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Magnetic Field Calculation of Helical Dipole Coils

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1. Introduction

The magnetic field of helical coils has been examined by several authors [1,2,3,4,5]. The aim of this paper is to give the expression of multipole of helical magnets for the coil design. In addition, the comparison between the analytical and numerical calculations is presented for the simple helical dipole coils.

2. The Magnetic Field of Helical Coils

The treatment of this section follows Morgan's [4] most closely. 3-dimensional Laplace's equation in circular cylindrical coordinates is as follows,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{1}$$

Since the winding is periodic in z with a pitch length L, the general solution is,

$$\phi_h(r,\theta,z) = \sum_{n=1}^{\infty}\left(c_n' I_n(nkr) + d_n' K_n(nkr)\right)\left\{a_n' \cos\left(n(\theta - kz)\right) + b_n' \sin\left(n(\theta - kz)\right)\right\} \tag{2}$$

where $k = 2\pi/L$, $I_n(nkr)$ is the modified Bessel function of the first kind of order n, and $K_n(nkr)$ is the modified Bessel function of the second kind of order n. Of the two, $I_n(nkr)$ is finite at r=0 and increases to the infinity with radius, and $K_n(nkr)$ is infinite at r=0 and decreases with radius. Then, the solution nearer the axis than conductors has the form with $d_n' = 0$, and the solution outside the conductors has the form with $c_n' = 0$.

Considering the form of the ascending series of $I_n(nkr)$,

$$I_n(nkr) = \sum_{j=0}^{\infty}\frac{1}{j!\,(n+j)!}\left(\frac{n\,k\,r}{2}\right)^{2j+n} \tag{3}$$

For the scalar potential nearer the axis than conductors of helical coil, we can define the following form,

$$\phi_h(r,\theta,z) = -B_{ref}\, r_0 \sum_{n=1}^{\infty}(n-1)!\left[\frac{2}{n\,k\,r_0}\right]^n I_n(n\,k\,r)\left\{-a_n \cos\left(n(\theta - k\,z)\right) + b_n \sin\left(n(\theta - k\,z)\right)\right\} \tag{4}$$

where we follow the European Definition for the multipole coefficients, a_n, b_n, which is different from the American Definition. The relationships between these coefficients are,

a_n (European) = $-\,a_{n-1}$ (American) : skew multipole coefficient of the 2n-pole
b_n (European) = b_{n-1}(American) : normal multipole coefficient of the 2n-pole

The asymptotic form for this scalar potential as $k\to 0$ ($L\to\infty$) is,

$$\mathrm{Lim}_{k\to 0}\left[\phi_h(r,\theta,z)\right] = \phi_{2d}(r,\theta) \tag{5}$$

$$\phi_{2d}(r,\theta) = -B_{ref}\, r_0 \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^n\left(-a_n \cos(n\theta) + b_n \sin(n\theta)\right) \tag{6}$$

where $\phi_{2d}(r, \theta)$ is the scalar potential nearer the axis than conductors of 2-dimensional non-spiral coil. From this scalar potential, the interior magnetic field of helical coil is,

$$\begin{cases} B_r(r,\theta,z) = -\dfrac{\partial \phi_h}{\partial r} = B_{ref}\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n k\, I_n'(n\,k\,r) \left\{-a_n \cos\left(n(\theta - k\,z)\right) + b_n \sin\left(n(\theta - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = -\dfrac{1}{r}\dfrac{\partial \phi_h}{\partial \theta} = B_{ref}\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n \dfrac{I_n(n\,k\,r)}{r} \left\{a_n \sin\left(n(\theta - k\,z)\right) + b_n \cos\left(n(\theta - k\,z)\right)\right\} \\ B_z(r,\theta,z) = -\dfrac{\partial \phi_h}{\partial z} = B_{ref}\, r_0 \sum_{n=1}^{\infty} (-k)\, n! \left[\dfrac{2}{n\,k\,r_0}\right]^n I_n(n\,k\,r) \left\{a_n \sin\left(n(\theta - k\,z)\right) + b_n \cos\left(n(\theta - k\,z)\right)\right\} \end{cases} \quad (7)$$

In this case,

$$B_y(r=0, \theta=0, z=0) = B_\theta(r=0, \theta=0, z=0) = B_{ref}\, b_1 \quad (8)$$

Then, $B_y(0,0,0)=B_{ref}$ with the definition of $b_1=1$. The above expressions are same with those given in the Appendix of a paper by J. Blewett et al., with the condition that $B_{ref}=B_0$, $b_1=1$, and other multipole coefficients are zero. On the both cases of helical and 2 dimensional non-spiral dipole coils, B_{ref} are the central dipole field, which are different. Similarly, the multipole coefficients, a_n, b_n, are different for both coils. However, the above expressions are beneficial for the comparison.

On the situation that the currents are confined to lie on the surface of a circular cylinder of radius a, the surface currents will give rise to a discontinuity of the components B_z, B_θ, at the interface of radius a, but the radial component B_r will pass continuously through this interface. Then, the exterior magnetic field of helical coil is,

$$\begin{cases} B_r(r,\theta,z) = B_{ref}\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n k\, \dfrac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)} K_n'(n\,k\,r) \left\{-a_n \cos\left(n(\theta - k\,z)\right) + b_n \sin\left(n(\theta - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = B_{ref}\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n \dfrac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)} \dfrac{K_n(n\,k\,r)}{r} \left\{a_n \sin\left(n(\theta - k\,z)\right) + b_n \cos\left(n(\theta - k\,z)\right)\right\} \\ B_z(r,\theta,z) = B_{ref}\, r_0 \sum_{n=1}^{\infty} (-k)\, n! \left[\dfrac{2}{n\,k\,r_0}\right]^n \dfrac{I_n'(n\,k\,a)}{K_n'(n\,k\,a)} K_n(n\,k\,r) \left\{a_n \sin\left(n(\theta - k\,z)\right) + b_n \cos\left(n(\theta - k\,z)\right)\right\} \end{cases} \quad (9)$$

The values of B_{ref}, a_n, b_n can be determined for the current element. Appling Ampere's law for a closed path in z=constant plane enclosing the current element at radius a, we can obtain the following equation,

$$\left(B_{\theta,out} - B_{\theta,in}\right)\Big|_{r=a} = \mu_0\, j_z\, \Delta a \quad (10)$$

Then, the normal multipoles B_n and the skew multipoles A_n, in addition, the normal multipole coefficients b_n and the skew multipole coefficient a_n are obtained with the Wronskian relation,

$$I_n(n\,k\,a)\, K_n'(n\,k\,a) - I_n'(n\,k\,a)\, K_n(n\,k\,a) = -\frac{1}{n\,k\,a} \quad (11)$$

as follows,

$$\begin{cases} A_n = B_{ref}\, a_n = \dfrac{\mu_0}{\pi} \dfrac{1}{2^n (n-1)!\, r_0} k\, a^2 (n\,k\,r_0)^n K_n'(n\,k\,a)\, \Delta a \displaystyle\int j_z \sin\left(n(\theta - k\,z)\right) d\theta \\ B_n = B_{ref}\, b_n = \dfrac{\mu_0}{\pi} \dfrac{1}{2^n (n-1)!\, r_0} k\, a^2 (n\,k\,r_0)^n K_n'(n\,k\,a)\, \Delta a \displaystyle\int j_z \cos\left(n(\theta - k\,z)\right) d\theta \end{cases} \quad (12)$$

With the definition of $b_1=1$, $B_{ref} = B_1$

Then, for four helical line currents with dipole symmetry,

helical line current #1 : current +I , radius a, angle ϕ ,
helical line current #2 : current - I , radius a, angle $\pi - \phi$,
helical line current #3 : current - I , radius a, angle $\pi + \phi$,
helical line current #4 : current +I , radius a, angle $-\phi$,

Bref, an , bn at z=0, are calculated as follows,

$$\begin{cases} A_n = B_{ref}\, a_n = \dfrac{\mu_0}{\pi} \dfrac{1}{2^n (n-1)!\, r_0} k\, a^2\, (n\, k\, r_0)^n\, K_n'(n\, k\, a)\, \Delta a \displaystyle\sum_{i=1}^{4} \dfrac{I_i}{a\, \Delta\theta\, \Delta a} \sin n\varphi_i\, \Delta\theta \\ B_n = B_{ref}\, b_n = \dfrac{\mu_0}{\pi} \dfrac{1}{2^n (n-1)!\, r_0} k\, a^2\, (n\, k\, r_0)^n\, K_n'(n\, k\, a)\, \Delta a \displaystyle\sum_{i=1}^{4} \dfrac{I_i}{a\, \Delta\theta\, \Delta a} \cos n\varphi_i\, \Delta\theta \end{cases} \quad (13)$$

where the following relation between the current and the current density is used with the real cross section S of the conductor and the projected cross section S_Z of the conductor on the z=constant plane.

$$j = \frac{I}{S} \quad (14)$$

$$S = S_Z \sin \alpha \quad (15)$$

where α is the pitch of the winding so that the relationships between the above-mentioned k and α are $k = 1/(a \tan \alpha)$. Then,

$$j_z = j \sin \alpha = \frac{I}{S} \sin \alpha = \frac{I}{S_Z} \quad (16)$$

Due to the dipole symmetry, An=0 for n=1, 2, 3, 4, ..., and Bn=0 for n=2, 4, 6, ... ,

$$B_n = \frac{4\mu_0}{\pi} I \frac{1}{2^n (n-1)!\, r_0} k\, a\, (n\, k\, r_0)^n\, K_n'(n\, k\, a) \cos n\varphi \quad (17)$$

for n=1, 3, 5, ... , ∞.

With the following relation,

$$K_n'(n\, k\, a) = -\left(K_{n-1}(n\, k\, a) + \frac{1}{k\, a} K_n(n\, k\, a)\right) \quad (18)$$

we can obtain the following expression,

$$B_n = -\frac{4\mu_0}{\pi} I \frac{1}{2^n (n-1)!\, r_0} (n\, k\, r_0)^n \left(k\, a\, K_{n-1}(n\, k\, a) + K_n(n\, k\, a)\right) \cos n\varphi \quad (19)$$

for n=1, 3, 5, ... , ∞.

Especially, for n=1,

$$B_{ref} = B_1 = -\frac{2\mu_0}{\pi} I\, k \left(k\, a\, K_0(k\, a) + K_1(k\, a)\right) \cos \varphi \quad (20)$$

Similarly, for four helical current blocks with dipole symmetry,

helical current block #1 : current density $+j_z$, radii a_1, a_2, limiting angles ϕ_1, ϕ_2,
helical current block #2 : current density $-j_z$, radii a_1, a_2, limiting angles $\pi-\phi_1$, $\pi-\phi_2$,
helical current block #3 : current density $-j_z$, radii a_1, a_2, limiting angles $\pi+\phi_1$, $\pi+\phi_2$,
helical current block #4 : current density $+j_z$, radii a_1, a_2, limiting angles $-\phi_1$, $-\phi_2$,

Bref = B1, bn are calculated as follows,

$$B_n = \frac{4\mu_0}{\pi} j_z \frac{1}{2^n (n-1)!\, r_0} (n k r_0)^n \frac{1}{n} (\sin n\varphi_2 - \sin n\varphi_1) \int_{a1}^{a2} k a^2 K_n'(n k a)\, da$$

$$= -\frac{4\mu_0}{\pi} j_z \frac{1}{2^n (n-1)!\, r_0} (n k r_0)^n \frac{1}{n} (\sin n\varphi_2 - \sin n\varphi_1) \times \int_{a1}^{a2} a \left(k a K_{n-1}(n k a) + K_n(n k a)\right) da \qquad (21)$$

for n=1, 3, 5, ... , ∞.

$$b_n = \frac{B_n}{B_1} \qquad (22)$$

3. Magnetic Field Calculation of Helical Coils with the Biot and Savart's Law

The treatment of this section follows Smythe's [6]. The conductor position (x_h, y_h, z_h) of the right-handed helical coil is described, using the angle ϕ , as follows,

$$x_h = a \cos(\varphi+\varphi_0),\ y_h = a \sin(\varphi+\varphi_0),\ z_h = a\, \varphi \tan \alpha, \qquad (23)$$

where a is the radius of helical coil, α is the pitch of the winding so that the relationships between the above-mentioned k and α are $k = 1/(a \tan \alpha)$. First of all, using the Biot and Savart's Law, we calculate the magnetic field of the following position (x_p, y_p, z_p) on the axis,

$$x_p = 0,\ y_p = 0,\ z_p = z, \qquad (24)$$

The components of the vector $\vec{R} = (R_x, R_y, R_z)$ from the helical coil to the position (x_p, y_p, z_p) on the axis, and the condutor element vector $\vec{ds} = (dx_h, dy_h, dz_h)$ are,

$$\begin{cases} R_x = x_p - x_h = -a \cos(\varphi+\varphi_0), \\ R_y = y_p - y_h = -a \sin(\varphi+\varphi_0), \\ R_z = z_p - y_h = z - a\, \varphi \tan \alpha, \end{cases} \qquad (25)$$

$$\begin{cases} dx_h = -a \sin(\varphi+\varphi_0)\, d\varphi, \\ dy_h = a \cos(\varphi+\varphi_0)\, d\varphi, \\ dz_h = a \tan \alpha\, d\varphi, \end{cases} \qquad (26)$$

The y component B_y of the magnetic field of the right-handed helical coil with the current I are,

$$B_y(z) = \frac{\mu_0 I}{4\pi} \int \frac{[\vec{ds} \times \vec{R}]_y}{|\vec{R}|^3} = - \frac{\mu_0 I}{4\pi} \frac{\tan\alpha}{a} \int_{\varphi_i}^{\varphi_f} \frac{\cos(\varphi+\varphi_0) + \left(\varphi - \frac{z}{a\tan\alpha}\right) \sin(\varphi+\varphi_0)}{\left\{1 + \left(\varphi - \frac{z}{a\tan\alpha}\right)^2 \tan^2\alpha\right\}^{3/2}} d\varphi \qquad (27)$$

The other components B_x, B_z are described similarly.

For the infinite long helical coil, $\phi i = -\infty$, $\phi f = \infty$, The y component B_y at z=0 is,

$$B_{y,z=0} = - \frac{\mu_0 I}{4\pi} \frac{\tan\alpha}{a} \int_{-\infty}^{\infty} \frac{\cos(\varphi+\varphi_0) + \varphi \sin(\varphi+\varphi_0)}{\left\{1 + \varphi^2 \tan^2\alpha\right\}^{3/2}} d\varphi$$

$$= - \frac{\mu_0 I}{4\pi} \frac{\tan\alpha}{a} \int_{-\infty}^{\infty} \left\{ \frac{\cos\varphi + \varphi \sin\varphi}{\left\{1 + \varphi^2 \tan^2\alpha\right\}^{3/2}} \cos\varphi_0 + \frac{-\sin\varphi + \varphi\cos\varphi}{\left\{1 + \varphi^2 \tan^2\alpha\right\}^{3/2}} \sin\varphi_0 \right\} d\varphi \qquad (28)$$

where the second term of integrand vanishes, as it is odd. Then,

$$B_{y,z=0} = - \frac{\mu_0 I}{4\pi} \frac{\tan\alpha}{a} \cos\varphi_0 \int_{-\infty}^{\infty} \frac{\cos\varphi + \varphi \sin\varphi}{\left\{1 + \varphi^2 \tan^2\alpha\right\}^{3/2}} d\varphi \qquad (29)$$

$$= - \frac{\mu_0 I}{2\pi a \tan\alpha} \left(\frac{1}{\tan\alpha} K_0\left(\frac{1}{\tan\alpha}\right) + K_1\left(\frac{1}{\tan\alpha}\right) \right) \cos\varphi_0$$

$$= - \frac{\mu_0 I}{2\pi} k \left(k\, a\, K_0(k\, a) + K_1(k\, a)\right) \cos\varphi_0$$

Then, for four helical line currents with dipole symmetry, the transeverse field magnitude at axis of the helical winding Bref is obtaned as follows,

$$B_{ref} = - \frac{\mu_0}{2\pi a \tan\alpha} \left(\frac{1}{\tan\alpha} K_0\left(\frac{1}{\tan\alpha}\right) + K_1\left(\frac{1}{\tan\alpha}\right) \right) \left\{ I\cos\varphi - I\cos(\pi-\varphi) - I\cos(\pi+\varphi) + I\cos(-\varphi) \right\}$$

$$= - \frac{2\mu_0 I}{\pi a \tan\alpha} \left(\frac{1}{\tan\alpha} K_0\left(\frac{1}{\tan\alpha}\right) + K_1\left(\frac{1}{\tan\alpha}\right) \right) \cos\varphi \qquad (30)$$

$$= - \frac{2\mu_0 I}{\pi} k \left(k\, a\, K_0(k\, a) + K_1(k\, a)\right) \cos\varphi$$

This result is the same with the above expression of Eq.(20) which is obtaned with the different method.

Similarly, for four helical current blocks with dipole symmetry,

$$B_{ref} = \int_{a_1}^{a_2} B_{ref}(a,\varphi)\, a\, da \int_{\varphi_1}^{\varphi_2} d\varphi$$

$$= - \frac{2\mu_0 j}{\pi} k \int_{a_1}^{a_2} \left(k\, a\, K_0(k\, a) + K_1(k\, a)\right) a\, da \left(\sin\varphi_2 - \sin\varphi_1\right) \qquad (31)$$

$$\cong - \frac{2\mu_0 j}{\pi} k\, \bar{a} \left(k\, \bar{a}\, K_0(k\, \bar{a}) + K_1(k\, \bar{a})\right) \Delta a \left(\sin\varphi_2 - \sin\varphi_1\right)$$

For the off-axis position (x_p, y_p, z_p),

$x_p = r\cos\theta,\ y_p = r\sin\theta,\ z_p = z,$ (32)

The components of the vector $R = (\vec{R}_x, R_y, R_z)$ from the helical coil to the position (x_p, y_p, z_p) on the axis, and the condutor element vector $ds = (dx_h, dy_h, dz_h)$ are,

$$\begin{cases} R_x = x_p - x_h = r\cos\theta - a\cos(\varphi+\varphi_0), \\ R_y = y_p - y_h = r\sin\theta - a\sin(\varphi+\varphi_0), \\ R_z = z_p - y_h = z - a\,\varphi\tan\alpha, \end{cases} \quad (33)$$

$$\begin{cases} dx_h = -a\sin(\varphi+\varphi_0)\,d\varphi, \\ dy_h = a\cos(\varphi+\varphi_0)\,d\varphi, \\ dz_h = a\tan\alpha\,d\varphi, \end{cases} \quad (34)$$

Then, the magnetic field of the off-axis position of the helical coilis experssed as follows,

$$B_x(r,\theta,z) = \frac{\mu_0 I}{4\pi}\int \frac{[\vec{ds}\times\vec{R}]_x}{|\vec{R}|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_{\varphi_i}^{\varphi_f} \frac{(z - a\tan\alpha\,\varphi)\,a\cos(\varphi+\varphi_0) - \left(r\sin\theta - a\sin(\varphi+\varphi_0)\right)a\tan\alpha}{\left\{r^2 + a^2 - 2ra\cos\left\{\theta-(\varphi+\varphi_0)\right\} + (z - a\tan\alpha\,\varphi)^2\right\}^{3/2}}\,d\varphi \quad (35)$$

$$B_y(r,\theta,z) = \frac{\mu_0 I}{4\pi}\int \frac{[\vec{ds}\times\vec{R}]_y}{|\vec{R}|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_{\varphi_i}^{\varphi_f} \frac{\left\{r\cos\theta - a\cos(\varphi+\varphi_0)\right\}a\tan\alpha + (z - a\tan\alpha\,\varphi)\,a\sin(\varphi+\varphi_0)}{\left\{r^2 + a^2 - 2ra\cos\left\{\theta-(\varphi+\varphi_0)\right\} + (z - a\tan\alpha\,\varphi)^2\right\}^{3/2}}\,d\varphi \quad (36)$$

$$B_z(r,\theta,z) = \frac{\mu_0 I}{4\pi}\int \frac{[\vec{ds}\times\vec{R}]_z}{|\vec{R}|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_{\varphi_i}^{\varphi_f} \frac{\left(r\sin\theta - a\sin(\varphi+\varphi_0)\right)\left(-a\sin(\varphi+\varphi_0)\right) - \left\{r\cos\theta - a\cos(\varphi+\varphi_0)\right\}a\cos(\varphi+\varphi_0)}{\left\{r^2 + a^2 - 2ra\cos\left\{\theta-(\varphi+\varphi_0)\right\} + (z - a\tan\alpha\,\varphi)^2\right\}^{3/2}}\,d\varphi \quad (37)$$

These equations can be calculated numerically, and are used for the comparison with the analitycal calculations.

4. Comparison between the analytical and numerical calculations

For four helical line currents with dipole symmetry of current +I , radius a, angle ϕ , with

Radius of helical line current $a = 0.05$ m,
Angle of helical line current $\phi = \pi/6$,
Current $I = -1\times10^5$ A,
Pitch length $L = 2$ m,
$k = 2\pi/L = 1/(a\tan\alpha) = \pi$,
Pitch of the winding $\alpha = \tan^{-1}(1/0.05\pi)$,
Reference radius for multipole $r_0 = 0.03$ m,

as shown in Fig.1-1 and Fig.1-2, the comparison between the analytical and numerical calculations is made. The numerical calculation is made for the helical coil with the infinite length, that is, $\phi i = -\infty$, $\phi f = \infty$ with the numerical integration of Eqs.(35)-(37), using Mathematica [7]. The results are shown in Fig.1-3 to Fig.1-10. The agreement between the analytical and numerical calculations is quite good. The multipole coefficients derived from the numerical calculation for each compoment of magnetic field at z=0 of the middle of the helical coil are shown in Table 1-1, with the value of B_{ref} = 1.41117 T for Bref of Eq.(7). In Table 1-1, the b-r means the normal multiple coefficients derived from the radial component of field B_r, the b-theta means that of B_θ, and the b-z means that of B_z. These multipole coefficients are equivalent to those shown in Table 1-3. In addition, The multipole coefficients of the non-helical dipole are shown for the comparison with those of the helical dipole in Table 1-3. The calculation of the non-heical dipole is made for the infinite length and the same current I with that of the helical dipole. For the helical coil with the length of one period L (= 2 m), furthermore, the multipoe coefficients derived from the numerical calculation for each compoment of magnetic field are shown in Table 1-2, with the value of B_{ref} = 1.41416 T for B_{ref} of Eq.(7).

At z=0 of the middle of the helical coil, the dipole field $B_{y,z=0}$ at the axis depends on the length of the coil, as shown in Figs.1-11 and 1-12. From the analytical and numerical methods for the infinite length, using Eq.(20) or (29), and the numerical method for the finite length of one period L (= 2 m), the dipole field $B_{y,z=0}$ at the axis are calculated, using Eq.(27), as follows,

$B_{y,z=0}$ = 1.41117 T (analitical calculation for the infinite length),
$B_{y,z=0}$ = 1.41115 T (numerical calculation for the infinite length),
$B_{y,z=0}$ = 1.41416 T (numerical calculation for the finite length of one period L=2 m),

The dipole field $B_{y,z=0}$ on the axis at z=0 of the middle changes periodically with the length of the helical dipole coil. This dependence on the length for $B_{y,z=0}$ of the heical dipole is different from that of the non-helical one.

Similarly, for four helical current blocks with dipole symmetry of current density +jz, radii a_1, a_2, limiting angles ϕ_1, ϕ_2, with

Inner radius of helical line current a_1 = 0.05 m,
Outer radius of helical line current a_2 = 0.06 m,
Inner angle of helical line current $\phi_1 = 0$,
Outer angle of helical line current $\phi_2 = \pi/3$,
Length L = 2 m,
Current I = -2×10^5 A,
Current density j_z = 347 A/mm^2,
$k = 2\pi/L = 1/(a \tan\alpha) = \pi$,
Pitch of the winding $\alpha = \tan^{-1}(1/0.05\pi)$,
Reference radius for multipole r_0 = 0.03 m,

as shown in Fig.2-1 and Fig.2-2, the comparison between the analytical and numerical calculations is made. The numerical calculation is made for the helical coil with the finite length of one period, using the code OPERA-3d [8] for the 3-dimensioinal magnetic field calculation. The results are shown in Fig.2-3 to Fig.2-8. The agreement between the analitical and numerical calculations is quite good. The multipoe coefficients derived from the numerical calculation for each compoment of the magnetic field at z=0 of the middle of the helical coil are shown in Table 2-1, with the value of for B_{ref} = 2.4627 T for B_{ref} of Eq.(7). These multipoe coefficients are equivalent to those shown in Table 2-2. In addition, The multipole coefficients of the non-helical dipole are shown for the comparison with those of the helical dipole in Table 2-2. The calculation of the non-heical dipole is made for the same current I and the same z-directional current density j_z with those of the heical dipole, but the current densitys j for both of the heical and non-heical dipoles are different.

At z=0 of the middle of the helical coil, the dipole field $B_{y,z=0}$ at the axis also depends on the length of the coil. For the infinite length, using Eq.(21), and for the finite length of one period L, the dipole field at the axis are calculated as follows,

$B_{y,z=0} = 2.45618$ T (analitical calculation for the infinite length),
$B_{y,z=0} = 2.4627$ T (numerical calculation for the finite length of one period L=2 m),

The dipole field at the axis, B_{ref} of Eq.(7) derived for the helical coil with the infinite length is constant, but the dipole field B_{ref} of this helical coil with the finite length depends on the position as shown in Fig.2-9. This means that Eq.(7) is not correct for the end portion of this helical coil.

5. Conclusion

An analytical expressions for the magnetic field and the multipole for the helical coils have been given. The expression of the multipoles for the helical coil is more effective to optimize the cross sectional configulation of the helical coil than those of the non-helical coil. Further investigation is needed for obtaining the expression of the multipole for the ends of a helical coil.

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Appendix. The Magnetic Field of 2-dimensional Dipole Coils

On the European Definition, the scalar potential nearer the axis is as follows, [9]

$$\phi(r,\theta) = -B_{ref}\, r_0 \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{r_0}\right)^n(-a_n \cos n\theta + b_n \sin n\theta) \quad \text{(A1)}$$

Similarly, the vector potential is as follows,

$$A_z(r,\theta) = -B_{ref}\, r_0 \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{r_0}\right)^n(a_n \sin n\theta + b_n \cos n\theta) \quad \text{(A2)}$$

The treatment of this section follows Meβ and P. Schmüser's [10]. For four helical line currents with dipole symmetry, of current $+I_z$, radius a, angle ϕ, at the position (r, θ) of $r < a$, the multipole expansion of the vector potential is as follows,

$$A_z(r,\theta,a,\varphi) = 2\frac{\mu_0 I_z}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n \cos n\varphi \cos n\theta \quad \text{(A3)}$$

where, n=1, 3, 5, ... , ∞

Then, the normal multipoles B_n, the multipole coefficients b_n, and with the definition of $b_1=1$, the dopole field B_{ref} are as follows,

$$B_n = -\frac{2\mu_0}{\pi} I_z \frac{1}{r_0}\left(\frac{r_0}{a}\right)^n \cos n\varphi \quad \text{(A4)}$$

$$B_1 = B_{ref} = -\frac{2\mu_0}{\pi} I_z \frac{\cos\varphi}{a} \qquad \text{(A5)}$$

$$b_n = \frac{B_n}{b_n} = \left(\frac{r_0}{a}\right)^{n-1} \frac{\cos n\varphi}{\cos\varphi} \qquad \text{(A6)}$$

Similarly, for <u>four</u> helical current blocks with dipole symmetry, of current density +j (=+j_z), radii a_1, a_2, limiting angles ϕ_1, ϕ_2, at the position (r, θ) of $r<a$, the multipole expansion of the vector potential is as follows,

$$A_z(r,\theta) = \int_{a_1}^{a_2} A_z(r,\theta,a,\varphi)\, a\, da \int_{\varphi_1}^{\varphi_2} d\varphi$$

$$= -2\frac{\mu_0 j}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} r^n \frac{1}{2-n}\left(a_2^{2-n} - a_1^{2-n}\right)\frac{1}{n}\left(\sin n\varphi_2 - \sin n\varphi_1\right)\cos n\theta \qquad \text{(A7)}$$

Then, the multipoles are as follows,

$$B_n(r_0) = -\frac{2\mu_0}{\pi} j \frac{1}{n}\frac{r_0^{n-1}}{2-n}\left\{\sin(n\varphi_2) - \sin(n\varphi_1)\right\}\left[a_2^{2-n} - a_1^{2-n}\right] \qquad \text{(A8)}$$

$$B_1 = B_{ref} = -\frac{2\mu_0}{\pi} j\ (\sin\varphi_2 - \sin\varphi_1)\ [a_2 - a_1] \qquad \text{(A9)}$$

$$b_n = \frac{B_n}{B_1} = \frac{1}{n}\frac{r_0^{n-1}}{2-n}\frac{\left\{\sin(n\varphi_2) - \sin(n\varphi_1)\right\}\left[a_2^{2-n} - a_1^{2-n}\right]}{(\sin\varphi_2 - \sin\varphi_1)\ [a_2 - a_1]} \qquad \text{(A10)}$$

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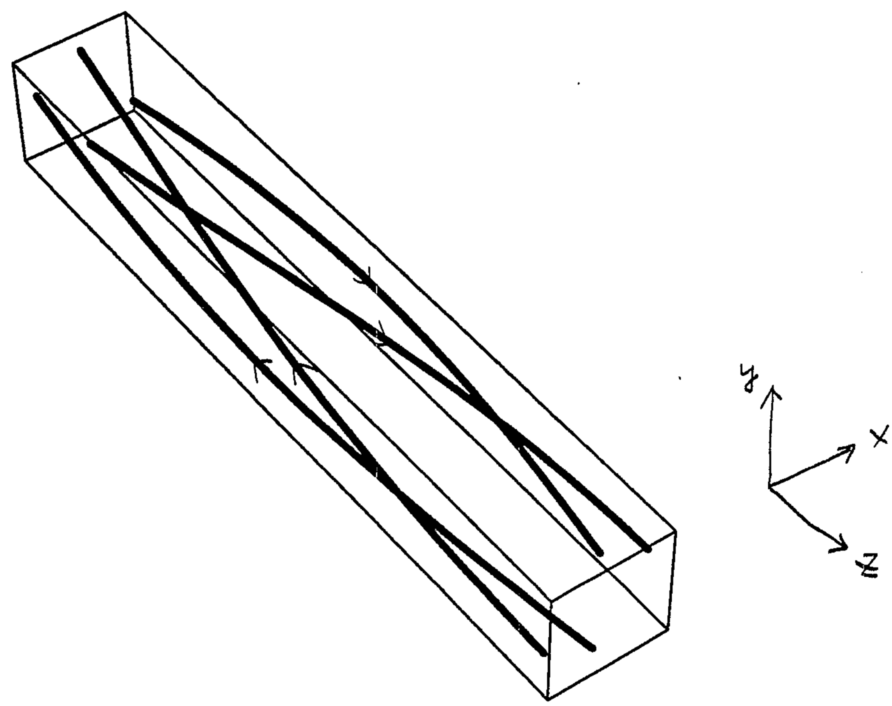

Fig.1-1 3-dimensional view of a helical dipole

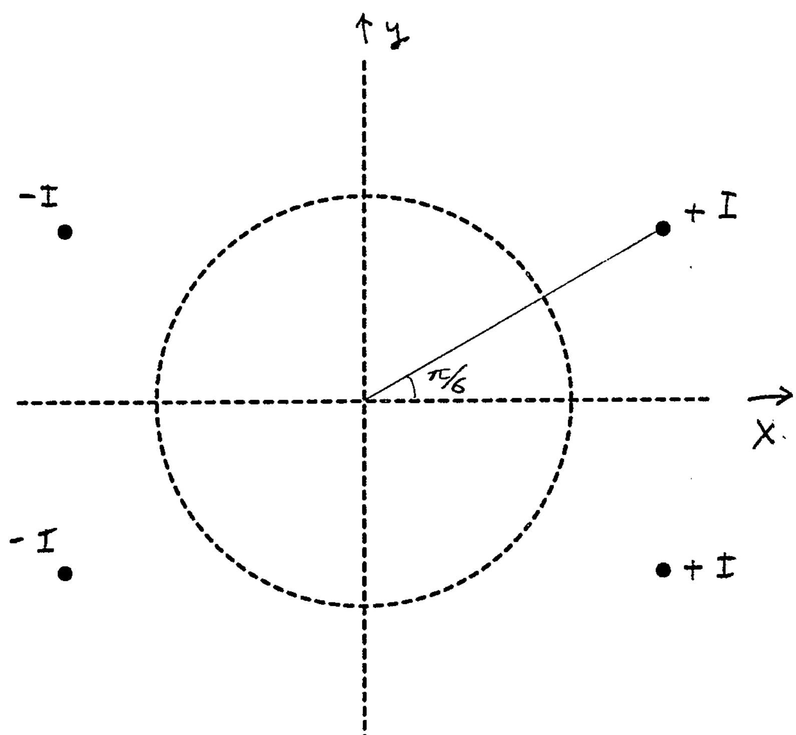

Fig.1-2 Four helical line currents with dipole symmetry

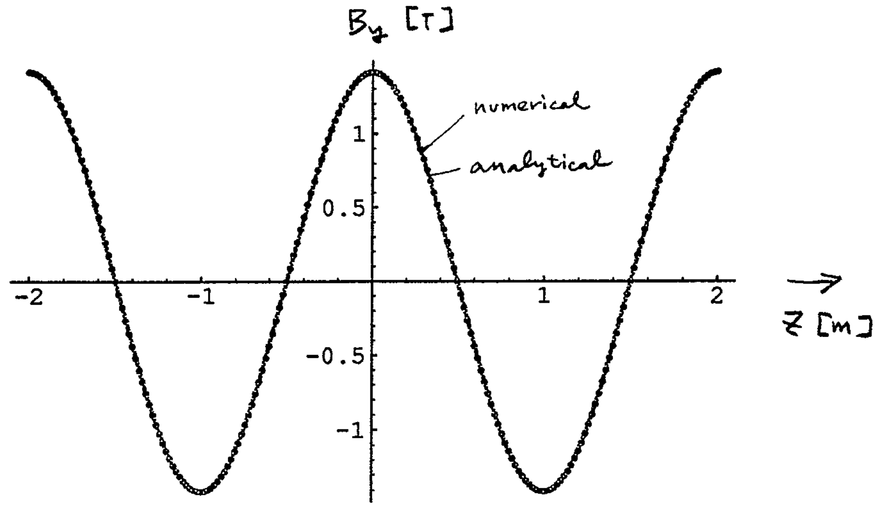

Fig.1-3 Distribution of By along the z axis

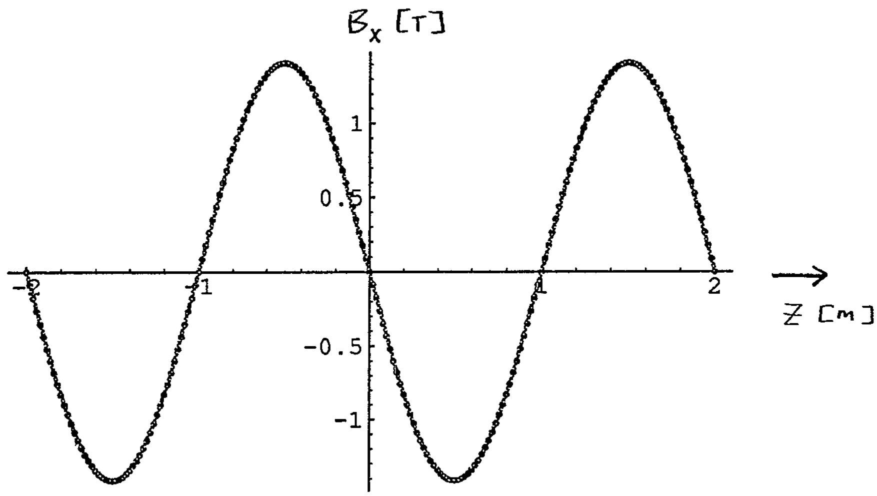

Fig.1-4 Distribution of Bx along the z axis

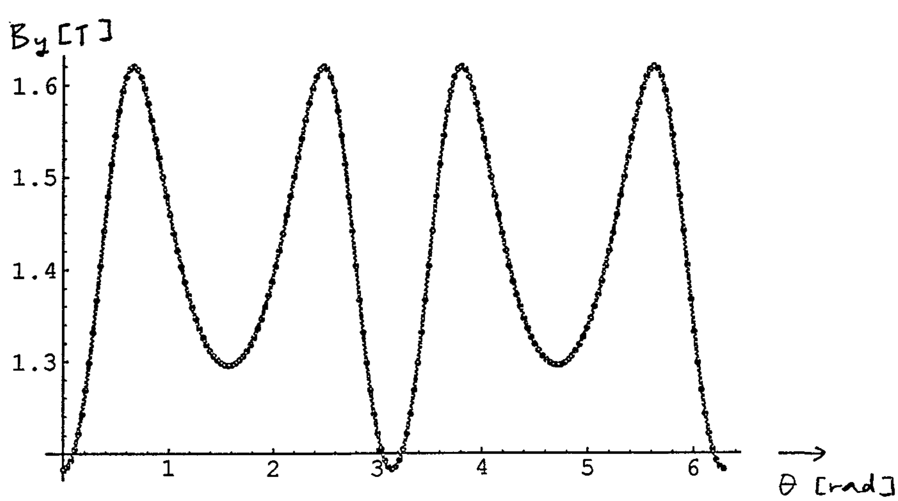

Fig.1-5 Distribution of By on the circle of radius r0=0.03 m at z = 0

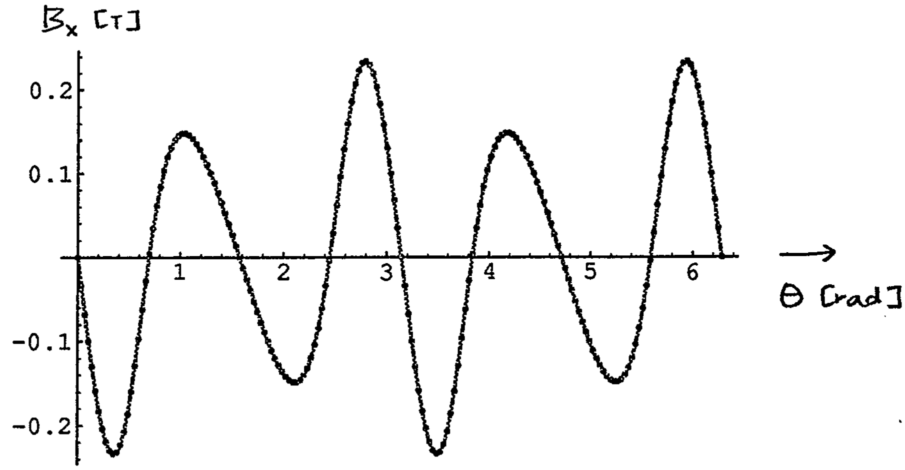

Fig.1-6 Distribution of Bx on the circle of radius r0=0.03 m at z = 0

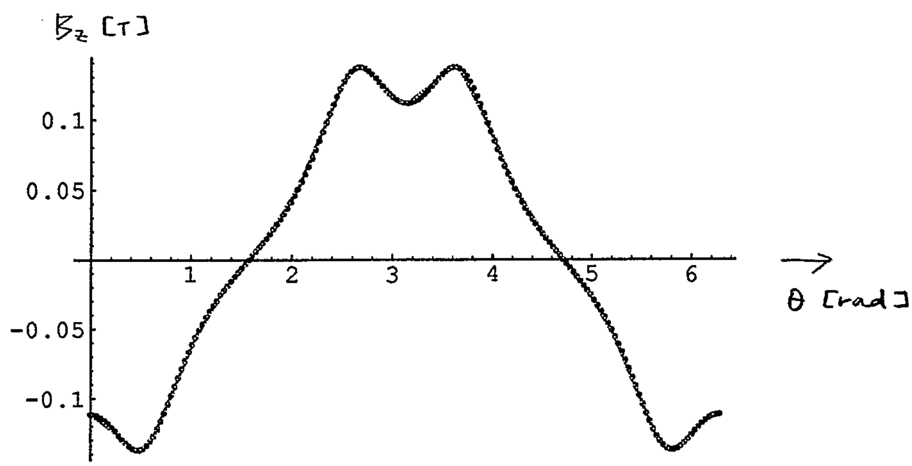

Fig.1-7 Distribution of Bz on the circle of radius r0=0.03 m at z = 0

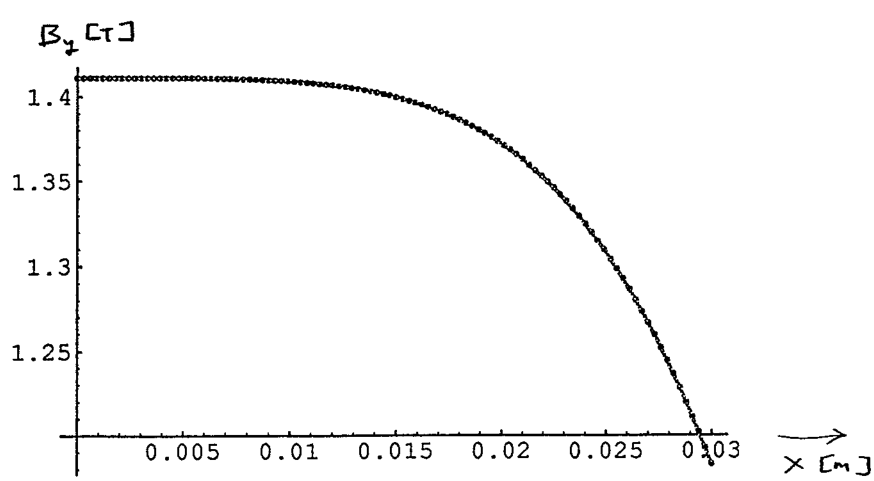

Fig.1-8 Distribution of By on the x axis at z = 0

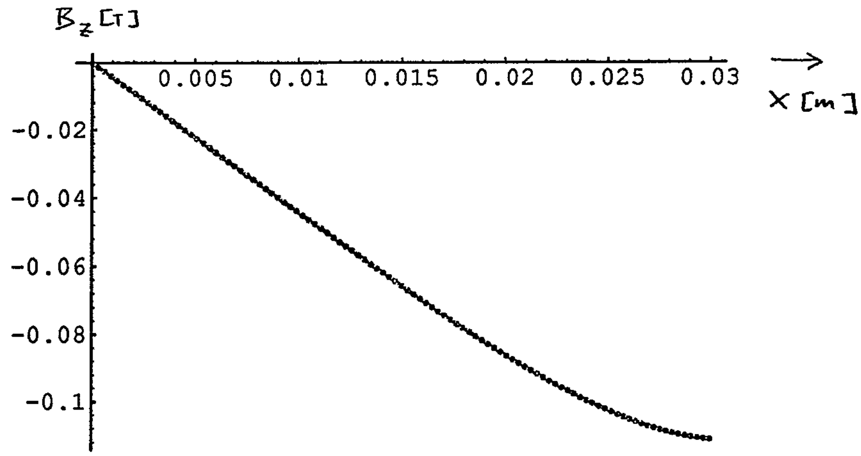

Fig.1-9 Distribution of Bz on the x axis at z = 0

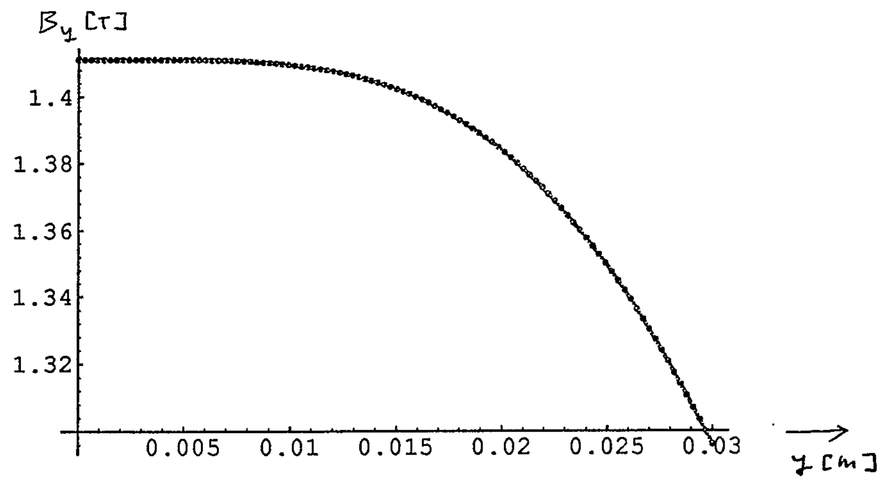

Fig.1-10 Distribution of By on the y axis at z = 0

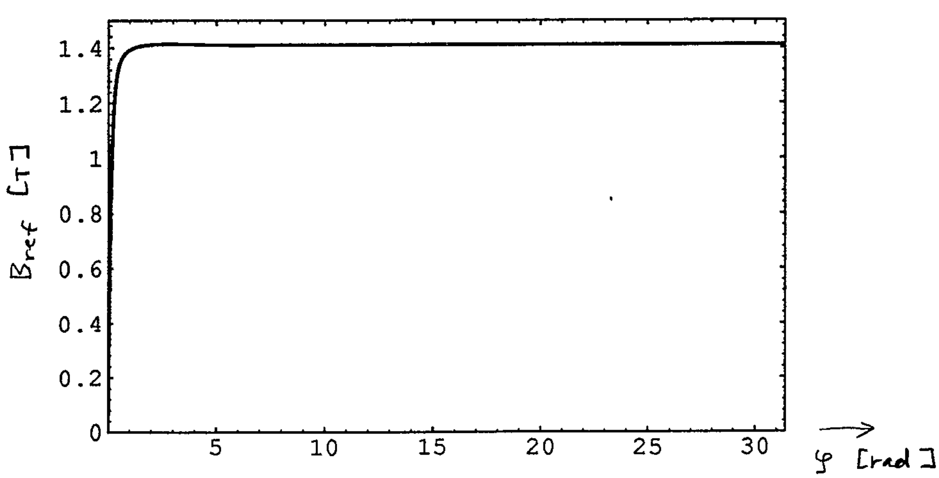

Fig.1-11 Length dependence of Bref at z = 0
(half length = a tan α $\phi = \phi / 2\pi$)

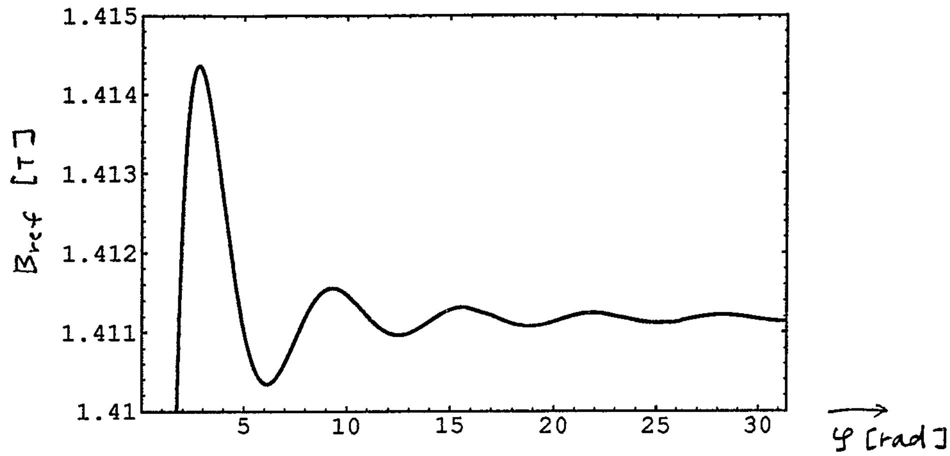

Fig.1-12 Length dependence of Bref at z = 0
(half length = a tan α $\phi = \phi / 2\pi$), enhanced of Fig.1-10

Table 1-1 Normal multipole coefficients derived from the numerical calculation for a infinitely long helical dipole

n	b-r	b-theta	b-z
1	0.999829	0.999829	0.999848
3	1.17709×10^{-10}	-2.85882×10^{-10}	2.30424×10^{-7}
5	-0.123864	-0.123864	-0.123864
7	-0.0438636	-0.0438636	-0.0438631
9	1.46902×10^{-10}	-1.72114×10^{-10}	5.28806×10^{-7}
11	0.00548344	0.00548344	0.00548422
13	0.0019353	0.0019353	0.00193623
15	4.85152×10^{-11}	-6.91257×10^{-11}	9.62863×10^{-7}
17	-0.000240154	-0.000240154	-0.000239159
19	-0.0000844321	-0.0000844323	-0.0000832915

Table 1-2 Normal multipole coefficients derived from the numerical calculation for a helical dipole with the length of one period L = 2 m

n	b-r	b-theta	b-z
1	0.999839	0.999843	0.997408
3	$5.56122\ 10^{-12}$	$1.69874\ 10^{-12}$	$-6.57707\ 10^{-10}$
5	-0.123603	-0.123603	-0.123603
7	-0.043771	-0.043771	-0.043771
9	$7.85488\ 10^{-12}$	$-9.24768\ 10^{-13}$	$-6.24984\ 10^{-10}$
11	0.00547187	0.00547187	0.00547187
13	0.00193122	0.00193122	0.00193122
15	$-1.5853\ 10^{-12}$	$-1.91519\ 10^{-12}$	$-5.70833\ 10^{-10}$
17	-0.000239648	-0.000239648	-0.000239648
19	-0.0000842539	-0.0000842539	-0.0000842544

Table 1-3 Normal multipole coefficients derived from the analytical calculation for the helical and non helical coils

n	b-helix	b-2d
(Bref)	1.41117	1.38564
1	1.	1.
3	0	0
5	-0.124337	-0.1296
7	-0.044193	-0.046656
9	0	0
11	0.00558607	0.00604662
13	0.00198621	0.00217678
15	0	0
17	-0.000251127	-0.000282111
19	-0.0000892971	-0.00010156

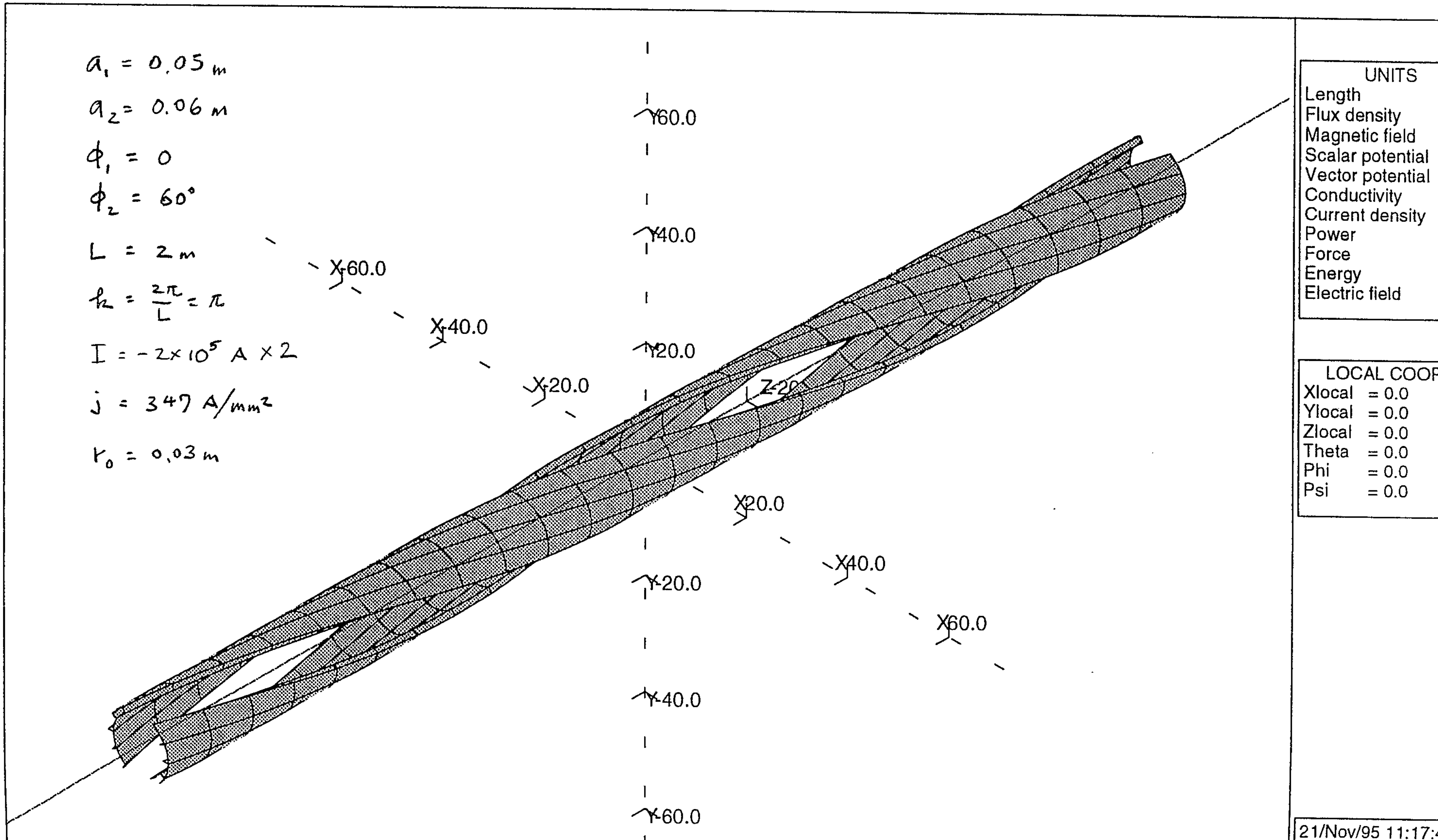

Fig. 2-1 3-dimensional view of a helical dipole

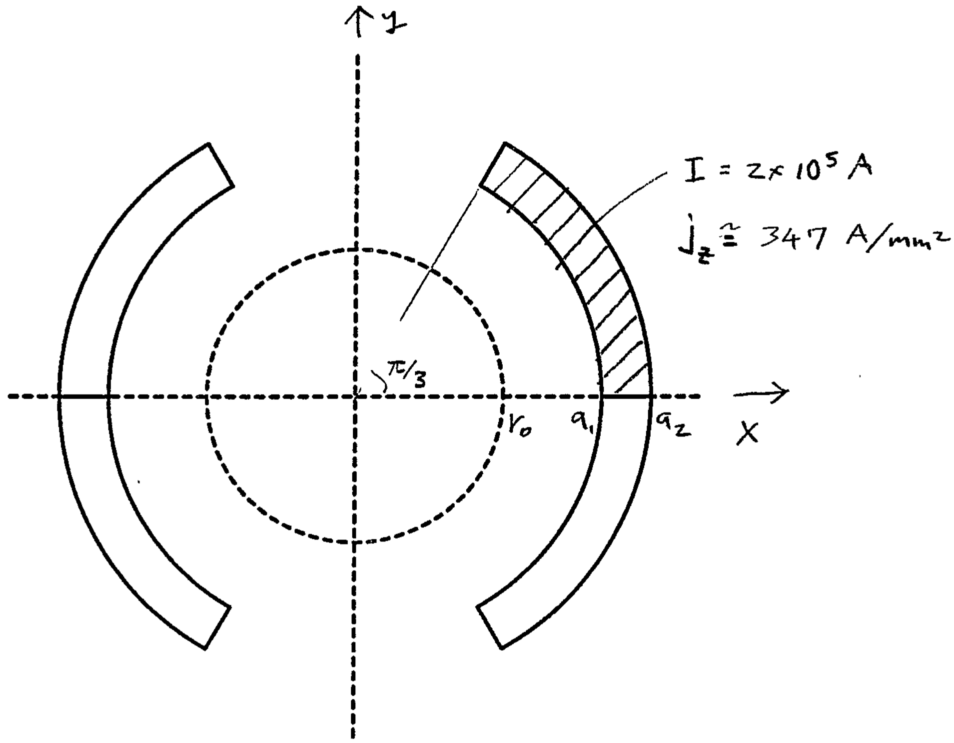

Fig.2-2 Cross section of a helical dipole

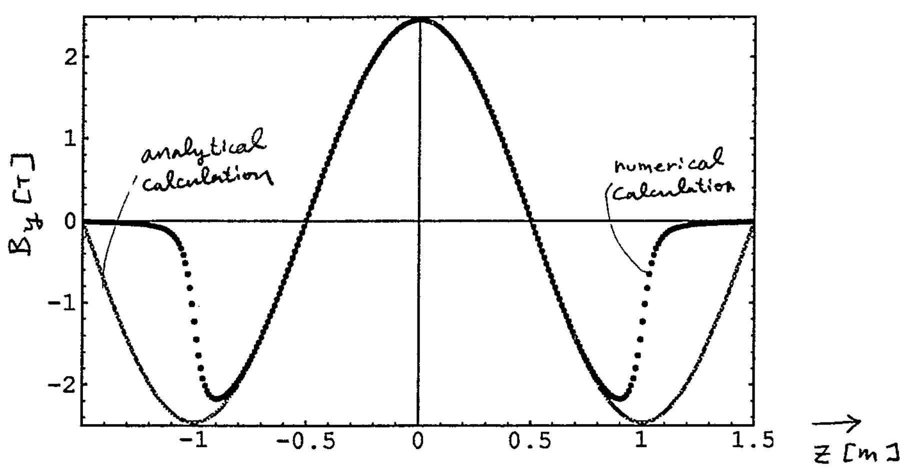

Fig.2-3 Distribution of By along the z axis

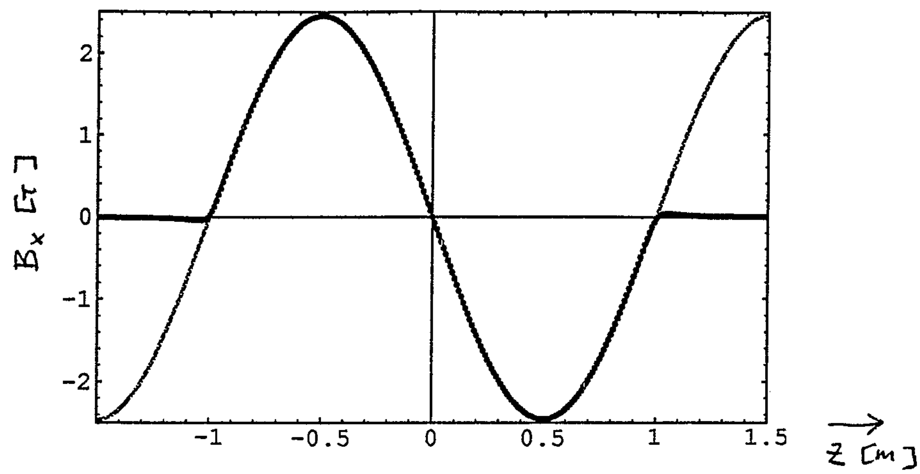

Fig.2-4 Distribution of Bx along the z axis

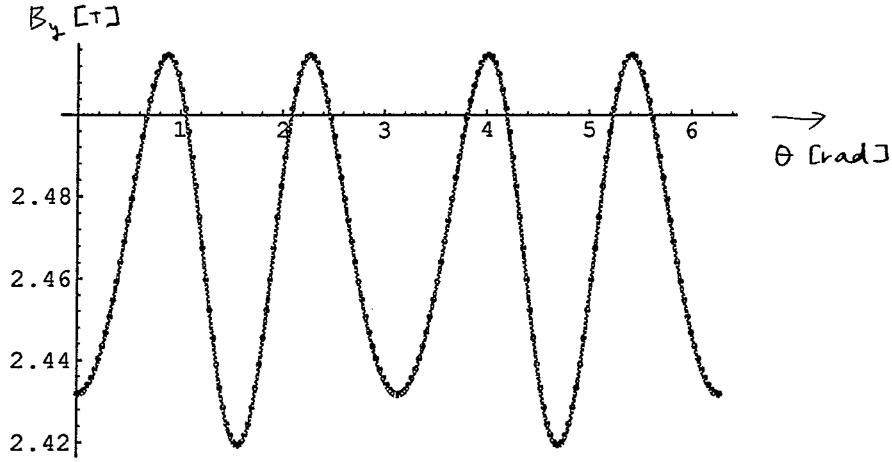

Fig.2-5 Distribution of By on the circle of radius r0=0.03 m at z = 0

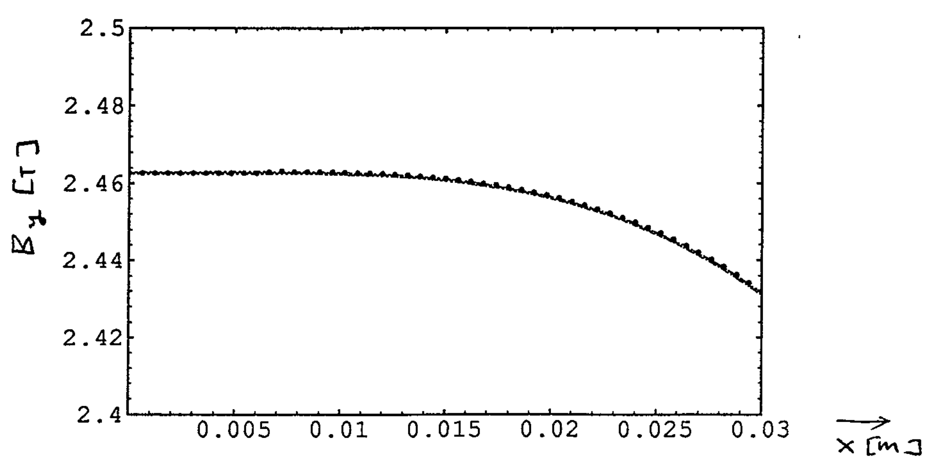

Fig.2-6 Distribution of By on the x axis at z = 0

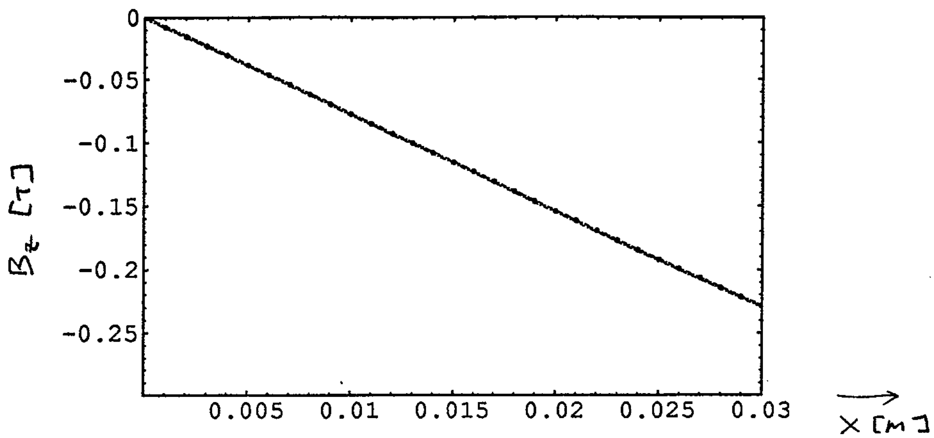

Fig.2-7 Distribution of Bz on the x axis at z = 0

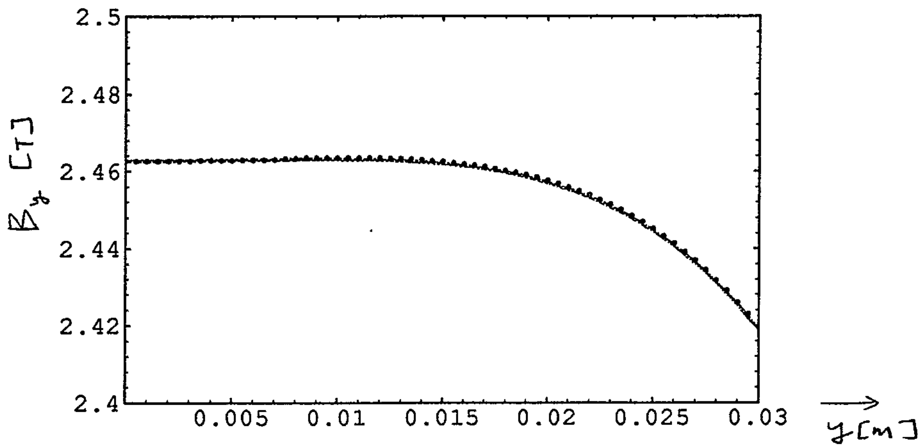

Fig.2-8 Distribution of By on the y axis at z = 0

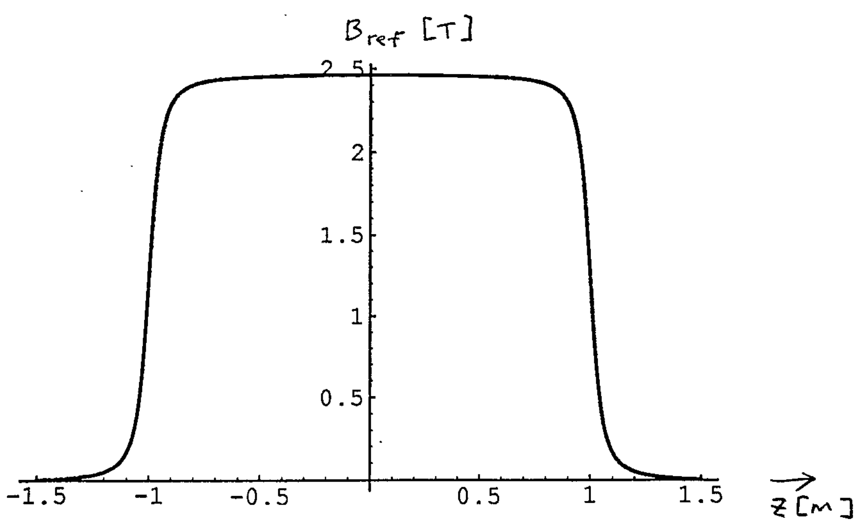

Fig.2-9 Distribution of Bref along the z axis

Table 2-1 Normal multipole coefficients derived from the numerical calculation

n	b-r	b-theta	b-z
1	1.00007	1.00007	0.997985
3	$2.131\ 10^{-6}$	$-1.00099\ 10^{-6}$	$-3.48087\ 10^{-7}$
5	-0.0172713	-0.0172738	-0.017302
7	0.00371776	0.00371768	0.00372466
9	$-5.2623\ 10^{-6}$	$-3.24599\ 10^{-6}$	$-7.38791\ 10^{-7}$
11	-0.00021577	-0.000220001	-0.000222324
13	0.0000631848	0.0000645796	0.0000570429
15	$-1.46767\ 10^{-6}$	$3.99728\ 10^{-6}$	$1.11228\ 10^{-6}$
17	$-4.26972\ 10^{-6}$	$-4.27167\ 10^{-6}$	$-4.07128\ 10^{-6}$
19	$4.42199\ 10^{-7}$	$2.20902\ 10^{-6}$	$3.01335\ 10^{-7}$

Table 2-2 Normal multipole coefficients derived from the analytical calculation for the helical and non helical coils

n	b-helix	b-2d
(Bref)	2.45618	2.4058
1	1.	1.
3	$1.36929\ 10^{-17}$	$1.4141\ 10^{-17}$
5	-0.0173424	-0.0182
7	0.00374336	0.00398657
9	$3.77454\ 10^{-19}$	$4.07687\ 10^{-19}$
11	-0.000224835	-0.000246199
13	0.00005935	0.0000658676
15	$1.14528\ 10^{-20}$	$1.28792\ 10^{-20}$
17	$-4.54038\ 10^{-6}$	$-5.17256\ 10^{-6}$
19	$1.301\ 10^{-6}$	$1.50127\ 10^{-6}$