BNL-227925-2025-TECH

CA/AP/722

Force Between Two Moving, Non-Accelerating Charges

S. Brooks

October 2024

Collider Accelerator Department

Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC), Nuclear Physics (NP)

Force Between Two Moving, Non-Accelerating Charges

Stephen Brooks

October 28, 2024

1 Non-Moving Charges

The electric potential and field from a stationary charge q_1 placed at the origin are

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}|} \qquad \text{and} \qquad \mathbf{E}(\mathbf{x}) = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}|^3}\mathbf{x}.$$

The force on another charge q_2 at position $\mathbf{x}$ is then

$$\mathbf{F}_2 = q_2(\mathbf{E} + \mathbf{v}_2 \times \mathbf{B}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}|^3}\mathbf{x},$$

where $\mathbf{B} = \mathbf{0}$ because the non-moving charge q_1 produces no magnetic field.

2 Transforming the Four-Potential

The electromagnetic four-potential is $A^\mu = (\frac{1}{c}V, \mathbf{A})$, which transforms as a four-vector under Lorentz transformations. Let primes ($'$) denote the rest frame of charge q_1. In this frame,

$$A'^\mu(x'^\mu) = \left(\frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|}, \mathbf{0}\right) = \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|}(1, \mathbf{0}).$$

If this charge is travelling at velocity $\mathbf{v}_1 = \boldsymbol{\beta}c$, the (inverse) Lorentz transformation is

$$x^\mu = \begin{bmatrix} \gamma & \gamma\boldsymbol{\beta}^T \\ \gamma\boldsymbol{\beta} & I + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T \end{bmatrix} x'^\mu,$$

where $\beta = |\boldsymbol{\beta}|$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. The same transformation gives $A^\mu(x^\mu)$ from $A'^\mu(x'^\mu)$, so

$$A^\mu(x^\mu) = \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|}(\gamma, \gamma\boldsymbol{\beta}).$$

The forward Lorentz transformation on $x^\mu = (ct, \mathbf{x})$ gives

$$x'^\mu = \left(\gamma ct - \gamma\boldsymbol{\beta}\cdot\mathbf{x}, -\gamma ct\boldsymbol{\beta} + \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}\right),$$

thus $\mathbf{x}' = -\gamma ct\boldsymbol{\beta} + \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}$.

3 Deriving the Fields

The field tensor is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where $\partial^\mu = (\frac{1}{c}\partial_t, -\nabla)$. The four-potential is a function of $\mathbf{x}'$, so first calculate the derivatives

$$\begin{aligned}
\partial^0 x'^i &= -\gamma\beta^i \\
\partial^j x'^i &= -\left(\delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right).
\end{aligned}$$

Next observe that

$$\partial^\mu|\mathbf{x}'| = \sum_i \frac{\partial|\mathbf{x}'|}{\partial x'^i}\partial^\mu x'^i = \sum_i \frac{x'^i}{|\mathbf{x}'|}\partial^\mu x'^i$$

and use the chain rule to get

$$\begin{aligned}
\partial^\mu A^\nu &= \frac{1}{4\pi\epsilon_0 c}\frac{-q_1}{|\mathbf{x}'|^2}(\gamma,\gamma\boldsymbol{\beta})^\nu \sum_i \frac{x'^i}{|\mathbf{x}'|}\partial^\mu x'^i \\
&= \frac{1}{4\pi\epsilon_0 c}\frac{-q_1}{|\mathbf{x}'|^3}(\gamma,\gamma\boldsymbol{\beta})^\nu \sum_i x'^i \left(-\gamma\beta^i, -\left(\mathbf{e}_i + \frac{\gamma-1}{\beta^2}\beta^i\boldsymbol{\beta}\right)\right)^\mu \\
&= \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|^3}(\gamma,\gamma\boldsymbol{\beta})^\nu \left(\gamma\mathbf{x}'\cdot\boldsymbol{\beta}, \mathbf{x}' + \frac{\gamma-1}{\beta^2}(\mathbf{x}'\cdot\boldsymbol{\beta})\boldsymbol{\beta}\right)^\mu.
\end{aligned}$$

The fields and forces on the particles will be evaluated at $t = 0$, so there are simpler expressions for

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}, \\
\mathbf{x}'\cdot\boldsymbol{\beta} &= \mathbf{x}\cdot\boldsymbol{\beta} + (\gamma-1)(\boldsymbol{\beta}\cdot\mathbf{x}) = \gamma\boldsymbol{\beta}\cdot\mathbf{x}, \\
\mathbf{x}' + \frac{\gamma-1}{\beta^2}(\mathbf{x}'\cdot\boldsymbol{\beta})\boldsymbol{\beta} &= \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} + \frac{\gamma-1}{\beta^2}(\gamma\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} \\
&= \mathbf{x} + \frac{(\gamma-1)(\gamma+1)}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} = \mathbf{x} + \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}.
\end{aligned}$$

This means at $t = 0$,

$$\partial^\mu A^\nu = \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|^3}(\gamma,\gamma\boldsymbol{\beta})^\nu(\gamma^2\boldsymbol{\beta}\cdot\mathbf{x}, \mathbf{x} + \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta})^\mu.$$

The electric field is

$$\frac{1}{c}E^i = F^{i0} = \partial^i A^0 - \partial^0 A^i$$

$$\begin{aligned}
\Rightarrow \quad \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}(\gamma(\mathbf{x} + \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}) - \gamma\boldsymbol{\beta}(\gamma^2\boldsymbol{\beta}\cdot\mathbf{x})) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}(\gamma\mathbf{x} + \gamma^3(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma^3(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\gamma\mathbf{x}.
\end{aligned}$$

The x component of the magnetic field is

$$B^x = F^{zy} = \partial^z A^y - \partial^y A^z$$

$$\begin{aligned}
\Rightarrow \quad B^x &= \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|^3}(\gamma\beta^y(z+\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x})\beta^z) - \gamma\beta^z(y+\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x})\beta^y)) \\
&= \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|^3}(\gamma\beta^y z - \gamma\beta^z y).
\end{aligned}$$

This is true cyclically $x \to y \to z \to x$, so

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0 c}\frac{q_1}{|\mathbf{x}'|^3}\gamma\boldsymbol{\beta}\times\mathbf{x}.$$

4 Evaluating the Force

Taking these results together, the force on charge q_2 at position $\mathbf{x}$, travelling at velocity $\mathbf{v}_2$ is

$$\begin{aligned}
\mathbf{F}_2 &= q_2(\mathbf{E}+\mathbf{v}_2\times\mathbf{B}) \\
&= \frac{1}{4\pi\epsilon_0 c}\frac{q_1 q_2}{|\mathbf{x}'|^3}(c\gamma_1\mathbf{x} + \mathbf{v}_2\times(\gamma_1\boldsymbol{\beta}_1\times\mathbf{x})) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}'|^3}\gamma_1(\mathbf{x}+\boldsymbol{\beta}_2\times(\boldsymbol{\beta}_1\times\mathbf{x})) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}'|^3}\gamma_1(\mathbf{x}+(\boldsymbol{\beta}_2\cdot\mathbf{x})\boldsymbol{\beta}_1 - (\boldsymbol{\beta}_2\cdot\boldsymbol{\beta}_1)\mathbf{x}),
\end{aligned}$$

where $\mathbf{x}' = \mathbf{x} + \frac{\gamma_1 - 1}{\beta_1^2}(\boldsymbol{\beta}_1\cdot\mathbf{x})\boldsymbol{\beta}_1$.

5 Check: using the E, B field transformation rules

In the rest frame of charge q_1,

$$\mathbf{E}'(\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\mathbf{x}'$$

and $\mathbf{B}' = \mathbf{0}$. The field transformation rules are

$$\begin{aligned}
\mathbf{E}_\parallel &= \mathbf{E}'_\parallel, \\
\mathbf{B}_\parallel &= \mathbf{B}'_\parallel, \\
\mathbf{E}_\perp &= \gamma(\mathbf{E}'_\perp - \mathbf{v}\times\mathbf{B}'), \\
\mathbf{B}_\perp &= \gamma\left(\mathbf{B}'_\perp + \frac{1}{c^2}\mathbf{v}\times\mathbf{E}'\right),
\end{aligned}$$

where $\parallel$ and $\perp$ denote the components parallel and perpendicular to $\mathbf{v}$ respectively.

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_\parallel + \mathbf{E}_\perp = \mathbf{E}'_\parallel + \gamma\mathbf{E}'_\perp \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\left(\frac{1}{\beta^2}(\mathbf{x}'\cdot\boldsymbol{\beta})\boldsymbol{\beta} + \gamma\left(\mathbf{x}' - \frac{1}{\beta^2}(\mathbf{x}'\cdot\boldsymbol{\beta})\boldsymbol{\beta}\right)\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\left(\gamma\mathbf{x}' - \frac{\gamma-1}{\beta^2}(\mathbf{x}'\cdot\boldsymbol{\beta})\boldsymbol{\beta}\right).
\end{aligned}$$

Using the formulae for $\mathbf{x}'$ in terms of $\mathbf{x}$ at $t=0$ from a previous section,

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\left(\gamma\left(\mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}\right) - \frac{\gamma-1}{\beta^2}(\gamma\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{x}'|^3}\gamma\mathbf{x}.
\end{aligned}$$

For the magnetic field,

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}_\| + \mathbf{B}_\perp = 0 + \gamma \frac{1}{c^2} \mathbf{v} \times \mathbf{E}' \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\mathbf{x}'|^3} \gamma \frac{1}{c^2} \mathbf{v} \times \mathbf{x}' \\
&= \frac{1}{4\pi\epsilon_0 c} \frac{q_1}{|\mathbf{x}'|^3} \gamma \boldsymbol{\beta} \times \mathbf{x},
\end{aligned}
$$

where we have used $\mathbf{v} \times \mathbf{x}' = \mathbf{v} \times \mathbf{x}$ since $\mathbf{v} \parallel \boldsymbol{\beta}$ and $\mathbf{x}'$ and $\mathbf{x}$ only differ by a multiple of $\boldsymbol{\beta}$.