numerically calculated with the Biot and Savart's Law.

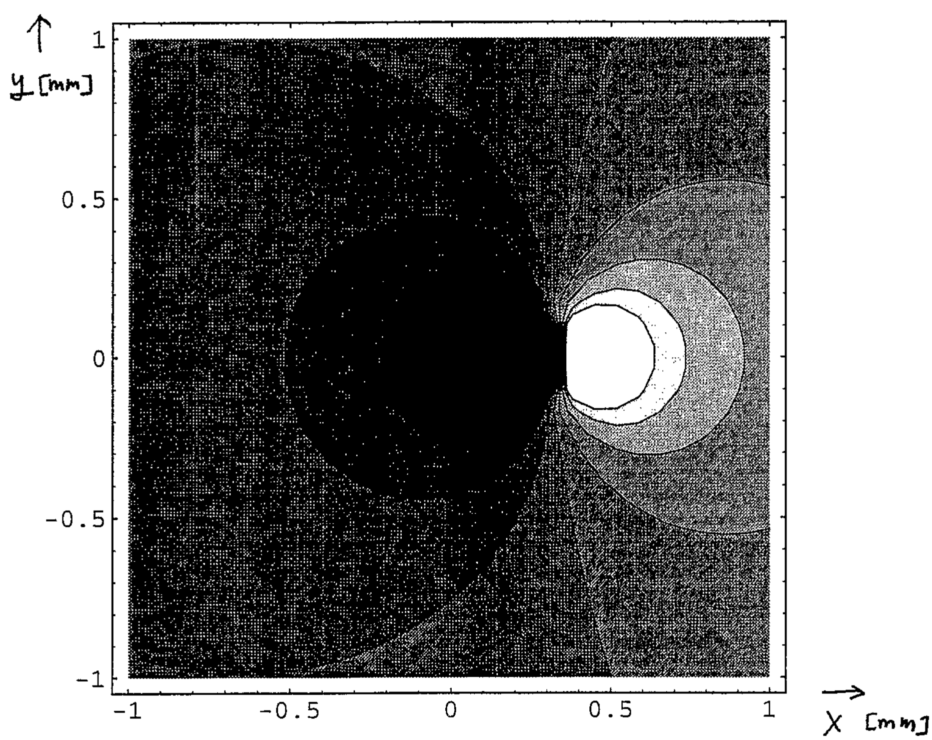

Fig.11 Contour plot of By at z=0 analytically calculated with Eqs.(25) and (26) to N=20.