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# Spin Rotation Matrices for Spin Tracking (10/95)


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October 1995


Collider Accelerator Department

**Brookhaven National Laboratory**



**U.S. Department of Energy**

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*Spin Note*

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# SPIN ROTATION MATRICES FOR SPIN TRACKING

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ALTERNATING GRADIENT SYNCHROTRON DEPARTMENT

BROOKHAVEN NATIONAL LABORATORY
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# Spin Rotation Matrices for Spin Tracking

A.Luccio

For spin trackingin a synchrotron[1] it is convenient to express the spin rotation in each machine element in matrix form

(1) $$\mathbf{S} = \hat{\mathbf{M}}\mathbf{S}_0$$

Start from the BMT equation [2]

(2) $$\frac{d\mathbf{S}}{dt} = \frac{e}{\gamma m}\mathbf{S} \times \mathbf{F}$$

with

(3) $$\mathbf{F} = (1+G\gamma)\mathbf{B}_\perp + (1+G)\mathbf{B}_\parallel + \left(G\gamma + \frac{\gamma}{1+\gamma}\right)\frac{\mathbf{E}\times\boldsymbol{\beta}}{c}$$

containing the magnetic field components transverse and longitudinal with respect to the velocity **v**.

Assume E = 0, and use vector identities to express the magnetic field *components* in terms of the magnetic field **B**

(4) $$(\mathbf{v}\times\mathbf{B}_\perp)\times\mathbf{v} = (\mathbf{v}\cdot\mathbf{v})\mathbf{B}_\perp - (\mathbf{v}\cdot\mathbf{B}_\perp)\mathbf{v} = v^2\mathbf{B}_\perp$$

Since $\mathbf{v}\times\mathbf{B}_\perp \equiv \mathbf{v}\times\mathbf{B}$

obtain

(5) $$\mathbf{B}_\perp = \frac{1}{v^2}(\mathbf{v}\times\mathbf{B})\times\mathbf{v}$$

Also

(6) $$\mathbf{v}\cdot\mathbf{B} = vB_\parallel, \quad \mathbf{B}_\parallel = \frac{\mathbf{v}}{v}B_\parallel$$

then

(7) $$\mathbf{B}_\parallel = \frac{1}{v^2}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}$$

After some readjustment

(8) $$\mathbf{F} = (1+G\gamma)\mathbf{B} - G(\gamma-1)\frac{1}{v^2}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}$$

Use a coordinate system that revolves around the accelerator[3,4]. The axis $z$ , longitudinal, is tangent to the "equilibrium" orbit, $x$ and $y$ are the displacements with respect to this orbit, radial and vertical, respectively. Call $s$ the longitudinal coordinate along the eq-orbit. In this coordinates, the derivative of a vector **a** is

(9) $$\frac{d\mathbf{a}}{ds} = \frac{da_x}{ds}\hat{\mathbf{x}} + a_x\frac{d\hat{\mathbf{x}}}{ds} + \frac{da_y}{ds}\hat{\mathbf{y}} + \frac{da_z}{ds}\hat{\mathbf{z}} + a_z\frac{d\hat{\mathbf{z}}}{ds}$$

with

(10) $$\frac{d\hat{\mathbf{x}}}{ds} = \frac{\hat{\mathbf{z}}}{\rho}, \quad \frac{d\hat{\mathbf{z}}}{ds} = -\frac{\hat{\mathbf{x}}}{\rho}$$

with $\rho$ the local curvature radius of the eq-orbit.

Because of Eqs. (9) and (10) the velocity is

(11) $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left[x'\,\hat{\mathbf{x}} + y'\,\hat{\mathbf{y}} + (1+\tfrac{x}{\rho})\hat{\mathbf{z}}\right]\frac{ds}{dt}$$

Primes denote differentiation with respect to $s$.

Make the assumption that in a machine element the magnetic field and the orbit don't change. If this is not the case, we will slice each element and write matrices for each slice.

Use Eqs. (9), (10), (11), and

$$\frac{e}{\gamma m} = \frac{v}{(B\rho)}$$

to rewrite Eq. (2) as follows

(12) $$\frac{d\mathbf{S}}{ds} = \mathbf{S}\times\mathbf{f}$$

with

(13) $$\mathbf{f} = \frac{h}{(B\rho)}\left[(1+G\gamma)\mathbf{B} - G(\gamma-1)(\mathbf{r}'\cdot\mathbf{B})\mathbf{r}'\right]$$

where

$$h = h(x, x', y') = \sqrt{x'^2 + y'^2 + (1 + x/\rho)^2}$$

and

$$\mathbf{r}' = \frac{\mathbf{v}}{v}$$

is the trajectory angle.

Using Eqs. (9) and (10), Eq. (12) is equivalent to three scalar diff. equations

$$\text{(14)} \qquad \begin{cases} S_x' = f_z S_y - (f_y - \frac{1}{\rho}) S_z \\ S_y' = f_x S_z - f_z S_x \\ S_z' = (f_y - \frac{1}{\rho}) S_x - f_x S_y \end{cases}$$

The system (14) yields three third order formally identical linear equations for the three components of the spin

$$\text{(15)} \qquad S''' + \omega^2 S' = 0$$

with

$$\text{(16)} \qquad \omega^2 = f_x^2 + (f_y - \frac{1}{\rho})^2 + f_z^2$$

The general integral of Eq. (15) is

$$S = C_1 + C_2 \cos \omega \delta s + C_3 \sin \omega \delta s$$

with $\delta s$ the orbit path length through the machine element along the eq-orbit. The constants of integration can be found as a function of the initial values of the spin components using $S$, $S'$, $S''$ and the original system. We obtain a matrix **M** [Eq. (1)] for the result

$$\text{(17)} \qquad \begin{pmatrix} 1-(B^2+C^2)c & ABc+Cs & ACc-Bs \\ ABc-Cs & 1-(A^2+C^2)c & BCc+As \\ ACc+Bs & BCc-As & 1-(A^2+B^2)c \end{pmatrix}$$

with

$$\begin{cases} c = 1 - \cos \omega \delta s \\ s = \sin \omega \delta s \end{cases}, \quad A = \frac{f_x}{\omega}, \quad B = \frac{f_y - 1/\rho}{\omega}, \quad C = \frac{f_z}{\omega}$$

Particular cases (to first order in $x$ and $y$)

1- Horizontal bend (field vertical). Rectangular Magnet.

$$B_x = 0, \quad B_y = B\rho/\rho, \quad B_z = 0$$

$$f_x = 0, \quad f_y - \frac{1}{\rho} = \frac{G\gamma}{\rho} - \frac{1+G\gamma}{\rho}\frac{x}{\rho}, \quad f_z = 0$$

$$A = 0, \quad B = 1, \quad C = 0$$

The matrix

$$\begin{pmatrix} \cos\delta\psi & 0 & \sin\delta\psi \\ 0 & 1 & 0 \\ -\sin\delta\psi & 0 & \cos\delta\psi \end{pmatrix} \tag{18}$$

represents a rotation of the spin around the vertical axis. The angle of rotation is

$$\delta\psi = \omega\,\delta s = \left[G\gamma - (1+G\gamma)\frac{x}{\rho}\right]\delta\theta$$

with $\delta\theta = \delta s/\rho$ the bend angle.

2- Quadrupole.

$$k_1 = -\frac{\partial B/\partial r}{(B\rho)}$$

$$B_x = k_1(B\rho)y, \quad B_y = k_1(B\rho)x, \quad B_z = 0$$

$$\frac{1}{\rho} = 0$$

Then

$$f_x = k_1(1+G\gamma)y, \quad f_y - 1/\rho = k_1(1+G\gamma)x, \quad f_z = 0$$

$$\omega = k_1(1+G\gamma)r, \quad r = \sqrt{x^2+y^2}$$

$$A = y/r, \quad B = x/r, \quad C = 0$$

The matrix is

$$\frac{1}{r^2}\begin{pmatrix} y^2 + x^2\cos\delta\psi & xy(1-\cos\delta\psi) & -xr\sin\delta\psi \\ xy(1-\cos\delta\psi) & x^2 + y^2\cos\delta\psi & yr\sin\delta\psi \\ xr\sin\delta\psi & -yr\sin\delta\psi & \cos\delta\psi \end{pmatrix} \tag{19}$$

with

$$\delta\psi = \omega\,\delta s$$

If $x = 0$ (no horizontal motion) Eq. (18) represents a rotation around the radial $x$. axis by an angle proportional to 1+Gγ. If $y = 0$, a rotation around the vertical $y$ axis by an angle proportional to Gγ.

3- Thin Siberian Snake.

In this device the field is in an horizontal plane at an angle $\phi$ with the $z$ axis.

$$B_x = B_{SN}\sin\phi, \quad B_y = 0, \quad B_z = B_{SN}\cos\phi$$

It is

$$f_x = \frac{1}{(B\rho)}(1+G\gamma)B_{SN}\sin\phi, \quad f_y = 0, \quad f_z = \frac{1}{(B\rho)}(1+G\gamma)B_{SN}\cos\phi$$
$$\omega = (1+G\gamma)B_{SN}$$
$$A = \sin\phi, \quad B = 0, \quad C = \cos\phi$$

producing the matrix

$$\text{(20)} \qquad \begin{pmatrix} 1-\cos^2\phi(1-\cos\delta\psi) & \cos\phi\sin\delta\psi & \sin\phi\cos\phi(1-\cos\delta\psi) \\ -\cos\phi\sin\delta\psi & \cos\delta\psi & \sin\phi\sin\delta\psi \\ \sin\phi\cos\phi(1-\cos\delta\psi) & -\sin\phi\sin\delta\psi & 1-\sin^2\phi(1-\cos\delta\psi) \end{pmatrix}$$

with the spin rotation angle

$$\delta\psi = \omega\,\delta s = (1+G\gamma)\frac{B_{SN}\delta s}{(B\rho)}$$

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**References**

[1] A.Luccio. Numerical Spin Tracking in a Synchrotron. Brookhaven Report BNL-52481, September 14, 1995.

[2] B.W. Montague. Polarized Beams in High Energy Storage Rings. Phys. Reports 113, No. 1 (1984) 1-96.

[3] S.Y. Lee. On the Polarized Beam Acceleration in Medium Energy Synchrotrons. Lecture note presented at RCNP-Kikuchi School on "Physics at Intermediate Energy", Osaka, Nov. 15-19, 1992.

[4] E.D. Courant and R.D. Ruth. The Acceleration of Polarized Protons in Circular Accelerators. Brookhaven Report BNL 51270/UC-28, September 12, 1980.