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Radial Steering Displacement Method to Measure Dispersion

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Abstract

The measurement of the dispersion function in a transfer line plays the crucial role for the development of an accurate optics model for the beam tracking simulation. In this technical note, we describe an experimental method to measure the dispersion value at multi-wire location MW063 in the NASA Space Radiation Laboratory (NSRL) transfer line. Finally, we use our MADX model to calculate the dispersion function at the same multi-wire location. The ultimate goal of this measurement and calculation is to understand the right value of dispersion function and the beam size (emittance) at the entrance of NSRL beam line.

1 Motivation

The transverse beam size along a beam line is determined using the projected beam emittance and the Courant-Snyder parameters along with the dispersion function. Dispersion and the projected emittance values play the major role to determine the beam size. Unknown dispersion can lead to overestimated values for the emittance [1]. In this note, we calculate the dispersion function at multi-wire location MW063 in the NASA Space Radiation Laboratory (NSRL) transport line. The measurement of the dispersion function at the given multi-wire location will help to compare the measured dispersion value with the theoretically calculated dispersion value from MADX [2] model of the NSRL transfer line.

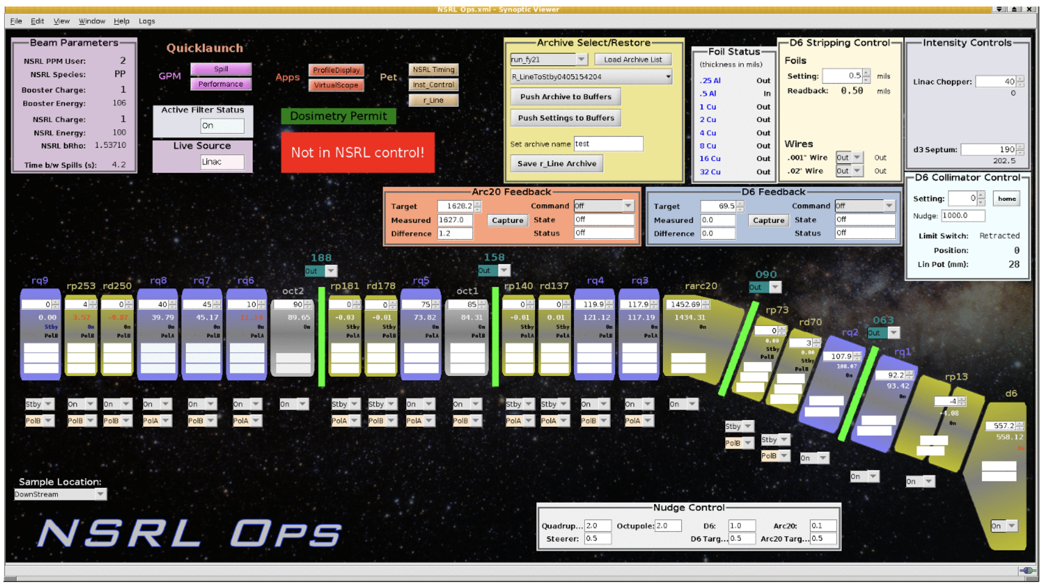

Figure 1: NSRL beam line with different magnets set up and multi-wire locations. Multi-wire MW063 is between quadrupoles magnets rq1 and rq2.

2 Dispersion

The standard formula to calculate the horizontal dispersion D_x is given by [3]

$$D_x = \frac{\Delta x}{(\Delta p/p_0)}, \tag{1}$$

where Δx is the measurement of the transverse beam displacement and $\Delta p/p_0$ is the momentum spread of the beam. Hence for the given transverse displacement, we can calculate the dispersion with stepping the beam momentum in a small range Δp for a given reference momentum value p_0. In the following subsections, we will discuss the horizontal dispersion value calculated from experiment using the radial steering displacement method in the booster and compare this value with the dispersion value given by beam line model using MADX.

2.1 Dispersion Calculation from the Measurement

Our goal is to measure the horizontal dispersion at multi-wire MW063 location in the NSRL beam line. Radial steering method [4] is used to run this experiment where we shift the radial steering function equally on both direction from zero in the booster ring. The details on the experimental procedures and scan via radial steering function adjustment is presented in Appendix B. Following Eq. (1), two main parameters we need to calculate from the experiment are: the transverse beam displacement Δx and momentum spread $\Delta p/p_0$. With the momentum change Δp in a small range, there will be transverse beam profile position change. For each horizontal beam profile data from the experiment, we can do Gaussian fit and calculate the mean position of the horizontal beam profile. For the corresponding change in the momentum value Δp, the corresponding change in the beam profile mean position change Δx can be calculated. With these two values, Dispersion can be calculated easily.

The momentum spread $\Delta p/p_0$ can be expressed as

$$\frac{\Delta p}{p_0} = \gamma_t^2 \frac{\Delta R}{R_0}. \tag{2}$$

The details on the derivation of the above equation is given in Appendix A. Further ΔR from the radial steering displacement method can be calculated using the following formula

$$\Delta R_{real} = \frac{(\gamma^2 - \gamma_{tr}^2(RF))}{(\gamma^2 - \gamma_{tr}^2(real))} \times \Delta R_{radial}. \tag{3}$$

Combining Eq. (2) and Eq. (3), $\Delta p/p_0$ can be calculated easily. The calculated horizontal dispersion values from the experimental data using radial steering displacement method is presented in Table 1.

We plot the dispersion values using error bars that shows the variability of dispersion values in our experiment. We use error propagation method [5] to calculate the error contributions to the measured dispersion values in our experiment. The details is presented in Appendix C.

Dispersion Calculation from Experiment					
RSD (mm)	ΔR (m)	X (m)	ΔX (m)	$\Delta p/p_0$	D_x (m)
-5	-0.00651	0.017348	0.014323	-0.003836	-3.73
-3	-0.00390	0.012045	0.00902	-0.002301	-3.91
-1	-0.00130	0.005760	0.00273	-0.000767	-3.55
+0	0.000	0.003024	0.000	0.00	
+3	-0.00390	-0.005935	-0.008959	0.002301	-3.86

RSD=Radial Steering Displacement, X = centroid of beam profile, ΔX = X(-5,-3,-1,+3)-X(+0).

Table 1: Dispersion calculation from the experimental data using radial steering function method.

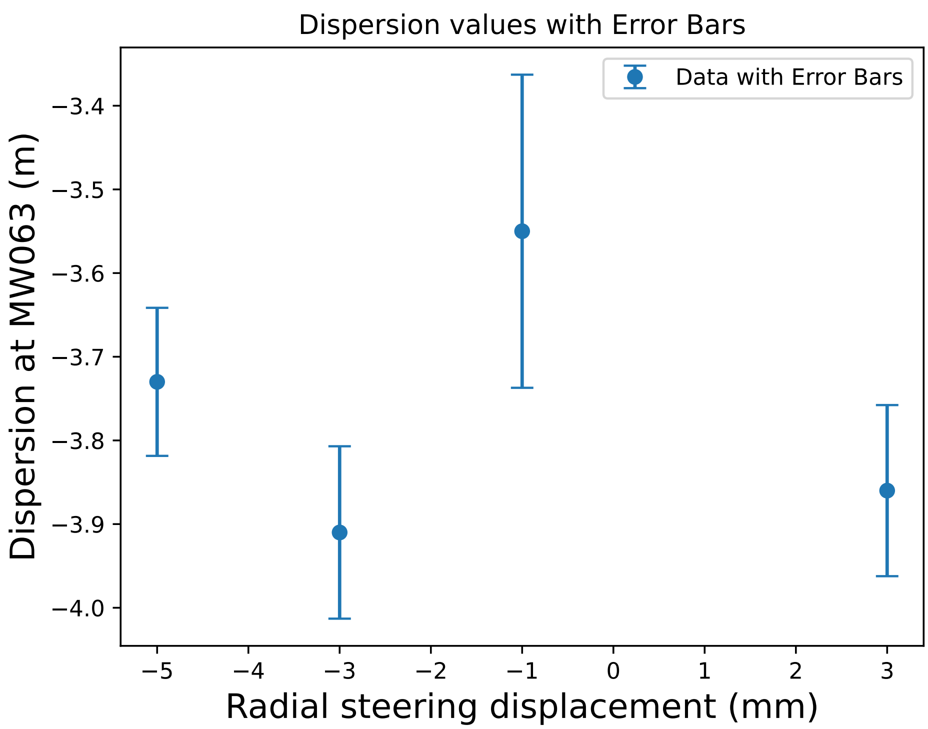

Figure 2: Dispersion values with error bars for the radial steering displacement method.

2.2 Dispersion Calculation from MADX Model

We use our MADX model of the NSRL beam line and calculate the dispersion at the same MW063 location. The plot of Courant-Snyder parameters and dispersion function starting at D3 septum magnet location to the multi-wire MW063 location is shown in Fig. 3. Initial Courant-Snyder parameter for

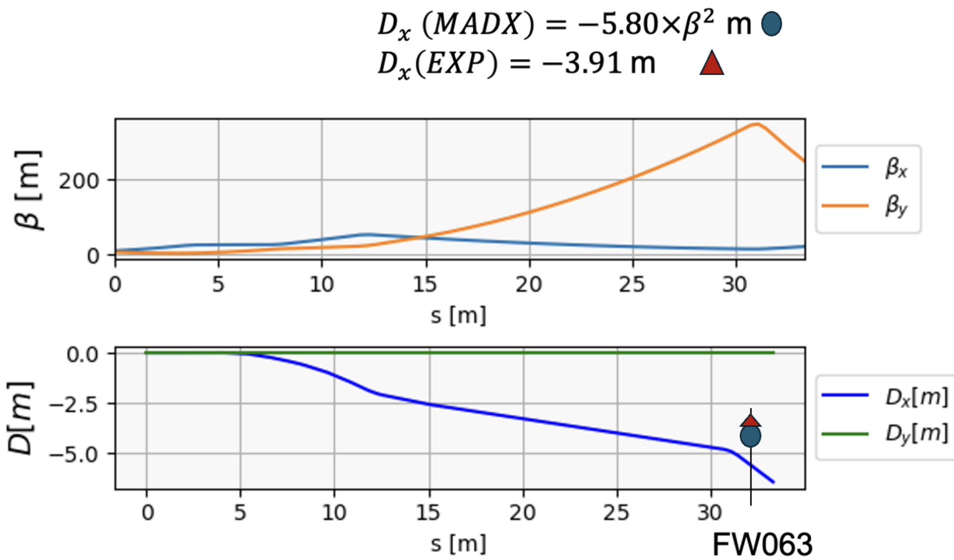

Figure 3: Courant-Snyder and dispersion function plot starting at D3 septum location in the booster to multi-wire MW063 location in the NSRL line.

the beam transport line starting at D3 septum location is chosen from the periodic condition in a booster ring. Then a new beam line is created in MADX starting at D3 septum location to the end of multi-wire location MW063, where we need to estimate the dispersion.

MADX calculates dispersion in terms of energy spread $\Delta E/E$, i.e. Dispersion $D_x(MADX) = \Delta X/(\Delta E/E)$. But the dispersion formula we have used in our calculation is $D_x = \Delta X/(\Delta p/p_0)$ and this is the right one. Hence we need a correction to the dispersion value given by MADX model. The relationship between $\Delta E/E$ and $\Delta p/p_0$ for the given energy is given by

$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p_0}, \tag{4}$$

where β = 0.87 for 1 GeV proton beam. The horizontal dispersion value calculated by MADX at MW063 location is -5.80 m (*you can see in Fig. 3*), and after using Eq. (4), horizontal dispersion value becomes $D_x(MADX) =$ -4.40 m (multiplying -5.80 m by β^2).

2.3 Dispersion Measurement using the Booster BPM Data

We want to calculate $\Delta p/p_0$ from the booster BPM data. The booster orbit data is recorded at the flattop for 5 different values of radial steering displacement (-5,-3,-1,+0,+1) mm, where +0 mm is the reference orbit. We obtained dispersion values for each booster BPM locations from the Booster MADX-lattice (horizontal tune value, Q_h=4.40). We calculate $\Delta p/p_0$ as an average over all booster BPM location as

$$\frac{\Delta p}{p_0} = \left\langle \frac{x_l - x_0}{D_l} \right\rangle, \tag{5}$$

where l is the location of BPM in the booster ring. $(x_l - x_0)$ is called the difference orbit and dispersion value D_l is obtained from MADX lattice file for the given set up at the corresponding BPM location. Figure 7 shows the booster horizontal orbit vs. BPM location around the ring, selected times sample the flattop where the radial steering is held constant and the extraction bumps turn on at 425ms. The BMP location are named as [A2,A6,A8,B4,B6,C2,C8,D2,D8,E2,E4,E6,E8,F2,F4,F8]. The corresponding dispersion values at those BPM locations from MADX are [1.880,3.657,1.894,3.056,3.490,2.025,1.982,2.148,2.309,1.959,2.927,3.121,2.104, 2.266,3.370,1.635] m. Finally, we calculate dispersion values at the multi-wire location MW063 in the NSRL beamline. Since the value of $\Delta p/p_o$ of the slowly extracted beam does not change much going towards MW063, once we get the values of $\Delta p/p_0$ from BPM measurement, we can easily calculate dispersion values as

$$D_{MW063} = \frac{\Delta x_{MW063}}{(\Delta p/p_0)_{BPM}}. \tag{6}$$

Table 2 has dispersion values calculated using the booster BPM data. The values of dispersion presented in Table 2 are markedly different than values presented in Table 1. This means the dispersion values calculated using the radial steering displacement method and using booster BPM data has significant difference. This difference in dispersion values can be caused by various errors associated with the experimental procedures and data processing.

Dispersion Calculation from Booster BPM data					
RSD (mm)	ΔR (m)	X (m)	ΔX (m)	$\Delta p/p_0$	D_x (m)
-5	-0.00651	0.017348	0.014323	-0.004258	-3.35
-3	-0.00390	0.012045	0.00902	-0.002889	-2.74
-1	-0.00130	0.005760	0.00273	-0.000993	-3.12
+0	0.000	0.003024	0.000	0.00	
+3	-0.00390	-0.005935	-0.008959	0.002654	-3.36

RSD=Radial Steering Displacement, X = centroid of beam profile, ΔX = X(-5,-3,-1,+3)-X(+0).

Table 2: Dispersion measurement using the booster BPM data.

3 Result and Discussion

The horizontal dispersion value given by MADX model is $D_x(MADX)$ = -4.40 m and the dispersion values from the experiment are -3.73 m, -3.91 m, -3.55 m, and -3.88 m respectively. The horizontal dispersion values from the model and experiment differ by -0.5 m ($D_x(MADX) - D_x(EXP)_{max} = -|4.40 - 3.91|$). The maximum measured dispersion value -3.91 m is about one standard deviation (1σ) away from the MADX model value -4.40 m. Dispersion values are negative since we take booster main magnet as a reference magnet and define positive dispersion in clockwise direction along the booster ring. From this experiment, we have a good understanding on how to improve the MADX model to better address the dispersion in the NSRL beam line.

Table 2 shows the dispersion values measured using the booster BPM data. $\Delta p/p_0$ values measured from booster BPM data differ from $\Delta p/p_0$ values measured from the radial steering displacement method giving the significant difference in the calculated dispersion values. In this document, we take dispersion values calculated from the radial steering displacement method and compare the dispersion value with the MADX model.

4 Future Work

Future work will focus on refining the MADX model to better align with the experimental dispersion measurements in the NSRL beam line. We plan to incorporate the insights gained from the radial steering displacement method to adjust model parameters and improve predictive accuracy. Further experiments with varied beam energies and lattice configurations will help determine whether the current discrepancy is consistent across different operational conditions.

5 Acknowledgment

The authors would like to thank Keith Zeno at Collider Accelerator Department at BNL for his suggestion on radial steering displacement calculation, and also thank Chuyu Liu for many useful discussion based on this experiment.

A Radial Displacement Calculation

The momentum spread of the beam $\Delta p/p_0$ can be calculated in many ways. The momentum spread makes the beam broader and it does not displace the beam. Once the transverse beam displacement is calculated for the given momentum spread, dispersion can be calculated easily [6]. Following the reference [7], we derive the formula of momentum spread in the booster synchrotron. The details on the transverse beam displacement using the radial steering function for the given momentum spread of the spills will be discussed in section 2.1.

Let us consider a circular orbit in a synchrotron with orbit radius R as a function of the particle momentum p and magnetic field B. This relationship can be expressed as

$$R = R(p, B). \tag{7}$$

For a synchronous particle with energy E_0 and momentum p_0 correspond to a given magnetic field B_0 to provide the given constant trajectory curvature ρ_0, the following relation holds:

$$p_0 = \rho_0 B_0. \tag{8}$$

For off-momentum particle, Eq. (8) becomes

$$p = \rho B. \tag{9}$$

From Eq. (8), you know the value of orbit radius R_0, whereas for the off-momentum particle, the value of orbit radius R is unknown. Let us try to find it.

If the magnetic field and particle momentum deviate from (p, B) by a small amount (dp, dB), then the change in the orbit radius differential becomes: $dR = R(p+dp, B+dB) - R(p, B)$. This expression can be expanded using Taylor expansion to the first order as

$$dR = \frac{\partial R}{\partial p} dp - \frac{\partial R}{\partial B} dB. \tag{10}$$

We use the minus sign in the above expression because the radius should decrease when the magnetic field increases. The momentum compaction factor in a ring can be defined as

$$\alpha_c = \frac{\Delta R/R_0}{\Delta p/p_0}. \tag{11}$$

The first coefficient in Eq. (10) is $\partial R/\partial p$. Using the definition of α_c, we can write the following expression

$$\frac{\partial R}{\partial p} = \alpha_c \frac{R}{p}. \tag{12}$$

The second coefficient in Eq. (10) can be expressed in the same way with the help of Eq. (9) as

$$\frac{\partial R}{\partial B} = \frac{\partial R}{\partial p}.\frac{\partial p}{\partial B} = \alpha_c \frac{R}{p}.\rho = \alpha_c \frac{R}{p}.\frac{p}{B} = \alpha_c \frac{R}{B}. \tag{13}$$

Substituting Eq. (12) and Eq. (13) in Eq. (10), we have

$$\frac{dR}{R} = \alpha_c \left(\frac{dp}{p} - \frac{dB}{B} \right) \tag{14}$$

Further, momentum compaction factor α_c is related with the transition energy γ_t and defined by [3]

$$\alpha_c = \frac{1}{\gamma_t^2}, \tag{15}$$

where γ_t is the transition energy for which the momentum compaction vanishes and this plays an important role in phase focusing. Hence, the momentum spread $\Delta p/p_0$ in Eq. (14) can be expressed as [8]

$$\frac{\Delta p}{p_0} = \gamma_t^2 \frac{dR}{R} + \frac{dB}{B}. \tag{16}$$

In our experiment, we flattened the booster main magnet (BMM) field at its flattop and the slope on BMM field using the spill servo constant starting at 500 ms. This condition modifies our Eq. (16) by eliminating the dB/B term,

$$\frac{\Delta p}{p_0} = \gamma_t^2 \frac{dR}{R_0}. \tag{17}$$

In a booster ring, $R_0 = L_0/2\pi = 32.11$ m, where $L_0 = 201.78$ m is the ring circumference and $\gamma_t = 4.4$ for the given set up of magnet currents for 1 GeV proton beam.

In a circular ring, time of travel τ is given by

$$\tau = \frac{L}{\beta c}, \tag{18}$$

where L is the ring circumference, β is the relativistic velocity and c is the speed of a light in a vacuum. Differentiating Eq.(18), we get

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta}. \tag{19}$$

Since $\Delta L/L = \alpha_c \Delta p/p_0$ and, $\Delta\beta/\beta = (1/\gamma^2)\ \Delta p/p_0$. Hence, Eq. (19) takes the form

$$\frac{\Delta\tau}{\tau} = \frac{\gamma^2 - \gamma_{tr}^2}{\gamma^2\gamma_{tr}^2}\frac{\Delta p}{p_0}. \tag{20}$$

Again, $\tau = 1/f$, where f is the revolution frequency. This gives, $\Delta\tau/\tau = -\Delta f/f$. Combining these relations with Eq. (17), Eq. (20), takes the form

$$\gamma^2\frac{\Delta f}{f} = -(\gamma^2 - \gamma_{tr}^2)\frac{\Delta R}{R_0}. \tag{21}$$

The RF system uses a value for γ_{tr}=4.832. This value is not correct for the NSRL setups, where horizontal tunes are kept below 4.4. Hence the radial steering function change by 1 mm does not correspond to radius change of 1 mm. There are no RF feedback loops on for NSRL cycle. Let us assume that the Booster main magnet is not ramping, that we already reached the flattop. The Booster RF system changes frequency (*df*) of the beam (the beam momentum), if we change the radial steering function ($\mathrm{dR_{radial}}$). RF system currently uses $\gamma_{tr}(RF) = 4.832$. Hence Eq. (21) to calculate ΔR_{radial} from the radial steering function can be expressed as

$$\gamma^2\frac{\Delta f}{f} = -(\gamma^2 - \gamma_{tr}^2(RF))\frac{\Delta R_{radial}}{R_0} \tag{22}$$

Similarly, the real radius change can be expresses as

$$\gamma^2\frac{\Delta f}{f} = -(\gamma^2 - \gamma_{tr}^2(real))\frac{\Delta R_{real}}{R_0}. \tag{23}$$

Now, comparing Eq. (22) and Eq. (23), the real radius change during the experiment can be expressed as [9]

$$\Delta R_{real} = \frac{(\gamma^2 - \gamma_{tr}^2(RF))}{(\gamma^2 - \gamma_{tr}^2(real))} \times \Delta R_{radial}. \tag{24}$$

As the radial function changes, it cause change in the magnetic field B and RF frequency. It eventually changes the beam energy. Hence γ value for each radial steering displacement may differ slightly from 2.066 for 1 GeV beam kinetic energy. We calculate the γ values for each measurements from *'bbrat'*. The pathway for this tool in the CAD control computer is: *StartUp/Specialtools/RFAps/bbrat*. We take the rf frequency at the time of beginning of flat-top and put the value in the tool gives us γ value for each radial steering measurements. The γ values for radial displacements -5 mm, -3 mm, -1 mm, 0 mm, and +3 mm are 2.0597,2.0615,2.0635,2.0645, and 2.0671 respectively.

Let us take an example, for radial steering displacement of 5 mm.
ΔR_{radial}= 5 mm
$\gamma_5 = 2.0597$
$\gamma_{tr}(real) = 4.35$
$\gamma_{tr}(RF)$= 4.832
$\Delta R_{real} = \Delta R_{radial} \times (\gamma_5^2 - \gamma_{tr}^2(RF))/(\gamma_5^2 - \gamma_{tr}^2(real))$
$\Delta R_{real} = 6.5$ mm.
It shows that the radial steering function change by 5 mm does not correspond to radius change of 5 mm, instead it is 6.5 mm.

B Experimental Procedure

B.1 Radial Steering Displacement Method

The following are the set up used in our experiment.

- We flattened (as much as we could) the Booster Main Magnet (BMM) field at its flattop which starts at 400 ms with the SpillServo zeroed.
- We set Radial steering function constant starting at 409 ms.
- We have extended the RF voltage till 500ms.
- We have sampled the Booster orbit between 410 ms - 500 ms (We have tried to flatten the horizontal as much as we could at 410 ms - 420 ms with the Radial Steering Function set to zero).
- We have set the extraction bumps to turn on at 425 ms.
- We made sure that the tune function is flat after 500ms, that is when drive sextupoles, spill servo, D3 septum turn on.

- There is no D6 foil.
- We put the slope on BMM field using the Spill Servo (constant) starting at 500 ms.

B.2 Scan via Radial Steering Function Adjustment

The only thing, that we change is the radial steering function after 409 ms (it is constant) in steps. We recorded data of the BMM field, Booster Horizontal Orbit, Revolution Frequency, Radial Steering, Radius Average, 302Spill.vs.time, and MW063 horizontal position. We did this scan for Radial Steering displacements equal to +3 mm, +0 mm, -1 mm, -3 mm, and -5 mm. +0 mm displacement is the reference radial steering measurement. The plots of different parameters are shown for radial steering displacement equal to -5 mm in the following figures.

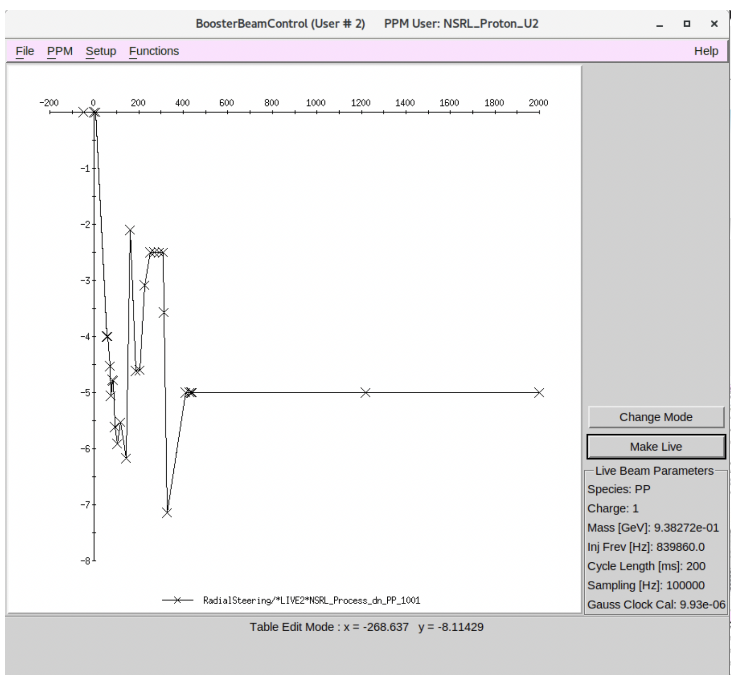

Figure 4: Radial steering function in the booster.

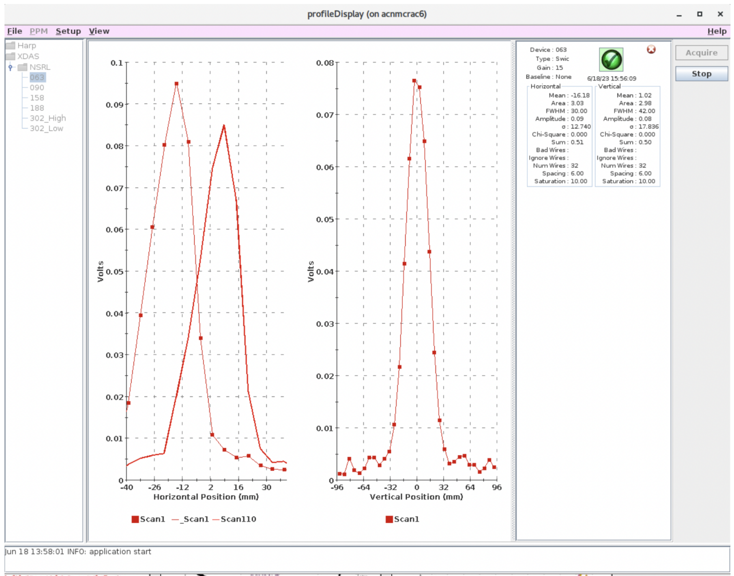

Figure 5: Horizontal and vertical beam profiles. The left figure shows overlapping of two horizontal profiles for two radial steering function adjustment showing beam position displacement. The right one is vertical beam profile.

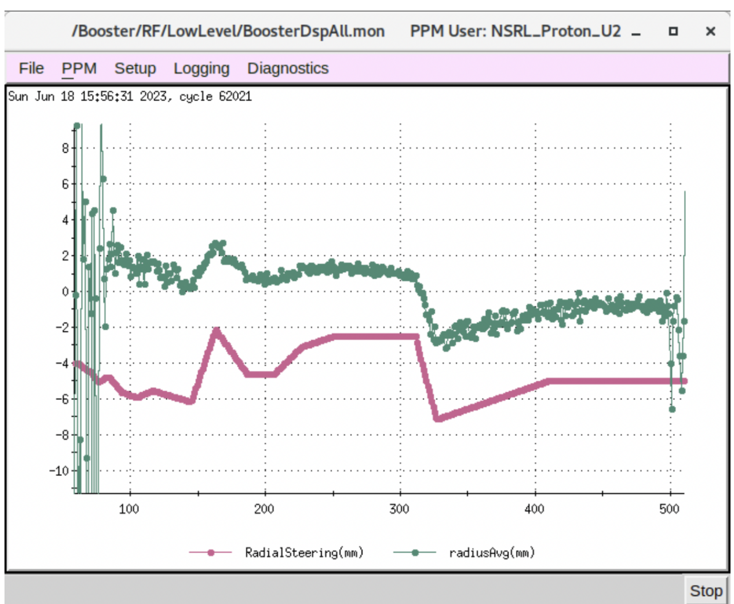

Figure 6: Radial steering and radius average in mm during the experiment.

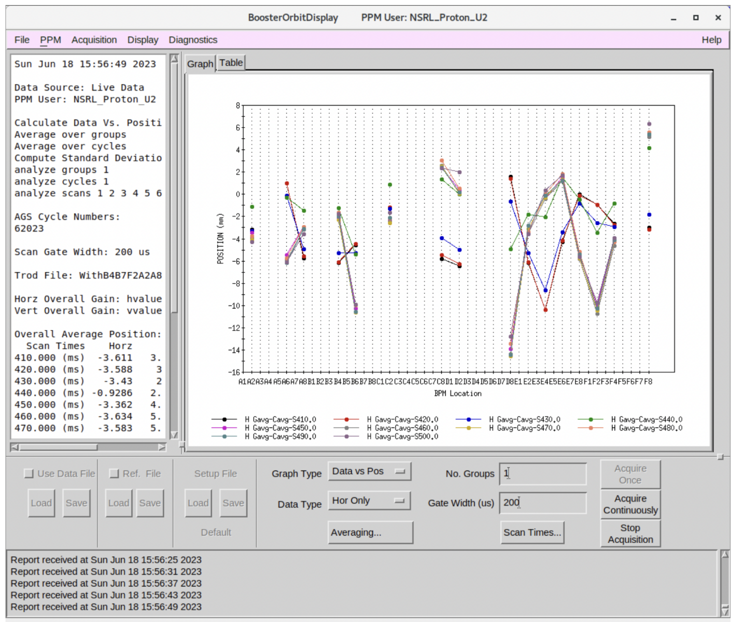

Figure 7: Example of Booster horizontal orbit vs. BPM location around the ring. Selected times sample the flattop; the radial steering is held constant; the extraction bumps turn on at 425 ms.

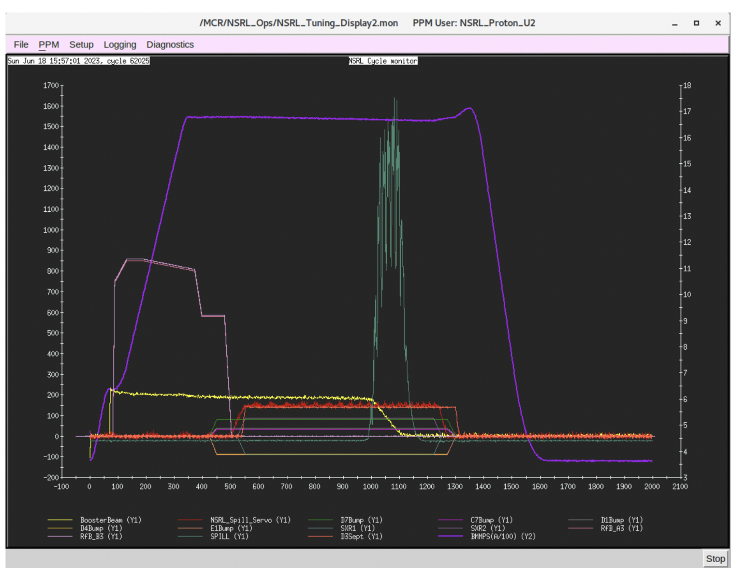

Figure 8: Booster monitor showing different beam parameters.

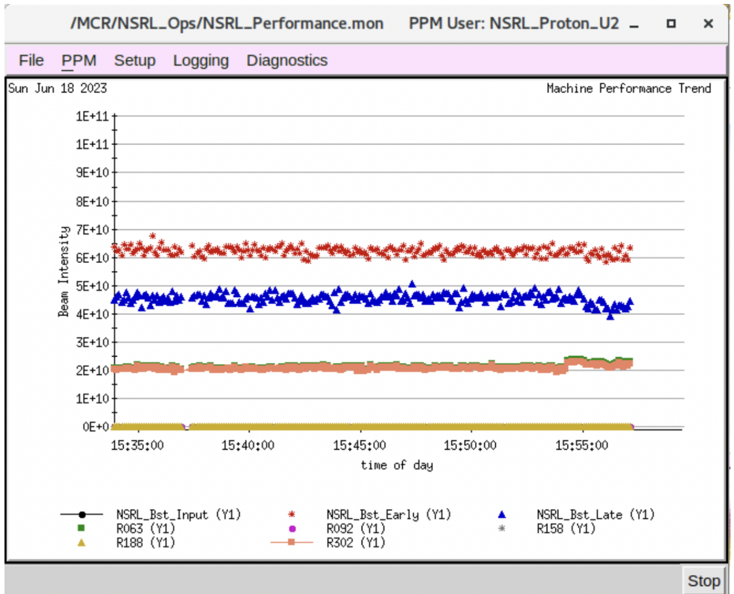

Figure 9: Beam intensity for the given time of day.

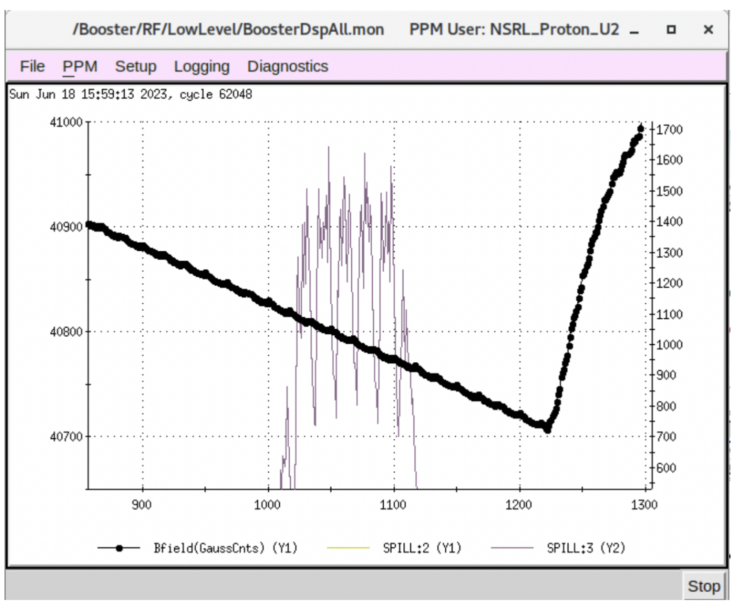

Figure 10: B field for the given spill servo function.

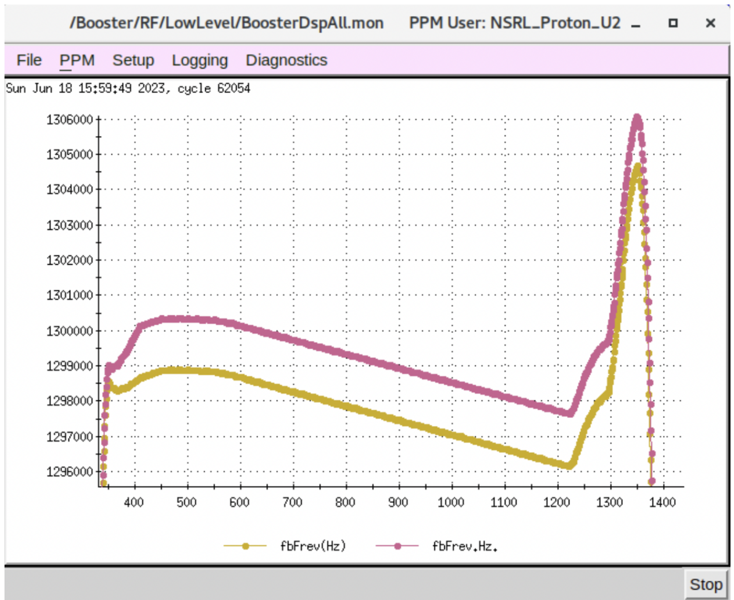

Figure 11: Revolution frequency in Hz. Two frequencies for the corresponding radial steering displacement are compared to show the frequency shift.

B.3 Dispersion Calculation from the Experimental Data

We take 1 GeV proton beam to run the experiment. During the radial steering function change, there is a change in the momentum of the spill structure. This variation is achieved by adding the RF system.

Following Eq. (17), we first calculate dp/p values for each case, where $(dp/p)_{real} = \gamma_{tr}^2(real)\Delta R_{real}/R_0$. Δx is the horizontal displacement of the beam centroid position which is measured at the multi-wire MW063 location for each radial steering displacements. We take 5 sets of data for radial steering displacements equal to -5mm, -3mm, -1mm, +0mm, and +3mm. We do Gaussian fitting of horizontal position of the beam profile for each radial steering displacement and the centroid value for each profile is calculated. ΔR_{real} is calculated using the formula given by Eq. (24).

C Error Bars on Dispersion Values

Define a function $z = f(x, y)$, the uncertainty in $z(\Delta z)$ can be calculated using the formula:

$$(\Delta z)^2 = \left(\frac{\partial f}{\partial x}\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\Delta y\right)^2. \tag{25}$$

Here, $\partial f/\partial x$ and $\partial f/\partial y$ are the partial derivative of f with respect to x and y, and Δx and Δy are the uncertainties in x and y, respectively. Following the same analysis, we calculate the uncertainty in dispersion value D_x in Eq. (1) as

$$(\Delta D_x)^2 = \left(\frac{\Delta(\Delta x)}{\Delta p/p_0}\right)^2 + \left(\frac{(\Delta x)}{(\Delta p/p_0)^2}\Delta(\Delta p/p_0)\right)^2. \tag{26}$$

Further, there exists some error in the measurement of $\Delta p/p_0$ given by Eq. (17). In this uncertainty in the measurement in $\Delta p/p_0$ is calculated in the following way using the standard error propagation method

$$\Delta\left(\frac{\Delta p}{p_0}\right)^2 = \left(2\gamma_t\ \frac{\Delta(\Delta R_{real})}{R_0}\ \Delta\gamma_t\right)^2 + \left(\gamma_t^2\ \frac{\Delta(\Delta R_{real})}{R_0}\right)^2. \tag{27}$$

Further we consider calculating uncertainty in ΔR calculation from the radial displacement measurement. ΔR is defined in Eq. (24). Uncertainty in R value depends on rf frequency and γ_t and can be calculated as

$$\frac{\Delta(\Delta R_{real})^2}{(\Delta R_{radial})^2} = \left(\frac{2\gamma\gamma_{tr}^2(rf)}{(\gamma^2 - \gamma_{tr}^2(real))^2}\Delta\gamma\right)^2 + \left(\frac{-2\gamma^2\gamma_{tr}(RF)}{(\gamma^2 - \gamma_{tr}^2(real))^2}\Delta\gamma_{tr}\right)^2 + \left(\frac{2\gamma\gamma_{tr}^2(real)}{(\gamma^2 - \gamma_{tr}^2(real))^2}\Delta\gamma_{tr}\right)^2. \tag{28}$$

Since $\Delta x = x - x_0$, where x is the centroid of the Gaussian beam profile and x_0 is the centroid of the reference Gaussian beam profile for +0 mm radial steering displacement. The uncertainty in Δx is calculated as

$$\Delta(\Delta x)^2 = (\Delta x)^2 + (\Delta x_0)^2. \tag{29}$$

Finally, we add up all these error propagation to calculate the uncertainty in each measured dispersion values. Our calculation shows that the uncertainty in the measured dispersion values are mainly caused by the error in the measurement of $\Delta p/p_0$. The plot of dispersion values with error bars for the given radial steering displacement is shown in Fig. 2.

All the data from the experiment and python script to calculate dispersion and error bars presented in this tech note can be found in the following BNL gitlab link:

`https://gitlab.pbn.bnl.gov/allusers/nsrl-beamline/-/tree/main/DispersionMW063?ref_type=heads`

References

[1] M Castellano, A Cianchi, and VA Verzilov. "Emittance and Dispersion Measurement at TTF". In: *Proc. of DIPAC*. 1999.

[2] H Grote and F Schmidt. "MAD-X-an upgrade from MAD8". In: *Proceedings of the 2003 Particle Accelerator Conference*. Vol. 5. IEEE. 2003, pp. 3497–3499.

[3] Helmut Wiedemann. *Particle accelerator physics*. Springer Nature, 2015.

[4] J Michael Brennan. *Rf beam control for the AGS Booster*. Tech. rep. Brookhaven National Lab., 1994.

[5] John Robert Taylor and William Thompson. *An introduction to error analysis: the study of uncertainties in physical measurements*. Vol. 2. Springer, 1982.

[6] N Tsoupas et al. *An online application to measue the dispersion function in AGS*. Tech. rep. Brookhaven National Lab.(BNL), Upton, NY (United States). Alternating ..., 2013.

[7] JM Kats. *Synchronous particle and bucket dynamics*. Tech. rep. Brookhaven National Lab.(BNL), Upton, NY (United States), 1988.

[8] Robert Gouiran et al. *A selection of formulae and data useful for the design of AG synchrotrons*. Tech. rep. CERN, 1970.

[9] Keith Zeno. *https://www.cadops.bnl.gov/People/zeno/wrkbk/radialex4.htm*. private communication.