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Maxwellian fields for a ring derived from multipole end fields

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1 Turning an End Field on its Side

A previous note derived the following expression for a magnetic field that is order n rotationally symmetric about the z axis:

$$\mathbf{B} = \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} ((n+j)\sin((n-1)\theta+\psi) + j\sin((n+1)\theta+\psi))f^{(2j)}(z) \\ ((n+j)\cos((n-1)\theta+\psi) - j\cos((n+1)\theta+\psi))f^{(2j)}(z) \\ r\sin(n\theta+\psi)f^{(2j+1)}(z) \end{bmatrix}.$$

Here, $r = \sqrt{x^2+y^2}$,

$$C_{nj} = \prod_{i=1}^{j} \frac{-1}{(n+2i)^2 - n^2} = \prod_{i=1}^{j} \frac{-1}{4(n+i)i} = \frac{(-\frac{1}{4})^j n!}{(n+j)!j!}$$

and $f(z)$ is proportional to the strength variation of the $2n$-pole along the z axis. $\psi = 0$ gives normal orientation and $\psi = \frac{\pi}{2}$ the skew orientation.

The idea of this note is to reinterpret (x, y) as the plane of a ring with superperiodicity n, making z the 'vertical' or out-of-plane axis. The vertical B_z component is oscillatory but can be supplemented by a constant field in that direction, producing something similar to 'field flutter' in cyclotrons, including the alternating gradients that keep the beam stable. Normally it is desired that the $z = 0$ ring plane only has field perpendicular to it, meaning $B_x = B_y = 0$ there. This can be guaranteed by making $f(z)$ an odd function, so that $f^{(2j)}(0) = 0$ for all j.

2 f with Order 1

The simplest case with $f(z)$ an odd polynomial is $f(z) = az$. As f'' and all higher derivatives are zero, only the $j = 0$ term is nonzero. Note that $C_{n0} = 1$ and then

$$\mathbf{B} = r^{n-1} \begin{bmatrix} n\sin((n-1)\theta+\psi)az \\ n\cos((n-1)\theta+\psi)az \\ r\sin(n\theta+\psi)a \end{bmatrix}.$$

The peak strength of the dipole field on the midplane is ar^n and the peak gradient is anr^{n-1}.

This field model has the advantage of being very simple. It is also the field of a scaling FFA with field index $k = n$ and a cell with sinusoidally-varying focussing (not a very efficient machine as the reverse bending nearly equals the bending). If more complex cells are desired, terms replacing n with $2n$, $3n$, etc. can add longitudinal harmonics to the periodic cell.

One disadvantage of this field model for larger beams is that it also has a sextupole $\frac{\mathrm{d}^2 B_z}{\mathrm{d}r^2}$ of peak strength $an(n-1)r^{n-2}$ that leads to nonlinear beam transport.

3 f with Order 3

The second simplest case is $f(z) = az + bz^3$. Now the $j = 0, 1$ terms are the only nonzero ones. $C_{n1} = \frac{-1}{4(n+1)}$ and

$$\mathbf{B} = r^{n-1} \begin{bmatrix} n\sin((n-1)\theta+\psi)(az+bz^3) \\ n\cos((n-1)\theta+\psi)(az+bz^3) \\ r\sin(n\theta+\psi)(a+3bz^2) \end{bmatrix}$$

$$+\frac{-1}{4(n+1)}r^{n+1} \begin{bmatrix} ((n+1)\sin((n-1)\theta+\psi)+\sin((n+1)\theta+\psi))6bz \\ ((n+1)\cos((n-1)\theta+\psi)-\cos((n+1)\theta+\psi))6bz \\ r\sin(n\theta+\psi)6b \end{bmatrix}.$$

The peak strength of the dipole field on the midplane is now $ar^n - \frac{6b}{4(n+1)}r^{n+2}$. The peak gradient is $anr^{n-1} - \frac{3b(n+2)}{2(n+1)}r^{n+1}$ and the peak sextupole $\frac{\mathrm{d}^2 B_z}{\mathrm{d}r^2}$ is $an(n-1)r^{n-2} - \frac{3b(n+2)}{2}r^n$. To make the sextupole zero at radius $r = R$, it is required that

$$an(n-1) = \frac{3b(n+2)}{2}R^2 \qquad \Rightarrow \qquad b = \frac{2n(n-1)}{3(n+2)R^2}a.$$

This makes the peak dipole field at radius $r = R$ equal to

$$aR^n - \frac{6}{4(n+1)}\frac{2n(n-1)}{3(n+2)R^2}aR^{n+2} = \left(1 - \frac{n(n-1)}{(n+1)(n+2)}\right)aR^n$$

and the peak gradient at $r = R$ equal to

$$anR^{n-1} - \frac{3(n+2)}{2(n+1)}\frac{2n(n-1)}{3(n+2)R^2}aR^{n+1} = \left(1 - \frac{n-1}{n+1}\right)anR^{n-1}.$$

4 General Case

Higher order polynomials may be useful for cancelling octupole and higher nonlinearities. For odd $f(z)$, the field on the midplane is

$$B_z(z=0) = \sum_{j=0}^{\infty} C_{nj} r^{n+2j} \sin(n\theta+\psi) f^{(2j+1)}(0).$$

The peak i^{th} radial derivative field is then

$$B_{z,peak}^{(i)} = \sum_{j=0}^{\infty} C_{nj} \frac{(n+2j)!}{(n+2j-i)!} r^{n+2j-i} f^{(2j+1)}(0).$$

If $f(z) = \sum_{k=0}^{K} a_{2k+1} z^{2k+1}$ then $f^{(2j+1)}(0) = (2j+1)! a_{2j+1}$ for $j \leq K$ and zero otherwise, giving

$$B_{z,peak}^{(i)} = \sum_{j=0}^{K} C_{nj} \frac{(n+2j)!}{(n+2j-i)!} r^{n+2j-i} (2j+1)! a_{2j+1}.$$

Typically a radius $r = R$ will be chosen at which several $(K+1)$ of the $B^{(i)}_{z,peak}$ are specified (e.g. nonzero gradient $i = 1$, some higher multipoles zero). The values of $R^n a_{2j+1}$ can be obtained by solving the linear system

$$B^{(i)}_{z,peak} = \sum_{j=0}^{K} \frac{(-\frac{1}{4})^j n!}{(n+j)!j!} \frac{(n+2j)!}{(n+2j-i)!} R^{2j-i}(2j+1)!(R^n a_{2j+1}).$$

The factor R^n can be too large for the computer precision, so is included in the coefficient value. The field can then be calculated with the rescaled formula

$$\mathbf{B} = \left(\frac{r}{R}\right)^n \sum_{j=0}^{\infty} C_{nj} r^{2j-1} \begin{bmatrix} ((n+j)\sin((n-1)\theta+\psi) + j\sin((n+1)\theta+\psi))R^n f^{(2j)}(z) \\ ((n+j)\cos((n-1)\theta+\psi) - j\cos((n+1)\theta+\psi))R^n f^{(2j)}(z) \\ r\sin(n\theta+\psi)R^n f^{(2j+1)}(z) \end{bmatrix}.$$