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In linac ring colliders like MeRHIC and eRHIC a modulation of the electron bunch can lead to a modulation of the beam beam tune shift and steering errors [1–3]. These modulations can lead to emittance growth. This note presents simple formulas to estimate these effects which generalize some previous results.

I. THEORY OF QUADRUPOLE GROWTH

Consider one dimensional motion in the x direction with bare tune Q. We use the evolution variable θ which updates by 2π each turn. The interaction point is at $\theta = 0$, the modulating beam beam force is denoted by $\epsilon(\theta)$ and the hamiltonian is given by

$$H(x,p,\theta) = \frac{p^2+Q^2x^2}{2} + \delta_p(\theta)\frac{Qx^2}{2}\epsilon(\theta), \tag{1}$$

where the periodic delta function is

$$\delta_p(\theta) = \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k).$$

The equations of motion are $dx/d\theta = \partial H/\partial p$ and $dp/d\theta = -\partial H/\partial x$. Do a canonical transformation to action angle variables with $x = \sqrt{2J/Q}\sin\psi$ and $p = \sqrt{2JQ}\cos\psi$ yielding

$$H(\psi,J,\theta) = QJ + \delta_p(\theta)J\sin^2\psi\epsilon(\theta). \tag{2}$$

Averaging over ψ and ϵ with $<\delta_p(\theta)>= 1/2\pi$ one finds $<H(J)>= QJ + J<\epsilon>/4\pi$ so the average beam beam tune shift is $<\delta Q>=<\epsilon>/4\pi$. We redefine the tune to be $Q \to Q+<\delta Q>$ and introduce the fluctuating part of the noise. $\delta\epsilon = \epsilon - <\epsilon>$. The hamiltonian becomes

$$H(\psi,J,\theta) = QJ + \delta_p(\theta)J\sin^2\psi\delta\epsilon(\theta). \tag{3}$$

The emittance growth is due to the modulation in $\delta\epsilon$ and the equations of motion are

$$\frac{dJ}{d\theta} = -\frac{\partial H}{\partial\psi} = -\delta_p(\theta)J\sin(2\psi)\delta\epsilon(\theta)$$

and

$$\frac{d\psi}{d\theta} = \frac{\partial H}{\partial J} = Q + \delta_p(\theta)\sin^2\psi\delta\epsilon(\theta).$$

The expansion parameter is $\delta\epsilon$. To zeroeth order $\psi_0(\theta) = Q\theta + \hat{\psi}$. To first order in $\delta\epsilon$

$$\psi_1(\theta) = Q\theta + \hat{\psi} + \int_0^\theta \delta_p(s)\sin^2(Qs+\hat{\psi})\delta\epsilon(s)ds \tag{4}$$

$$\approx Q\theta + \hat{\psi} - \frac{1}{2}\int_0^\theta \delta_p(s)\cos(2Qs+2\hat{\psi})\delta\epsilon(s)ds. \tag{5}$$

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The evolution of the action is given by

$$\frac{d\ln J}{d\theta} = -\delta_p(\theta)\sin(2\psi)\delta\epsilon(\theta) \tag{6}$$

$$\approx -\delta_p(\theta)\sin(2\psi_1(\theta))\delta\epsilon(\theta) \tag{7}$$

$$\approx -\delta_p(\theta)\sin(2Q\theta+2\hat{\psi})\delta\epsilon(\theta) + \delta_p(\theta)\cos(2Q\theta+2\hat{\psi})\delta\epsilon(\theta)\int_0^\theta \delta_p(s)\cos(2Qs+2\hat{\psi})\delta\epsilon(s)ds \tag{8}$$

$$\approx -\delta_p(\theta)\sin(2Q\theta+2\hat{\psi})\delta\epsilon(\theta) + \frac{1}{2}\frac{d}{d\theta}\left[\int_0^\theta \delta_p(s)\cos(2Qs+2\hat{\psi})\delta\epsilon(s)ds\right]^2. \tag{9}$$

Integrating equation (9) one finds $J(\theta) = J(0)\exp(u(\theta))$ where

$$u(\theta) = -\int_0^\theta \delta_p(s)\sin(2Qs+2\hat{\psi})\delta\epsilon(s) + \frac{1}{2}\left[\int_0^\theta \delta_p(s)\cos(2Qs+2\hat{\psi})\delta\epsilon(s)ds\right]^2.$$

As long as u is small compared to one, which is necessary for eq (8), the first term on the right hand side dominates. Hence u is effectively the sum of a large number of uncorrelated random variables and therefore nearly a gaussian random variable. Notice that

$$< u(\theta) >= \left\langle \frac{1}{2}\left[\int_0^\theta \delta_p(s)\cos(2Qs+2\hat{\psi})\delta\epsilon(s)ds\right]^2 \right\rangle \approx \sigma_u^2/2$$

Now since $< \exp(u) >= \exp(< u > +\sigma_u^2/2)$ for a gaussian, we have $< J(\theta) >= J(0)\exp(2 < u(\theta) >)$ and the problem is to evaluate $< u(\theta) >$. Start by using the periodic delta function to turn the integral to a sum and define $\delta\epsilon(2\pi k) = \delta\epsilon_k$ so that

$$< 2u_n > = \sum_{k=0}^{n}\sum_{m=0}^{n}\cos(4\pi Qk+2\hat{\psi})\cos(4\pi Qm+2\hat{\psi}) < \delta\epsilon_k\delta\epsilon_m > \tag{10}$$

$$\approx \sum_{k=0}^{n}\sum_{m=0}^{n}\frac{1}{2}\cos(4\pi Q[k-m]) < \delta\epsilon_k\delta\epsilon_m > \tag{11}$$

$$\approx n\sum_{m=-\infty}^{\infty}\frac{1}{2}\cos(4\pi Qm) < \delta\epsilon_k\delta\epsilon_{m+k} > \tag{12}$$

where $\delta\epsilon_k$ is assumed to be stationary noise. As a simple model take $< \delta\epsilon_k\delta\epsilon_{m+k} >= \sigma_\epsilon^2\exp(-\alpha|m|)$. In this case the sum can be done resulting in

$$\ln\left\langle\frac{J(\theta)}{J(0)}\right\rangle =< 2u_n >= n\frac{\sigma_\epsilon^2}{2}\frac{1-e^{-2\alpha}}{1+e^{-2\alpha}-2\cos(4\pi Q)e^{-\alpha}} \tag{13}$$

II. COMPARISON WITH SIMULATIONS

For simulation purposes I take a one turn matrix followed by a thin lens beam beam kick,

$$x_n(k+1) = \cos\psi_n x_n(k) + \sin\psi_n p_n(k) \tag{14}$$

$$p_n(k+1) = \cos\psi_n p_n(k) - \sin\psi_n x_n(k) + \epsilon_k x_n(k+1) \tag{15}$$

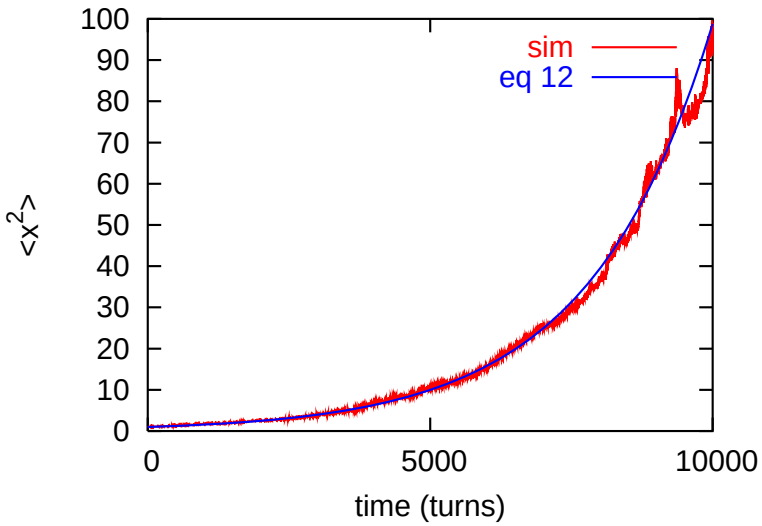

FIG. 1: Comparison of simulation and equation 12. The correlation suppression factor is 0.06.

where k is the time like variable and each of the n particles has a slightly different phase advance to create filamentation. The beam beam kick ϵ_k is normalized as above. Comparison of a simulation and equation (13) is shown in Fig 1. The correlation time of the noise was $\alpha = 0.1$ and $Q = 0.2$ resulting in a growth rate only 0.06 of what it would be for white noise of the same amplitude. As one can see from the figure the agreement is very good over many e-folding times. Of course in a real beam the electron kick is not purely linear so the growth would be reduced. In any case the formula seems reliable for an e-folding time.

III. GROWTH DUE TO STEERING ERRORS

Along with variations in electron beam intensity and emittance there can also be steering errors. We start by writing the one turn map in the linear approximation

$$x(k+1) = \cos\psi x(k) + \sin\psi x'(k)\beta^* \tag{16}$$
$$x'(k+1) = \cos\psi x'(k) - \sin\psi x(k)/\beta^* - \epsilon[x(k+1) - y(k+1)]/\beta^* \tag{17}$$

where $y(k)$ is the electron beam offset, β^* is the beta function at the crossing point, and the beam beam tune shift is $\epsilon/4\pi$ as before. We neglect the beam beam tune shift and define $z(k) = x(k) + i\beta^* x'(k)$ then

$$z(k+1) = e^{-i\psi} z(k) + i\epsilon y(k+1). \tag{18}$$

Set $U(k) = z(k)\exp(ik\psi)$ yielding

$$U(k+1) = U(k) + i\epsilon y(k+1)\exp(i(k+1)\psi), \tag{19}$$

so that

$$U(k) = U_0 + i\epsilon \sum_{m=0}^{k} y(m)\exp(im\psi). \tag{20}$$

Taking averages

$$< |U(k)|^2 > = |U_0|^2 + \epsilon^2 \sum_m \sum_\ell < y(m)y(\ell) > \exp(i(m-\ell)\psi) \tag{21}$$
$$\approx |U_0|^2 + k\epsilon^2 \sum_{m=-\infty}^{\infty} < y(m+\ell)y(\ell) > \cos(m\psi) \tag{22}$$

Taking $< y(m+\ell)y(\ell) >= \sigma_y^2 \exp(-\alpha|m|)$ one finds

$$2\epsilon_{rms}\beta^* =< |U(k)|^2 >= |U_0|^2 + k(\epsilon\sigma_y)^2 \frac{1 - e^{-2\alpha}}{1 + e^{-2\alpha} - 2\cos(2\pi Q)e^{-\alpha}}, \tag{23}$$

where $\epsilon_{rms} =< x^2 > /\beta^*$ is the rms geometric emittance of the ion beam.

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[1] C. Montag, RHIC/AP/143, 2004 also C. Montag NIM A, Volume 581, p581, 2007
[2] See Eduard Pozdeyev's MeRHIC discussion September 7, 2006.
[3] Y. Hao, V. Litvinenko, C. Montag, E. Pozdeyev, V. Ptitsyn, PAC07, p1865 (2007).