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Maxwellian Fields for a Ring derived from Multipole End Fields

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1 Turning an End Field on its Side

A previous note derived the following expression for a magnetic field that is order n rotationally symmetric about the z axis:

$$\mathbf{B} = \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} ((n+j)\sin((n-1)\theta+\psi) + j\sin((n+1)\theta+\psi))f^{(2j)}(z) \\ ((n+j)\cos((n-1)\theta+\psi) - j\cos((n+1)\theta+\psi))f^{(2j)}(z) \\ r\sin(n\theta+\psi)f^{(2j+1)}(z) \end{bmatrix}.$$

Here, $r = \sqrt{x^2 + y^2}$,

$$C_{nj} = \prod_{i=1}^{j} \frac{-1}{(n+2i)^2 - n^2} = \prod_{i=1}^{j} \frac{-1}{4(n+i)i} = \frac{(-\frac{1}{4})^j n!}{(n+j)! j!}$$

and f(z) is proportional to the strength variation of the 2*n*-pole along the z axis. $\psi = 0$ gives normal orientation and $\psi = \frac{\pi}{2}$ the skew orientation.

The idea of this note is to reinterpret (x, y) as the plane of a ring with superperiodicity n, making z the 'vertical' or out-of-plane axis. The vertical B_z component is oscillatory but can be supplemented by a constant field in that direction, producing something similar to 'field flutter' in cyclotrons, including the alternating gradients that keep the beam stable. Normally it is desired that the z = 0 ring plane only has field perpendicular to it, meaning $B_x = B_y = 0$ there. This can be guaranteed by making f(z) an odd function, so that $f^{(2j)}(0) = 0$ for all j.

2 f with Order 1

The simplest case with f(z) an odd polynomial is f(z) = az. As f'' and all higher derivatives are zero, only the j = 0 term is nonzero. Note that $C_{n0} = 1$ and then

$$\mathbf{B} = r^{n-1} \begin{bmatrix} n \sin((n-1)\theta + \psi)az \\ n \cos((n-1)\theta + \psi)az \\ r \sin(n\theta + \psi)a \end{bmatrix}.$$

The peak strength of the dipole field on the midplane is ar^n and the peak gradient is anr^{n-1} .

This field model has the advantage of being very simple. It is also the field of a scaling FFA with field index k = n and a cell with sinusoidally-varying focussing (not a very efficient machine as the reverse bending nearly equals the bending). If more complex cells are desired, terms replacing n with 2n, 3n, etc. can add longitudinal harmonics to the periodic cell.

One disadvantage of this field model for larger beams is that it also has a sextupole $\frac{d^2 B_z}{dr^2}$ of peak strength $an(n-1)r^{n-2}$ that leads to nonlinear beam transport.

3 f with Order 3

The second simplest case is $f(z) = az + bz^3$. Now the j = 0, 1 terms are the only nonzero ones. $C_{n1} = \frac{-1}{4(n+1)}$ and

$$\mathbf{B} = r^{n-1} \begin{bmatrix} n \sin((n-1)\theta + \psi)(az + bz^3) \\ n \cos((n-1)\theta + \psi)(az + bz^3) \\ r \sin(n\theta + \psi)(a + 3bz^2) \end{bmatrix} \\ + \frac{-1}{4(n+1)} r^{n+1} \begin{bmatrix} ((n+1)\sin((n-1)\theta + \psi) + \sin((n+1)\theta + \psi))6bz \\ ((n+1)\cos((n-1)\theta + \psi) - \cos((n+1)\theta + \psi))6bz \\ r \sin(n\theta + \psi)6b \end{bmatrix}.$$

The peak strength of the dipole field on the midplane is now $ar^n - \frac{6b}{4(n+1)}r^{n+2}$. The peak gradient is $anr^{n-1} - \frac{3b(n+2)}{2(n+1)}r^{n+1}$ and the peak sextupole $\frac{d^2B_z}{dr^2}$ is $an(n-1)r^{n-2} - \frac{3b(n+2)}{2}r^n$. To make the sextupole zero at radius r = R, it is required that

$$an(n-1) = \frac{3b(n+2)}{2}R^2 \implies b = \frac{2n(n-1)}{3(n+2)R^2}a.$$

This makes the peak dipole field at radius r = R equal to

$$aR^{n} - \frac{6}{4(n+1)} \frac{2n(n-1)}{3(n+2)R^{2}} aR^{n+2} = \left(1 - \frac{n(n-1)}{(n+1)(n+2)}\right) aR^{n}$$

and the peak gradient at r = R equal to

$$anR^{n-1} - \frac{3(n+2)}{2(n+1)} \frac{2n(n-1)}{3(n+2)R^2} aR^{n+1} = \left(1 - \frac{n-1}{n+1}\right) anR^{n-1}.$$

4 General Case

Higher order polynomials may be useful for cancelling octupole and higher nonlinearities. For odd f(z), the field on the midplane is

$$B_z(z=0) = \sum_{j=0}^{\infty} C_{nj} r^{n+2j} \sin(n\theta + \psi) f^{(2j+1)}(0).$$

The peak i^{th} radial derivative field is then

$$B_{z,peak}^{(i)} = \sum_{j=0}^{\infty} C_{nj} \frac{(n+2j)!}{(n+2j-i)!} r^{n+2j-i} f^{(2j+1)}(0).$$

If $f(z) = \sum_{k=0}^{K} a_{2k+1} z^{2k+1}$ then $f^{(2j+1)}(0) = (2j+1)! a_{2j+1}$ for $j \leq K$ and zero otherwise, giving

$$B_{z,peak}^{(i)} = \sum_{j=0}^{K} C_{nj} \frac{(n+2j)!}{(n+2j-i)!} r^{n+2j-i} (2j+1)! a_{2j+1}.$$

Typically a radius r = R will be chosen at which several (K+1) of the $B_{z,peak}^{(i)}$ are specified (e.g. nonzero gradient i = 1, some higher multipoles zero). The values of $R^n a_{2j+1}$ can be obtained by solving the linear system

$$B_{z,peak}^{(i)} = \sum_{j=0}^{K} \frac{(-\frac{1}{4})^{j} n!}{(n+j)! j!} \frac{(n+2j)!}{(n+2j-i)!} R^{2j-i} (2j+1)! (R^{n}a_{2j+1}).$$

The factor \mathbb{R}^n can be too large for the computer precision, so is included in the coefficient value. The field can then be calculated with the rescaled formula

$$\mathbf{B} = \left(\frac{r}{R}\right)^n \sum_{j=0}^{\infty} C_{nj} r^{2j-1} \left[\begin{array}{c} ((n+j)\sin((n-1)\theta+\psi) + j\sin((n+1)\theta+\psi))R^n f^{(2j)}(z) \\ ((n+j)\cos((n-1)\theta+\psi) - j\cos((n+1)\theta+\psi))R^n f^{(2j)}(z) \\ r\sin(n\theta+\psi)R^n f^{(2j+1)}(z) \end{array} \right].$$