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Multipole Field Expansion of Helical Magnet

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Recently three dimensional field calculation has been extensively performed by T. Tominaka and M. Okamura on the helical magnet for the RHIC spin snake and rotator. One of the main concerns of field calculation is multipole field components which may have the different values from two dimensional cases. In this report, the analytical formula of multipole field expansion are given to investigate the numerical results of 3 D calculation. The betatron tune shift and local chromaticity due to the helical magnet field, are represented.

1. Multipole expansion

Near the axis of helical magnet, the magnetic field can be expressed in the cylindrical coordinate system(r,θ,s) as [1],

$$B_r = -\sum_{n=1,3,\cdots}^{\infty} \sum_{m=1,2,\cdots}^{\infty} G_{n,m}\omega_m I'_n(\omega_m r)\sin(n\theta - \omega_m s) \tag{1.1}$$

$$B_\theta = -\sum_{n=1,3,\cdots}^{\infty} \sum_{m=1,2,\cdots}^{\infty} nG_{n,m}\frac{1}{r} I_n(\omega_m r)\cos(n\theta - \omega_m s) \tag{1.2}$$

$$B_s = +\sum_{n=1,3,\cdots}^{\infty} \sum_{m=1,2,\cdots}^{\infty} G_{n,m}\omega_m I_n(\omega_m r)\cos(n\theta - \omega_m s) \tag{1.3}$$

where the coefficients are defined by

$$G_{n,m} = n!\left(\frac{2}{\omega_m}\right)^n B_{n,m} \tag{1.4}$$

and

$$\omega_m = \frac{(2m-1)}{L}\pi, \quad L = \frac{\lambda}{2} \tag{1.5}$$

λ is a wavelength of helical magnet. I is the modified Bessel function and its derivative is

$$I'_n(\omega_m r) = I_{n-1}(\omega_m r) - \frac{n}{\omega_m r} I_n(\omega_m r) \tag{1.6}$$

Leaving the fundamental term, namely $n=m=1$ term, the following well known representation of helical field[2)] are obtained.

$$\omega_1 = \frac{\pi}{L}, \quad G_{1,1} = \frac{2L}{\pi} B_{1,1} \tag{1.7}$$

$$B_r = -2B_{1,1} I'_1\left(\frac{\pi r}{L}\right) \sin\left(\theta - \frac{\pi s}{L}\right) \tag{1.8}$$

$$B_\theta = -2B_{1,1} \frac{L}{\pi r} I_1\left(\frac{\pi r}{L}\right) \cos\left(\theta - \frac{\pi s}{L}\right) \tag{1.9}$$

$$B_s = 2B_{1,1} I_1\left(\frac{\pi r}{L}\right) \cos\left(\theta - \frac{\pi s}{L}\right) \tag{1.10}$$

Now we will expand the magnetic field including the higher order terms of n. Concerning the longitudinal components, $m=1$ terms are assumed to be dominant. Then

$$B_r = -\sum_{n=1,3,\cdots}^{\infty} G_{n,1} \omega_1 I'_n(\omega_1 r) \sin(n\theta - \omega_1 s) \tag{1.11}$$

$$B_\theta = -\sum_{n=1,3,\cdots}^{\infty} n G_{n,1} \frac{1}{r} I_n(\omega_1 r) \cos(n\theta - \omega_1 s) \tag{1.12}$$

$$B_s = \sum_{n=1,3,\cdots}^{\infty} G_{n,1} \omega_1 I_n(\omega_1 r) \cos(n\theta - \omega_1 s) \tag{1.13}$$

where

$$\omega_1 = \frac{2\pi}{\lambda}, \quad G_{n,1} = n!\left(\frac{2}{\omega_1}\right)^n B_{n,1} \tag{1.14}$$

The modified Bessel function can be expanded as,

$$I_n(\omega_1 r) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\frac{\omega_1 r}{2}\right)^{2k+n} \tag{1.15}$$

$$I'_n(\omega_1 r) = I_{n-1}(\omega_1 r) - \frac{n}{\omega_1 r} I_n(\omega_1 r)$$

$$= \sum_{k=0}^{\infty} \frac{(2k+n)}{2k!(k+n)!} \left(\frac{\omega_1 r}{2} \right)^{2k+n-1} \tag{1.16}$$

The results of field expansion up to fourth orders of $\omega_1 r$ are,

$$B_r = -\frac{\omega_1}{2} \Big[G_{1,1} \left(1 + \frac{3}{8} \omega_1^2 r^2 + \frac{5}{192} \omega_1^4 r^4 \right) \sin(\theta - \omega_1 s)$$

$$+ G_{3,1} \left(\frac{1}{8} \omega_1^2 r^2 + \frac{5}{384} \omega_1^4 r^4 \right) \sin(3\theta - \omega_1 s)$$

$$+ G_{5,1} \left(\frac{1}{384} \omega_1^4 r^4 \right) \sin(5\theta - \omega_1 s) \Big] \tag{1.17}$$

$$B_\theta = -\frac{\omega_1}{2} \Big[G_{1,1} \left(1 + \frac{1}{8} \omega_1^2 r^2 + \frac{1}{192} \omega_1^4 r^4 \right) \cos(\theta - \omega_1 s)$$

$$+ G_{3,1} \left(\frac{1}{8} \omega_1^2 r^2 + \frac{1}{128} \omega_1^4 r^4 \right) \cos(3\theta - \omega_1 s)$$

$$+ G_{5,1} \left(\frac{1}{384} \omega_1^4 r^4 \right) \cos(5\theta - \omega_1 s) \Big] \tag{1.18}$$

$$B_s = \frac{\omega_1^2 r}{2} \Big[G_{1,1} \left(1 + \frac{1}{8} \omega_1^2 r^2 + \frac{1}{192} \omega_1^4 r^4 \right) \cos(\theta - \omega_1 s)$$

$$+ G_{3,1} \left(\frac{1}{24} \omega_1^2 r^2 + \frac{1}{384} \omega_1^4 r^4 \right) \cos(3\theta - \omega_1 s)$$

$$+ G_{5,1} \left(\frac{1}{1920} \omega_1^4 r^4 \right) \cos(5\theta - \omega_1 s) \Big] \tag{1.19}$$

In the limit of $\omega_1 = 0$, namely for the straight magnet,

$$\operatorname*{limit}_{\omega_1 \to 0} B_r = -\left[B_{1,1} \sin\theta + 3B_{3,1} r^2 \sin 3\theta + 5B_{5,1} r^4 \sin 5\theta \right] \tag{1.20}$$

$$\operatorname*{limit}_{\omega_1 \to 0} B_\theta = -\left[B_{1,1} \cos\theta + 3B_{3,1} r^2 \cos 3\theta + 5B_{5,1} r^4 \cos 5\theta \right] \tag{1.21}$$

$$\operatorname*{limit}_{\omega_1 \to 0} B_s = 0 \tag{1.22}$$

Comparing these results with the 2 dimesional expansion coefficients of,

$$B_\theta = B_0 \sum_{n=1,3}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(b_n \cos(n\theta) + a_n \sin(n\theta)\right)$$
$$B_r = B_0 \sum_{n=1,3}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(-a_n \cos(n\theta) + b_n \sin(n\theta)\right) \tag{1.23}$$

we get the following relations

$$B_{11} = -B_0 b_1$$
$$B_{31} = -\frac{1}{3}\frac{B_0 b_3}{r_0^2} \tag{1.24}$$
$$B_{51} = -\frac{1}{5}\frac{B_0 b_5}{r_0^4}$$

Inserting these results in the equations (1.17), (1.18) and (1.19), the fields are represented as follows.

$$B_r = B_0\left[b_1\left(1 + \frac{3}{8}(\omega_1 r)^2 + \frac{5}{192}(\omega_1 r)^4\right)\sin(\theta - \omega_1 s)\right.$$
$$+ b_3\left(\left(\frac{r}{r_0}\right)^2 + \frac{5}{48}(\omega_1 r_0)^2\left(\frac{r}{r_0}\right)^4\right)\sin(3\theta - \omega_1 s)$$
$$\left. + b_5\left(\frac{r}{r_0}\right)^4 \sin(5\theta - \omega_1 s)\right] \tag{1.25}$$

$$B_\theta = B_0\left[b_1\left(1 + \frac{1}{8}\omega_1^2 r^2 + \frac{1}{192}\omega_1^4 r^4\right)\cos(\theta - \omega_1 s)\right.$$
$$+ b_3\left(\left(\frac{r}{r_0}\right)^2 + \frac{1}{16}(\omega_1 r_0)^2\left(\frac{r}{r_0}\right)^4\right)\cos(3\theta - \omega_1 s)$$
$$\left. + b_5\left(\frac{r}{r_o}\right)^4 \cos(5\theta - \omega_1 s)\right] \tag{1.26}$$

$$B_s = -\omega_1 r B_0\left[b_1\left(1+\frac{1}{8}\omega_1^2 r^2+\frac{1}{192}\omega_1^4 r^4\right)\cos(\theta-\omega_1 s)\right.$$

$$+b_3\left(\frac{1}{3}\left(\frac{r}{r_0}\right)^2+\frac{1}{48}(\omega_1 r_0)^2\left(\frac{r}{r_0}\right)^4\right)\cos(3\theta-\omega_1 s)$$

$$\left.+\frac{b_5}{5}\left(\frac{r}{r_0}\right)^4\cos(5\theta-\omega_1 s)\right] \tag{1.27}$$

At the radial position $r = r_0$, reference radius, the fields are

$$B_r = B_0\left[b_1\left(1+\frac{3}{8}(\omega_1 r_0)^2+\frac{5}{192}(\omega_1 r_0)^4\right)\sin(\theta-\omega_1 s)\right.$$

$$+b_3(1+\frac{5}{48}(\omega_1 r_0)^2)\sin(3\theta-\omega_1 s)$$

$$\left.+b_5\sin(5\theta-\omega_1 s)\ \right] \tag{1.28}$$

$$B_\theta = B_0\left[b_1\left(1+\frac{1}{8}(\omega_1 r_0)^2+\frac{1}{192}(\omega_1 r_0)^4\right)\cos(\theta-\omega_1 s)\right.$$

$$+b_3\left(1+\frac{1}{16}(\omega_1 r_0)^2\right)\cos(3\theta-\omega_1 s)$$

$$\left.+b_5\cos(5\theta-\omega_1 s)\right] \tag{1.29}$$

$$B_s = -\omega_1 r_0 B_0\left[b_1\left(1+\frac{1}{8}(\omega_1 r_0)^2+\frac{1}{192}(\omega_1 r_0)^4\right)\cos(\theta-\omega_1 s)\right.$$

$$+\frac{b_3}{3}\left(1+\frac{1}{16}(\omega_1 r_0)^2\right)\cos(3\theta-\omega_1 s)$$

$$\left.+\frac{b_5}{5}\cos(5\theta-\omega_1 s)\right] \tag{1.30}$$

The vertical magnetic field is given as

$$B_z = B_r\sin(\theta)+B_\theta\cos(\theta) \tag{1.31}$$

and can be expanded as

$$B_z = B_0 \sum_{m=0,2,4....}^{\infty} [\beta_{zm}(r)\cos m\theta + \alpha_{zm}(r)\sin m\theta] \qquad (1.32)$$

and the coefficients are derived with use of the above results. They are

$$\beta_{z0}(r) = b_1\left[1 + \frac{1}{4}\omega_1^2 r^2 + \frac{1}{64}\omega_1^4 r^4\right]\cos(\omega_1 s)$$

$$\beta_{z2}(r) = \left[\left(\frac{b_3}{\omega_1^2 r_0^2} - \frac{b_1}{8}\right)\omega_1^2 r^2 + \left(\frac{b_3}{12\omega_1^2 r_0^2} - \frac{b_1}{96}\right)\omega_1^4 r^4\right]\cos(\omega_1 s)$$

$$\beta_{z4}(r) = \left[\left(-\frac{b_3}{48\omega_1^2 r_0^2} + \frac{b_5}{\omega_1^4 r_0^4}\right)\omega_1^4 r^4\right]\cos(\omega_1 s)$$

$$\begin{aligned} \alpha_{z0}(r) &= \beta_{z0}(r)\tan(\omega_1 s) \\ \alpha_{z2}(r) &= \beta_{z2}(r)\tan(\omega_1 s) \\ \alpha_{z4}(r) &= \beta_{z4}(r)\tan(\omega_1 s) \end{aligned} \qquad (1.33)$$

$$(\omega_1 = 2\pi / \lambda)$$

where $b_i(i = 1,3,\cdots)$ is 2D multipole field coefficients defined by

$$B_z = B_0 \sum_{m=0,2,4....}^{\infty} \left(\frac{r}{r_0}\right)^m (b_{m+1}\cos m\theta + a_{m+1}\sin m\theta) \qquad (1.34)$$

and skew components $a_1, a_3, ... = 0$ are assumed to be 0 in the derivation.

At $r = r_0$ the coefficients becomes as

$$\beta_{z0}(r_0) = b_1\left[1 + \frac{1}{4}\omega_1^2 r_0^2 + \frac{1}{64}\omega_1^4 r_0^4\right]\cos(\omega_1 s)$$

$$\beta_{z2}(r_0) = \left[b_3 - \frac{b_1}{8}\omega_1^2 r_0^2 + \left(\frac{1}{12}b_3 - \frac{b_1}{96}\omega_1^2 r_0^2\right)\omega_1^{\ 2} r_0^{\ 2}\right]\cos(\omega_1 s)$$

$$\beta_{z4}(r_0) = \left[b_5 - \frac{b_3}{48}\omega_1^2 r_0^2\right]\cos(\omega_1 s)$$

$$\alpha_{z0}(r_0)=\beta_{z0}(r_0)\tan(\omega_1 s)$$
$$\alpha_{z2}(r_0)=\beta_{z2}(r_0)\tan(\omega_1 s) \qquad (1.35)$$
$$\alpha_{z4}(r_0)=\beta_{z4}(r_0)\tan(\omega_1 s)$$

We shoud notice the sextupole component of

$$b_3-\frac{b_1}{8}\omega_1{}^2r_0{}^2 \qquad (1.36)$$

which has the additional term of $-\frac{b_1}{8}\omega_1{}^2r_0{}^2$ comparing with the normal dipole magnet. In the design of helical magnet, this fact should be kept in mind. Even when one designs the magnet ideally and the sextupole component is zero in 2 dimensional configulation, the real helical magnet has the intrinsic sextupole component given in above equation, originated from the structure of helical shape.

At the limit of $\omega_1=0$, coefficients at s=0, center of helical magnet, just equal to 2D coefficients as expected.

$$\beta_{z0}(r_0)=b_1$$
$$\beta_{z2}(r_0)=b_3$$
$$\beta_{z4}(r_0)=b_5$$

$$\alpha_{z0}(r_0)=0$$
$$\alpha_{z2}(r_0)=0$$
$$\alpha_{z4}(r_0)=0$$

Similarily horizontal field B_x can be expanded as

$$B_x=B_0\sum_{m=0,2,4....}^{\infty}[\beta_{xm}(r)\cos m\theta+\alpha_{xm}(r)\sin m\theta] \qquad (1.37)$$

$$\beta_{x0}(r)=-b_1\left[1+\frac{1}{4}\omega_1^2r^2+\frac{1}{64}\omega_1^4r^4\right]\sin(\omega_1 s)$$

$$\beta_{x2}(r)=-\left[\left(\frac{b_1}{8}+\frac{b_3}{(\omega_1 r_0)^2}\right)\omega_1^2r^2+\left(\frac{b_1}{96}+\frac{b_3}{12\omega_1^2r_0^2}\right)\omega_1^4r^4\right]\sin(\omega_1 s)$$

$$\beta_{x4}(r) = -\left[\left(\frac{b_3}{48\omega_1^2 r_0^2} + \frac{b_5}{(\omega_1 r_0)^4}\right)\omega_1^4 r^4\right]\sin(\omega_1 s)$$

$$\begin{aligned} \alpha_{x0}(r) &= -\beta_{x0}(r)\cot(\omega_1 s) \\ \alpha_{x2}(r) &= -\beta_{x2}(r)\cot(\omega_1 s) \\ \alpha_{x4}(r) &= -\beta_{x4}(r)\cot(\omega_1 s) \end{aligned} \qquad (1.38)$$

At the limit of $\omega_1 \to 0$

$$\begin{aligned} &\beta_{x0}(r) = \beta_{x2}(r) = \beta_{x4}(r) = 0 \\ &\alpha_{x0}(r) = b_1 \\ &\alpha_{x2}(r) = b_3(r/r_0)^2 \\ &\alpha_{x4}(r) = b_5(r/r_0)^4 \end{aligned} \qquad (1.39)$$

The longitudinal components of helical field are given as follows.

$$B_s = B_0 \sum_{m=0,2,4....}^{\infty}[\beta_{sm}(r)\cos(m+1)\theta + \alpha_{sm}(r)\sin(m+1)m\theta] \qquad (1.34)$$

$$\beta_{s0}(r) = -b_1\left[\omega_1 r + \frac{1}{8}(\omega_1 r)^3 + \frac{1}{192}(\omega_1 r)^5\right]\cos(\omega_1 s)$$

$$\beta_{s2}(r) = -\frac{b_3}{\omega_1^2 r_0^2}\left[\frac{1}{3}(\omega_1 r)^3 + \frac{1}{48}(\omega_1 r)^5\right]\cos(\omega_1 s)$$

$$\beta_{s4}(r) = -\frac{b_5}{\omega_1^4 r_0^4}\left[\frac{1}{5}(\omega_1 r)^5\right]\cos(\omega_1 s)$$

$$\begin{aligned} \alpha_{s0}(r) &= \beta_{s0}(r)\cdot\tan(\omega_1 s) \\ \alpha_{s2}(r) &= \beta_{s2}(r)\cdot\tan(\omega_1 s) \\ \alpha_{s4}(r) &= \beta_{s4}(r)\tan\cdot(\omega_1 s) \end{aligned} \qquad (1.40)$$

$$\underset{\omega_1 \to 0}{Limit}\, \beta_{sm}(r) = \underset{\omega_1 \to 0}{Limit}\, \alpha_{sm}(r) = 0$$

In the case of straight magnet, the longitudinal field is obviously vanished.

2. Comparison of 3D calculated results and analytical ones.

The magnetic fields in helical magnet are calculted numerically with use of the TOSCA program on the BNL slotted type magnet. The spiral angle is 345 degrees and the length is 2.30 m. The details of 3D results will be reported elsewhere[3]. Here the multipole components and reference field are tabulated in Table 1, and compared with the analytical results. In Table 1, B_{2D} means the 2 dimensional magnet, namely the infinite wavelength helical magnet, , and B_r, B_θ and B_z means the real helical type magnet and their multipole components are derived with B_r, B_θ and B_z fields representations. See the equations (1.28), (1.29) and (1.35). Between 2D and 3D results, the reference fields are different. Then multipole components are normalized as the dipole components are unity in each case. In the Table, upper values are from numerical calculation by TOSCA program and lower ones are by the analytical formula given in previous section. Both are calculated at the center of helical structure, s =0.

Table 1 Comparison of analytical and 3D calculated results

	B_{2D}	B_r	B_θ	B_z
B_{ref} (Tesla)	2.8259	2.8865	2.8822	2.8843
n=0 (Dipole)	1.0	1.0	1.0	1.0
n=2 (Sextupole)	- 0.00219	- 0.00217 - 0.00218	- 0.00217 - 0.00219	- 0.00291 - 0.00300
n=4 (Decapole)	0.000372	0.000374 0.000371	0.000374 0.000372	0.000376 0.000372

Both results are well in agreement with each other, and the sextupole components of vertical field representation, are largely different from the 2D case, as explained in the previous section. From the beam dynamics point of view in the storage ring, the sextupole components in the vertical field representation much affects the betatron tune shift and chromaticity.

3. The betatron tune shift and local chromaticity due to the helical magnet

To get the betatron tune shift of proton beam due to the helical magnet, we need the field gradient in the helical magnet. With use of the relation of derivatives of

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial r}\cos\theta - \frac{1}{r}\sin\theta\frac{\partial B_z}{\partial \theta}$$

the field gradient is represented as

$$\frac{\partial B_z}{\partial x} = B_0 b_1\left[\frac{1}{2}\omega_1^2 x + \frac{1}{16}\omega_1^4\left(x^3 + xy^2\right)\right]\cos(\omega_1 s)$$

$$+B_0\left[\left(\frac{2b_3}{r_0^2} - \frac{b_1}{4}\omega_1^2\right)(x\cos(\omega_1 s) - y\sin(\omega_1 s))\right.$$

$$+B_0\left(\frac{b_3\omega_1^2}{6r_0^2} - \frac{b_1\omega_1^4}{48}\right)\left(2x^3\cos(\omega_1 s) - (y^3 + 3yx^2)\sin(\omega_1 s)\right)\ ]$$

$$+B_0\left[\left(-\frac{b_3\omega_1^2}{12r_0^2} + \frac{4b_5}{r_0^4}\right)\left((x^3 - 3xy^2)\cos(\omega_1 s) + (y^3 - 3x^2 y)\sin(\omega_1 s)\right)\right]$$

(3.1))

or it can be rearranged as

$$\frac{1}{B_0}\frac{\partial B_z}{\partial x} = \left[\left(\frac{2b_3}{r_0^2} + \frac{1}{4}b_1\omega_1^2\right)x + \left(\frac{b_1\omega_1^4}{48} + \frac{b_3\omega_1^2}{4r_0^2} + \frac{4b_5}{r_0^4}\right)x^3\right.$$

$$\left. + \left(\frac{b_1\omega_1^4}{16} + \frac{b_3\omega_1^2}{4r_0^2} - \frac{12b_5}{r_0^4}\right)xy^2\right]\cos(\omega_1 s)$$

$$+\left[-\left(\frac{2b_3}{r_0^2} - \frac{1}{4}b_1\omega_1^2\right)y + \left(\frac{b_1\omega_1^4}{48} - \frac{b_3\omega_1^2}{4r_0^2} + \frac{4b_5}{r_0^4}\right)y^3\right.$$

$$\left. + \left(\frac{b_1\omega_1^4}{16} - \frac{b_3\omega_1^2}{4r_0^2} - \frac{12b_5}{r_0^4}\right)x^2 y\right]\sin(\omega_1 s) \qquad (3.2)$$

In the limit of straight magnet, it becomes as

$$\frac{\partial B_z}{\partial x} = B_0\left[\frac{2b_3}{r_0^2}x + \frac{4b_5}{r_0^4}\left(x^3 - 3xy^2\right)\right] \tag{3.3}$$

which is in agreement with 2 dimensional magnet case.

If we select the linear term only, the field gradient is given as

$$\frac{1}{B_0}\frac{\partial B_z}{\partial x} = \left(\frac{2b_3}{r_0{}^2} + \frac{1}{4}b_1\omega_1{}^2\right)x\cos(\omega_1 s) + \left(-\frac{2b_3}{r_0{}^2} + \frac{1}{4}b_1\omega_1{}^2\right)y\sin(\omega_1 s) \tag{3.4}$$

Similarily the second derivative is given as follows.

$$\begin{aligned}\frac{1}{B_0}\frac{\partial^2 B_z}{\partial x^2} = &\left[\left(\frac{2b_3}{r_0^2} + \frac{1}{4}b_1\omega_1^2\right) + \left(\frac{b_1\omega_1^4}{16} + \frac{3b_3\omega_1^2}{4r_0^2} + \frac{12b_5}{r_0^4}\right)x^2\right.\\ &\left.+\left(\frac{b_1\omega_1^4}{16} + \frac{b_3\omega_1^2}{4r_0^2} - \frac{12b_5}{r_0^4}\right)y^2\right]\cos(\omega_1 s)\\ &+\left[\frac{b_1\omega_1^4}{8} - \frac{b_3\omega_1^2}{2r_0^2} - \frac{24b_5}{r_0^4}\right]xy\sin(\omega_1 s)\end{aligned} \tag{3.5}$$

which is again in agreement with the 2 D magnet case as follows.

$$\lim_{\omega_1\to 0}\frac{\partial^2 B_z}{\partial x^2} = B_0\left[\frac{2b_3}{r_0^2} + \frac{12b_5}{r_0^4}\left(x^2 - y^2\right)\right] \tag{3.6}$$

The linear term is given as

$$\frac{1}{B_0}\frac{\partial^2 B_z}{\partial x^2} = \left(\frac{2b_3}{r_0{}^2} + \frac{1}{4}b_1\omega_1{}^2\right)\cos(\omega_1 s) \tag{3.7}$$

The betatron tune shift $\Delta\nu$ due to the field gradient of helical coil is given by

$$\Delta\nu = \frac{1}{4\pi}\int_0^{\lambda}\frac{\beta(s)}{B\rho}\frac{\partial B_z}{\partial x}ds \tag{3.8}$$

where $\beta(s)$ means the betatron function and the field gradient is given in equation (3.4).
The closed orbit in the helical coil is given by

$$x = x_0 + x_0' s - \frac{B_0}{B\rho}\frac{1}{\omega_1^2}(1-\cos(\omega_1 s))$$
$$y = y_0 + \frac{B_0}{B\rho}\frac{1}{\omega_1^2}\sin(\omega_1 s) + (y_0' - \frac{B_0}{B\rho}\frac{1}{\omega_1})s \tag{3.9}$$

where x_0, x_0', y_0, y_0' mean the initial values at the entrance point of the helical coils. Inserting these equations in (3.4) and (3.8) and performing the integration in one helical coil gives the following result

$$\Delta \nu = \frac{1}{4}\frac{\beta_{av}}{B\rho}\frac{B_0^2}{B\rho\omega_1}(b_1 - \frac{4b_3}{\omega_1^2 r_0^2}) \tag{3.10}$$

Here the betatron function in the helical coil is assumed to be constant as β_{av}.
The magnetic field strengths of 4 helical coils are selected as

$$|B_3| = |B_2|, \quad |B_4| = |B_1| \tag{3.11}$$

then the final results of the tune shift is

$$\Delta \nu = -\frac{2\beta_{av}}{(B\rho)^2}\frac{1}{\omega_1^3 r_0^2}(b_3 - \frac{b_1\omega_1^2 r_0^2}{4})(B_1^2 + B_2^2) \tag{3.12}$$

which is different from the formula by M. Syphers.[4] The difference is originated from the sextupole component due to the helical structure.
The local chromaticity is given by

$$\Delta \xi = \frac{1}{4\pi}\int \frac{\beta(s)}{B\rho} D(s) \frac{\partial^2 B_z}{\partial x^2} ds \tag{3.13}$$

where $D(s)$ shows the dispersion function at the helical magnet.
The second derivative of vertical magnetic field in the helical coil is independent on the beam orbit as in the equation (3.7), then if we assume the betatron function and dispersion function in the helical coil are assumed to be constant, the local chromaticity is almost zero.

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