BNL-103665-2014-TECH

AGS/RHIC/SN 039;BNL-103665-2014-IR

# Angles from Spin Matrices

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October 1996

Collider Accelerator Department

**Brookhaven National Laboratory**

**U.S. Department of Energy**

USDOE Office of Science (SC)

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*Spin Note*

**AGS/RHIC/SN No. 039**

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October 8, 1996

# ANGLES FROM SPIN MATRICES *

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October 1, 1996

## 1. Introduction

In spin tracking [1] we use $3 \times 3$ matrices. The spin is represented by a three dimensional real unitary vector. Spin matrices can be build from a theoretical expression [2] or from a magnetic field map of the magnetic device in consideration. For a helical snake, an expression for the matrix derived from a synthetic field [3] to zeroth order was given in Ref. [4].

We want to review briefly how a spin rotation matrix is constructed, to find handy expressions for the spin precession and spin flip angle and the orientation of the spin precession axis, to be used in tracking.

## 2. Rotation in Ordinary Space

A vector $\vec{S}$ rotates (precesses) around a fixed direction $\vec{b}$ in the ordinary cartesian space. The instantaneous variation of the vector $d\vec{S}$ is perpendicular to $\vec{b}$. $\vec{S}$ forms with $\vec{b}$ a constant angle. We write this condition as

$$\frac{d\vec{S}}{d\psi} = \vec{S} \times \vec{b} \tag{1}$$

where $\psi$ is some angle. By performing the vector product in Eq. (1), the equation can be written explicitly as

$$\frac{d\vec{S}}{d\psi} = \hat{\imath}(S_1 b_3 - S_3 b_2) + \hat{\jmath}(S_3 b_1 - S_1 b_3) + \hat{k}(S_1 b_2 - S_2 b_1) \tag{2}$$

with $\hat{\imath}, \hat{\jmath}, \hat{k}$ three unitary vectors directed along the cartesian axes. $b_1, b_2, b_3$ and $S_1, S_2, S_3$ are the components of the two vectors along the axes, respectively.

Performing a second derivative on Eq. (1), using the equation again and doing a little algebra, obtain

$$\vec{S''} = -\omega^2 \vec{S} + \vec{b}(\vec{b} \cdot \vec{S}) \tag{3}$$

with the definition

$$\omega^2 = b_1^2 + b_2^2 + b_3^2 \tag{4}$$

The dot product in Eq. (3) is a constant -indeed it gives the angle between $\vec{S}$ and $\vec{b}$-. This can be explictly verified by a non believer by performing the derivative and

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substitute the expression for $\vec{S'}$ from Eq. (2). Let's see:

$$\begin{aligned} K &= \vec{b}\cdot\vec{S} \\ K' &= b_1S_1' + b_2S_2' + b_3S_3' \\ &= b_1(S_1b_3 - S_3b_2) + b_2(s_3b_1 - S_1b_3) + b_3(S_1b_2 - S_2b_1) = 0 \\ K &= const. \end{aligned}$$

Write again Eq. (3) as follows

$$\vec{S''} + \omega^2\vec{S} = \vec{P} \tag{5}$$

It is a vector equation, equivalent to three scalar equations, all formally identical. Only the constant term on the r.h,s. is different from equation to equation. This term is $\vec{P} = K\vec{b}$. The general integral of Eq. (5) can be written as follows

$$\vec{S} = \vec{P} + \vec{Q}\sin\varphi + \vec{R}\cos\varphi \tag{6}$$

with the precession angle $\varphi = \omega\Delta\psi$

The three constant vectors $\vec{P}, \vec{Q}, \vec{R}$ can be found from the initial conditions. This done, the general solution can be written

$$\vec{S} = \vec{S}_0 + \vec{S''}_0\frac{1-\cos\varphi}{\omega^2} + \vec{S'}_0\frac{\sin\varphi}{\omega} \tag{7}$$

Let us now write the three components of the solution, and make it in a matrix form. The matrix connects the components of the vector before and after the rotation. The matrix $R$ is given in Ref. [2]. It can be written as the sum of a unitary matrix $I$, a symmetric matrix $W$ and an anti symmetric matrix $A$ as follows

$$R = I\ \cos\varphi + W\ \frac{1-\cos\varphi}{\omega^2} + A\ \frac{\sin\varphi}{\omega} \tag{8}$$

The individual matrices are

$$W = \begin{pmatrix} b_2^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b^1 & b_3b_2 & b_3^2 \end{pmatrix},\quad A = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} \tag{9}$$

### 3. Case of Spin Rotation

The vector $\vec{S}$ is now a spin vector and the environment is an accelerator. Make the usual convention for accelerator coordinates and take as axis No. 1 the radial axis ($x$), as axis No. 2 the vertical axis ($y$), and as axis No. 3 the longitudinal axis ($z$). The Spin precesses around some effective magnetic field in each element of the machine. The theory [2] based on the BMT equation for the spin, shows how the components of the vector $\vec{b}$ are related to the field and to the energy of the particle.

An important angle is the spin flipping angle $\mu$, i.e. the angle formed between the final $\vec{S}_1$ and initial $\vec{S}_0$ orientation of the spin. The absolute value of this angle can be calculated from the scalar product

$$\cos\mu = \vec{S}_1 \cdot \vec{S}_0 \tag{10}$$

By definition the spin flipping angle cannot be larger than $\pi$.

Let us also define two angles for the orientation of $\vec{b}$ as follows. The latitude $\theta$ is the angle between the vector and the horizontal $x, z$ plane. The azimuth $\phi$ is the angle formed by the projection of $\vec{b}$ on the horizontal plane with $z$. For instance, the axis of a Siberian snake for RHIC is located in the horizontal plane $\theta = 0$ and forms with $z$ an angle of $\pm 45^0$

A spin rotation is defined by three angles, $\mu, \theta, \phi$. We want to derive them from the elements of any spin matrix, e.g. a matrix calculated from a field map. Using the latitude and azimuth, the components of the precession axis are

$$\begin{aligned} b_1 &= \omega\cos\theta\sin\phi \\ b_2 &= \omega\sin\theta \\ b_3 &= \omega\cos\theta\cos\phi \end{aligned} \tag{11}$$

To write an explicit expression for the matrices of Eq. (8) let us use some compact notation

$$\begin{matrix} c = \cos\varphi \\ s = \sin\varphi \end{matrix}, \quad \begin{matrix} u = \cos\theta \\ v = \sin\theta \end{matrix}, \quad \begin{matrix} p = \cos\phi \\ q = \sin\phi \end{matrix} \tag{12}$$

and write

$$W = \begin{pmatrix} u^2q^2 & uvq & u^2pq \\ (.) & v^2 & uvp \\ (.) & (.) & u^2p^2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & up & -v \\ -(.) & 0 & uq \\ -(.) & -(.) & 0 \end{pmatrix} \tag{13}$$

Explicit expressions of the matrix elements are

$$\begin{matrix} R_{11} = c + u^2q^2(1-c) & R_{12} = u\,[vq(1-c)+p] & R_{13} = u^2pq(1-c) - vs \\ R_{21} = u\,[vq(1-c)-ps] & R_{22} = c + v^2(1-c) & R_{23} = u\,[vp(1-c)+qs] \\ R_{31} = u^2pq(1-c) + vs & R_{32} = u\,[vp(1-c)-qs] & R_{33} = c + u^2p^2(1-c) \end{matrix} \tag{14}$$

The angles can be calculated by combining the matrix elements in many different ways. Not all combinations are suitable for computers in all cases. In our case, with a precession axis on the horizontal plane, $v \approx 0$, and the total precession angle $\approx 180^0$, or $s \approx 0$, one should avoid all expressions that carry $v$ or $s$ in the denominator.

A suitable chain of calculation is

$$\begin{aligned} \tan 2\phi &= \tfrac{2W_{13}}{W_{33}-W_{11}} \\ \tan 2\theta &= -\tfrac{A_{12}A_{13}\sin\phi}{A_{12}A_{23}-A_{13}^2\sin\phi\cos\phi} \\ \cos\varphi &= \tfrac{A_{12}A_{23}}{W_{13}-1} \\ \cos\mu &= (R\vec{S}_0)\cdot\vec{S}_1 \end{aligned}$$

Note that the use of the arctan function[‡] in the calculation of the axis angles allows one to find the angles in the full circle $0^0 - 360^0$, while the precession angle $\varphi$ will be only defined between $0^0$ and $180^0$. It is also useful to note that if the initial orientation of the spin is $(0,1,0)$ (vertical), we obtain

$$\cos\mu = W_{22}$$

## 4. Particular Cases

If the axis of precession lies in the horizontal plane, the matrices of Eq. (9) become

$$W = \frac{1}{2}\begin{pmatrix} 1-\cos 2\phi & 0 & \sin 2\phi \\ 0 & 0 & 0 \\ \sin 2\phi & 0 & 1+\cos 2\phi \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \cos\phi & 0 \\ -\cos\phi & 0 & \sin\phi \\ 0 & -\sin\phi & 0 \end{pmatrix} \tag{15}$$

In this case, the vertical ($y$) component of the spin $\vec{S}$ is not affected by the $W$ or $A$ matrices, but only by the term $I\cos\mu$ in Eq. (8).

If, moreover, the axis is at $\phi = 45^0$ or $= 180^0 + 45^0$, we have

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and if the axis is at $\phi = -45^0$ or $= 180^0 - 45^0$

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad A = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Finally, for a $\mu = 180^0$ flipping, with axis in the horizontal plane, at $\pm 45^0$, or $\mp 135^0$, respectively (RHIC snakes), it is

$$R = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & -1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}$$

## 5. References


1. A.U.Luccio *Numerical Spin Tracking in a Synchrotron. Computer Code SPINK - Examples (RHIC).* Rept. BNL-52481, Upton, NY, September 1995. Also: Spin Note AGS/RHIC/SN No.011. Also: Proc. Adriatico Conf. Trieste, Italy, Dec. 1995.
2. A.U.Luccio *Spin Rotation Matrices for spin Tracking* Rept.BNL-62371, AGS/AD/96-1, Upton, October30, 1995
3. J.P.Blewett and R.Chasman *Orbits and Fields in the Helical Wiggler.* J. Ap. Phys. 48 (1977) p.2692
4. M.J.Syphers *Spin Motion through Helical Dipole Magnets.* Spin Note AGS/RHIC/SN No 020, Upton, NY,


[‡] function atan2, in Fortran