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Resonance trajectories and island tune spectra in the tune domain

S. Peggs

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S. Peggs (BNL), T. Satogata (JLab)

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Abstract

Island tune spectra are introduced, beginning with curves of resonance trajectories plotted in the (Q_x, Q_I) tune domain plane (where Q_x is horizontal tune and Q_I is island tune) for many resonance orders N. A slice in the (Q_x, Q_I) plane delivers a Q_I-spectrum at a fixed horizontal tune, with one spectral line for every appropriate value of N.

Resonance trajectory plots represent unique fingerprints of simple or complex one-turn maps. They are presented – possibly for the first time in the literature – for single-sextupole and single-octupole maps, over large but limited ranges of Q_x. These two single-element maps have only one independent control parameter – horizontal tune Q_x.

Each spectral line represents a vulnerability to accidental magnet power supply ripple, and to intrinsic tune modulation at the synchrotron tune Q_s. Hadron accelerators like RHIC and the HSR that pass through transition see the synchrotron tune chirp from a maximum value down to zero and back up again, crossing many spectral lines, possibly with significant damage. The proximity of Q_s to a Q_I line during long-term storage potentially decreases the beam lifetime.

Contents

1 Introduction and nomenclature

In general a one-turn horizontal map advances the normalized phase space co-ordinates x and x' that are measured in a Poncaré surface of section according to

$$x(t) = a(t)\,\sin(\phi(t)) \tag{1}$$
$$x'(t) = a(t)\,\cos(\phi(t)) \tag{2}$$

where t is the integer turn number. In the limit of small amplitude $a \to 0$ the amplitude is constant

$$a(t) = a_0 \tag{3}$$

and the phase advances according to

$$\phi(t) = \phi_0 + 2\pi Q_x \tag{4}$$

where the base horizontal tune Q_x is usually taken to be an irrational number.

A chain of resonance islands can appear in phase space when nonlinearities such as sextupoles or octupoles are present. In this case a fixed point location (x_{fp}, x'_{fp}) maps *exactly* back onto itself after N turns, with

$$x(t) = a_{fp}\,\sin(\phi_{fp}) \tag{5}$$
$$x'(t) = a_{fp}\,\cos(\phi_{fp}) \tag{6}$$

for integer values

$$t = t_0, \quad t_0 + N, \quad t_0 + 2N, \quad ... \tag{7}$$

The N fixed points in a chain have amplitudes and phases

$$a_{fp,j} \approx a_{fp} \tag{8}$$
$$\phi_{fp,j} \approx \phi_{fp} + j\,(2\pi/N) \tag{9}$$

where the island index number j ranges from 1 to N. Each island has an approximately identical width Δa_{fp}, so that a trajectory at launched at phase ϕ_{fp} with an amplitude a_{init} in the range

$$a_{fp} - \Delta a_{fp} < a_{init} < a_{fp} + \Delta a_{fp} \tag{10}$$

is forever trapped in the island, with a (long term average) tune of exactly

$$Q = M/N \tag{11}$$

where M is also integer.

Trajectories launched close to a fixed point perform small linear oscillations with amplitude and phase modulation amplitudes Δa and $\Delta\phi$, where

$$a(t) \approx a_{fp} + \Delta a\,\sin(2\pi Q_I t) \tag{12}$$
$$\phi(t) \approx \phi_{fp} + \Delta\phi\,\cos(2\pi Q_I t) \tag{13}$$

The island Q_I is the focus of attention in this note.

Most discussions of resonances and resonance islands work in the phase space domain, for example focusing on how the real-space quantities a_{fp} and Δa_{fp} vary with control parameters such as nonlinear magnet strength. By contrast, this note works almost exclusively in the (Q_x, Q_I) tune domain. This perspective has advantages and disadvantages, just as the frequency domain has pros and cons in other more general applications. The principal advantage of the tune domain is that it is natural when considering AC dipole modulation and tune modulation, for example through power supply ripple at harmonics of the line frequency, or through chromatic tune modulation at the synchrotron tune Q_s.

2 Resonance trajectories

Consider a one-turn map with a single nonlinear magnet containing sextupole and octupole components:

```
C = cos(2*pi * Qx)
S = sin(2*pi * Qx)
for t in range(1,turn_max):
     x1  =  C*x + S*xp
     xp1 = -S*x + C*xp
     x = x1
     xp = xp1 - b2*x*x     # Sextupole kick
     xp = xp1 - b3*x*x*x   # Octupole  kick
```

Superficially this map has three control parameters – Q_x, b_2 and b_3. In the single-sextupole and single-octupole cases (when either b_3 or b_2 is zero) the absolute value of the non-zero b does not matter, since normalized phase space can be rescaled to a situation in which

$$(b_2, b_3) = (1, 0) \quad \text{or} \tag{14}$$
$$= (0, 1) \tag{15}$$

This scaling is discussed in [1]. After re-scaling phase space co-ordinates (x, x') there is only one independent control parameter in the single-sextupole and single-octupoles cases – the horizontal base tune Q_x.

The single-sextupole map is closely related to the Hénon map, while the single-octupole map is similar to the cubic Hénon map. Such Hénon maps continue to be explored more generally and fundamentally [2].

Multiple resonance island chains may be present for a particular value of Q_x, each with its own resonance order N, and each with its own fixed point amplitude $a_{fp,N}$ and island width $\Delta a_{fp,N}$. How do the island tunes $Q_{i,N}$ vary with Q_x?

2.1 Single-sextupole map

Figure 1 answers this question numerically for the single-sextupole case, over the base tune range

$$0.1 < Q_x < 0.25 \tag{16}$$

for resonances of order N from 5 to 17. For each particular pair of values of Q_x and N a test particle is launched within a resonance island, and tracked for one turn. The resonance island fixed point finder `fpFinder.py` is then used in concert with the single turn map (applied N times) to locate the island center, and to derive the island tune from the linear matrix describing small oscillations about the center [3, 4].

The island tune for $N = 5$ reaches a maximum of almost 0.1, while all resonances have maximum island tunes of 0.03 or more. These values are significantly larger than most synchrotron tunes in the hadron collider world, for example

$$Q_s \approx 0.002 \quad \text{RHIC} \tag{17}$$
$$\approx 0.01 \quad \text{HSR} \tag{18}$$

The situation in which these tunes are commensurate

$$Q_s \approx Q_I \tag{19}$$

is potentially dangerous, because then tune modulation driven through non-zero chromaticity destroys regular motion in the resonance islands [3, 5, 6, 7]. However, the size of the region of phase space that is destroyed is of order $\Delta a_{fp,N}$, which usually decreases as resonance order N increases.

Resonance trajectories for two values of N may cross in the tune domain, because their resonance island chains are usually at quite different amplitudes a_{fp}. Resonance trajectories cross in the tune domain without island chains crossing in the phase space domain.

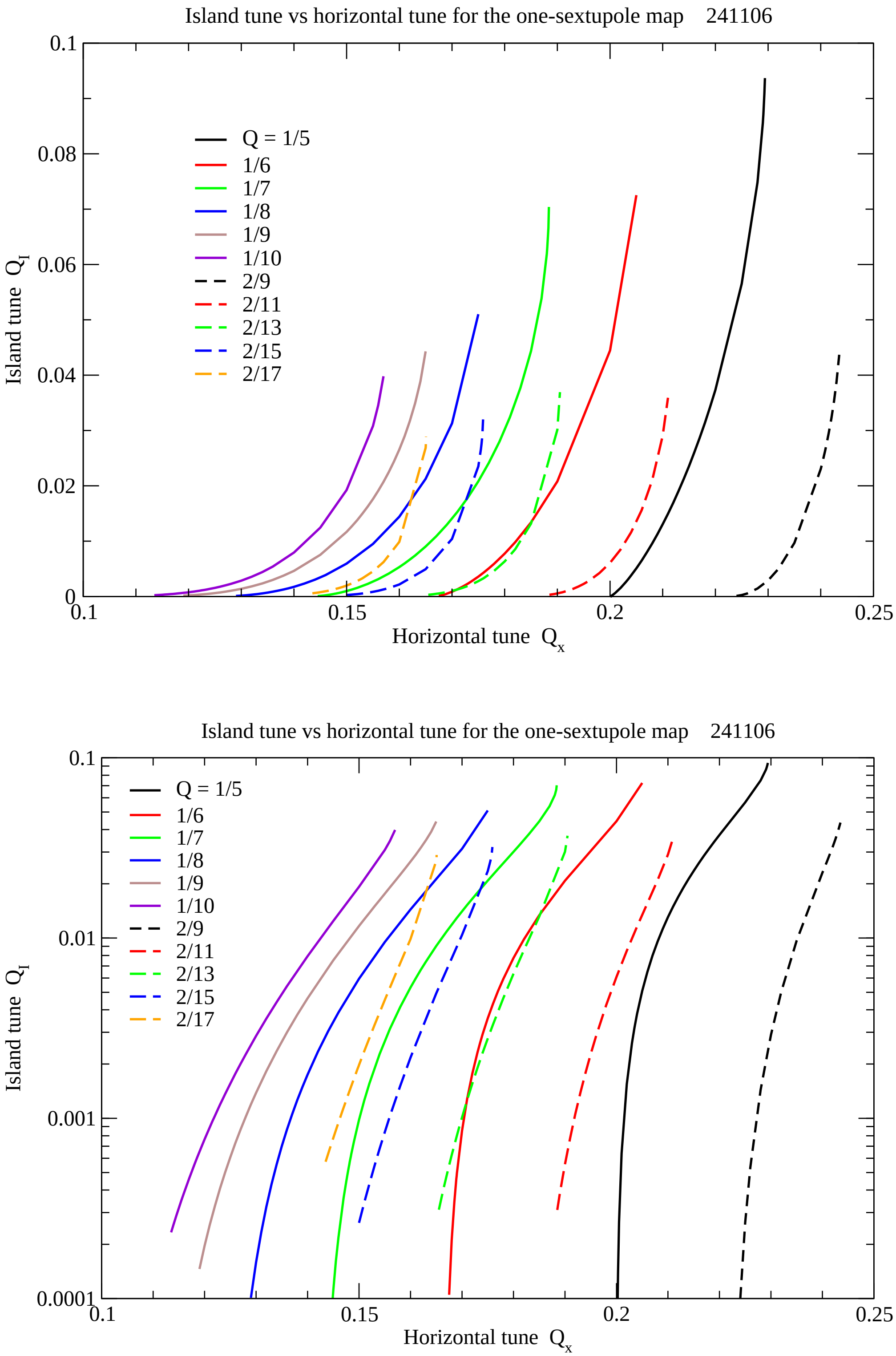

Figure 1: Resonance trajectories over a base tune Q_x range from 0.1 to 0.25 for a single-sextupole map, for the leading resonances of order N from 5 to 17. Island tunes almost as large as $Q_I = 0.1$ are observed. TOP: linear-linear. BOTTOM: log-linear.

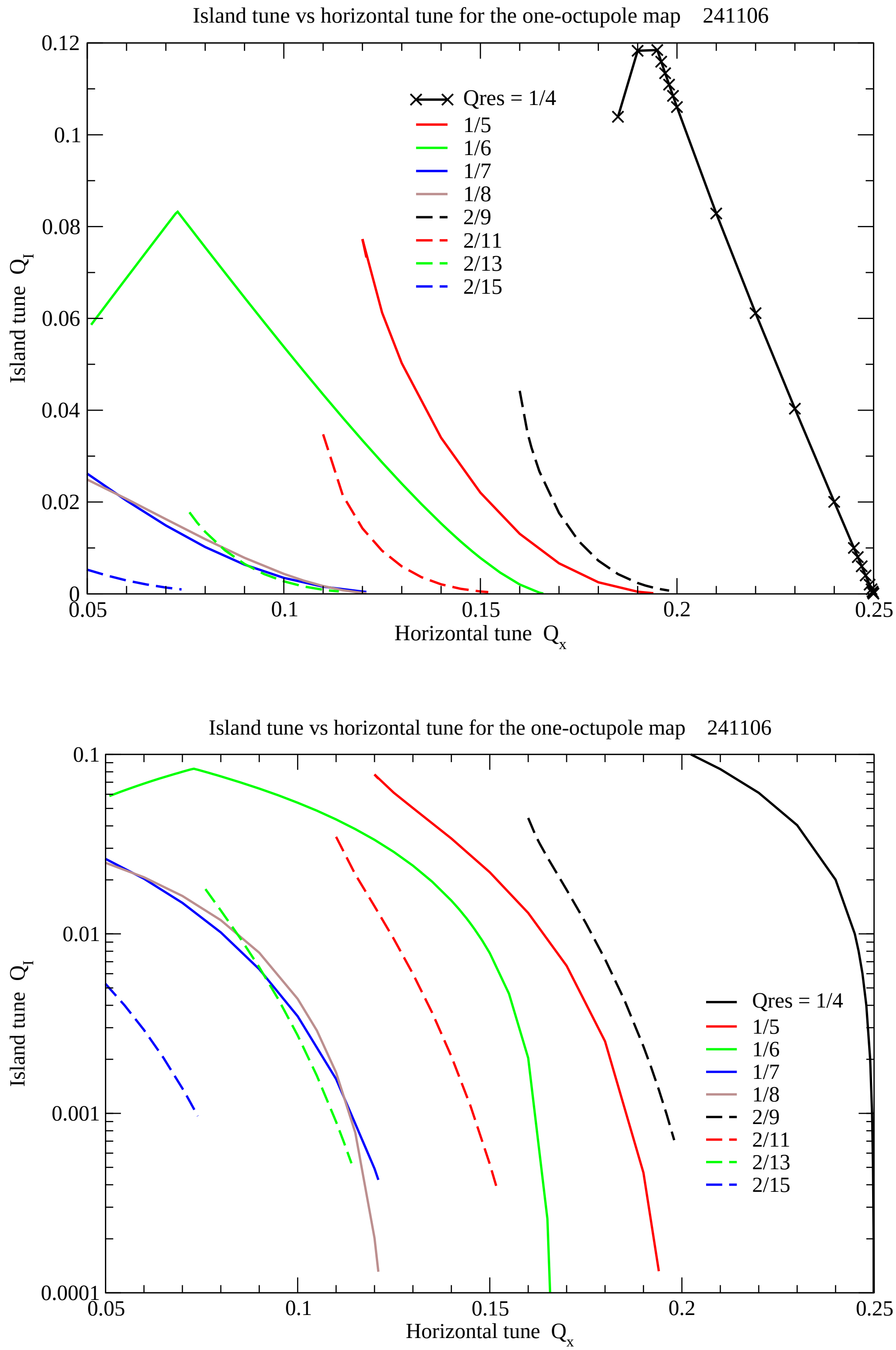

Figure 2: Resonance trajectories over a base tune Q_x range from 0.05 to 0.25 for a single-octupole map, for leading resonances of order N from 4 to 15. Island tunes as large as $Q_I = 0.12$ are observed. TOP: linear-linear. BOTTOM: log-linear.

2.2 Single-octupole map

Figure 2 shows resonance trajectories for the single-octupole case, over the tune range

$$0.05 \;<\; Q_x \;<\; 0.25 \tag{20}$$

for resonance orders from 4 to 15. The $N = 4$ curve is remarkable linear, with a largest island tune value of $Q_I = 0.12$. This linearity is consistent with first order perturbation theory [7], which predicts that

$$Q_I \;\sim\; |Q_x - 0.25| \tag{21}$$

Perhaps this strong linearity is maintained in realistic situations – for example when multipole octupole families are excited in RHIC. Of all the resonance trajectories plotted, only the $N = 4$ single-octupole trajectory is amenable to first order theory – not even $N = 3$ is amenable in the single-sextupole case.

The $N = 4$ and $N = 6$ cases are unusual in showing that Q_I turns over and decreases when Q_x becomes small enough. Turn over happens when resonance islands move out in amplitude to meet the dynamic aperture boundary, beyond which regular motion is not possible.

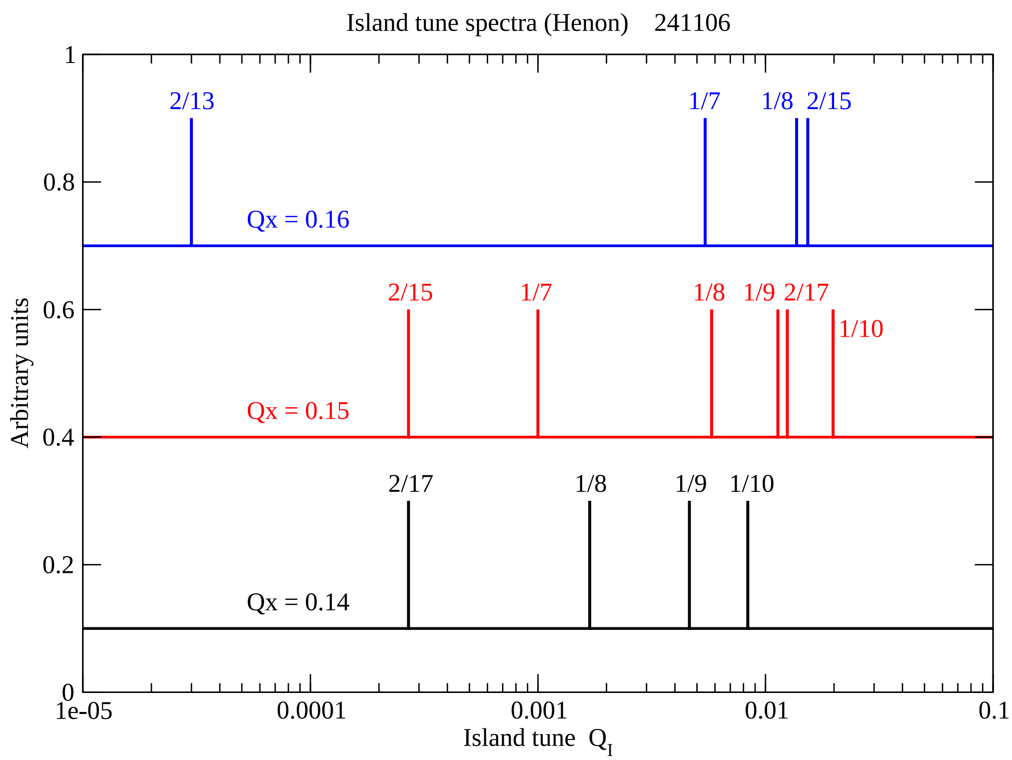

Figure 3: Island tune spectra under a single-sextupole map: BLACK $Q_x = 0.14$, RED $Q_x = 0.15$, BLUE: $Q_x = 0.16$. The height of each spectral line is identical, and does not reflect the strength of the resonance.

3 Implications

3.1 Island tune spectra

A spectrum of island tunes exists for each particular base tune, as shown for three Q_x values under a single-sextupole map in Figure 3. Phase space dynamics are vulnerable to magnet power supply ripple at frequencies

$$f_{modulation} \;=\; f_{\rm rev}\, Q_I \tag{22}$$

for each spectral line [7]. For example, resonance island chains with tunes of about $Q_I = 0.0008$ are vulnerable to 60 Hz modulation sources in RHIC, which has a revolution frequency of $f_{rev} = 78$ kHZ.

It would be useful to associate the height of each spectral line with a measure of its importance. For example, the height could be scaled by the island width Δa_{fp} at that value of Q_x, or perhaps some other sensitivity factor chosen for it relevance to dipole or quadrupole modulations. Nonetheless, all spectral lines have the same height in Figure 3.

3.2 Transition crossing

Longitudinal oscillations are an important and intrinsic source of tune modulation, with a tune shift of

$$\Delta Q_x \;=\; \chi_x \, a_\delta \, \cos(2\pi Q_s t) \tag{23}$$

where χ is chromaticity and a_δ is the amplitude of the off-momentum parameter $\Delta p/p$. For example, this leads to a tune modulation amplitude of

$$\chi_x \, a_\delta \;=\; 3 \times 0.001 \;=\; 0.003 \tag{24}$$

using typical values for RHIC.

The synchrotron tune Q_s decreases to zero as transition is crossed in RHIC or the HSR, as illustrated in Figure 4 for one second. Far away from transition the synchrotron tune reaches a maximum value of 0.002 in RHIC, and about 0.01 in HSR. All resonances with island tune values smaller than these maxima are vulnerable to tune modulation, because Q_s is chirped across Q_I on its way down to zero. Then Q_s re-crosses Q_I again as transition is left behind. The HSR appears to be more vulnerable than RHIC, because of its relatively large maximum Q_s value – more, stronger, resonances are crossed around transition.

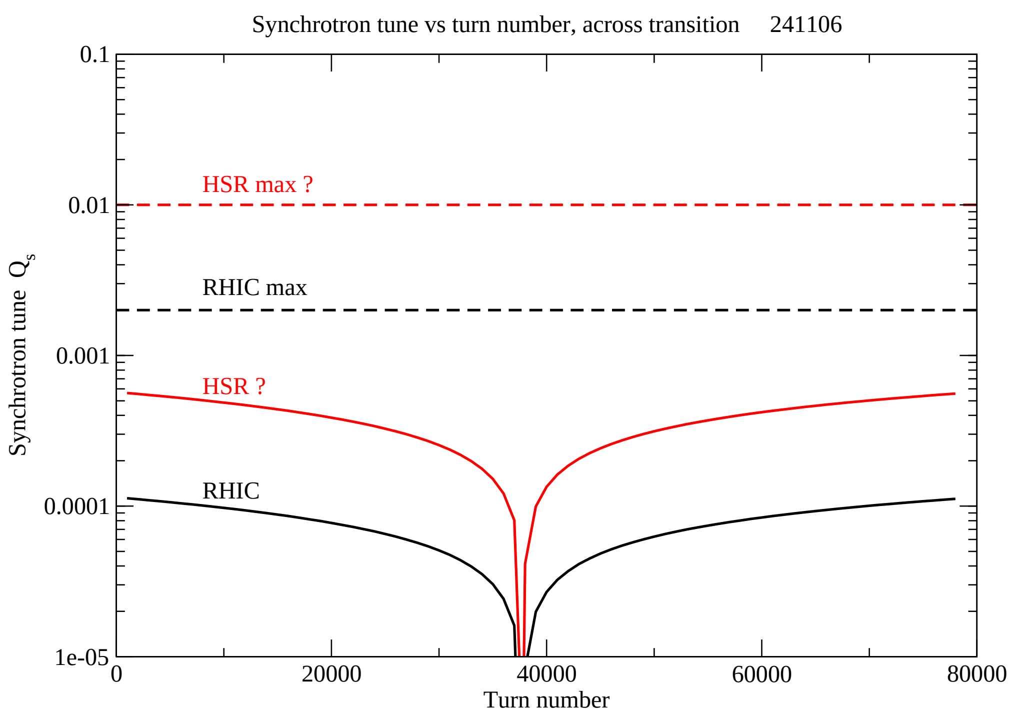

Figure 4: The synchrotron tune versus turn number for one second at transition in RHIC or the HSR.

3.3 Taking a fingerprint – deriving resonance trajectories for any map

A resonance trajectory plot exists in the (Q_x, Q_I) plane for complicated one-turn maps (such as RHIC and HSR), as well as for simple single-element maps. While the `fpFinder` software greatly simplifies the generation of plots like Figures 1 and 2, a higher level of automation is desirable for the more general case. Automation is challenging, however, because:

1. Many one-turn pushes need to be performed – compute-time may become an issue.
2. The initial guess of a location in (x, x') space has to be sufficiently close to the island center fixed point, in order to achieve convergence. This is more difficult for small island widths (for example at large values of N), even with human intervention.
3. A plan to integrate `fpFinder` (or its fundamental algorithm) with a standard tracking tool such as BMAD or Xtrack has not yet been presented.

4 Conclusions

A plot of resonance trajectories in the (Q_x, Q_I) tune domain is a unique fingerprint of a one-turn map – simple or complex – that is under study. At fixed horizontal tune a slice in the (Q_x, Q_I) plane delivers a Q_I-spectrum, with one spectral line for every value of resonance order N. Each spectral line represents a vulnerability to accidental magnet power supply ripple, and to intrinsic tune modulation at the synchrotron tune Q_s. Hadron accelerators like RHIC and the HSR that pass through transition see the synchrotron tune chirp from a maximum value down to zero and back up again, crossing many spectral lines, possibly with significant damage. The proximity of Q_s to a Q_I line during long-term storage potentially decreases the beam lifetime.

Taking a fingerprint is currently a semi-automated process, with challenges for more complete automation. Such automation is desirable, if the twin perspectives of resonance trajectories and Q_I-spectra are to find more general application.

Acknowledgements

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References

[1] See Exercises 9.6 and 9.8 in reference [6].

[2] S. Anastassiou. 2021. Complicated behavior in cubic Hénon maps. *Theoretical and Mathematical Physics,* **207***(2): 575-578.*

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[4] See appendix C in reference [7].

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