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Y. Kan

November 2024

Electron-Ion Collider  
**Brookhaven National Laboratory**

**U.S. Department of Energy**  
USDOE Office of Science (SC), Nuclear Physics (NP)

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# On the Electromagnetic Field of a Focusing Charged Particle Beam and Its Two-Dimensional Approximation

Yi-Kai Kan<sup>1,\*</sup> and Ji Qiang<sup>2</sup>

<sup>1</sup>Brookhaven National Laboratory, Upton, New York, USA

<sup>2</sup>Lawrence Berkeley National Laboratory, Berkeley, California, USA

## Abstract

The analytical expressions for the electromagnetic potential generated from a focusing charged particle beam are indispensable in various beam physics problems. In this article, we review the theory in detail and point out the necessary assumptions made in the derivation.

## 1. Introduction

The theoretical understanding of the electromagnetic potential generated from a charged particle beam is crucial to many studies, *e.g.*, space-charge and beam-beam interaction. While the physics problem can be fully described by the inhomogeneous electromagnetic wave equation, it is generally not easy to solve when a charge source with a complicated distribution is encountered. Therefore, the quasistatic approximation (model) is widely applied in the beam physics community; the electrostatic field of the particle beam is first solved in the beam's rest-frame, and the corresponding field in the lab-frame is then derived by using the Lorentz transformation. Under the quasistatic approximation, the analytical expression for the electromagnetic field of a rigid Gaussian bunch can be derived and can be found in many literature [1, 2, 3].

However, in the beam-beam studies, a focusing beam usually needs to be considered, which leads to some critical phenomena, like the longitudinal beam-beam effects [4, 5]. The expression of the electromagnetic potential of a focusing beam is usually given by replacing the constant transverse beam sizes in the three-dimensional (3D) potential of

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\*ykan@bnl.gov

a rigid beam with an  $s$ -dependent one [6, 7]. However, the legitimation of this generalization has not been rigorously discussed.

On the other hand, in the community of beam-beam studies, a two-dimensional (2D) formulation is often used for modeling the potential of a particle beam rather than solving the 3D inhomogeneous wave equation; a particle beam is chunked into thin slices along the longitudinal direction, and the field from each slice is calculated by solving the 2D Poisson equation. The 2D potential for a single slice in a Gaussian beam and its closed-form formula have been derived and applied in the literature [6, 8]. This 2D potential should be derivable from the 3D potential under certain approximations. While the parameters characterizing the 2D approximation were discussed in some literature [9, 10], how the 2D potential can be derived from the 3D potential was not rigorously demonstrated.

In this article, we try to provide a rigorous derivation of the 3D potential from a focusing beam and its 2D approximation. We first formalize the notion of the quasistatic approximation through a formulation based on the inhomogeneous wave equation. After that, based on the discussed quasistatic model, we derive the 3D potential of a Gaussian-distributed focusing beam and its 2D approximation.

## 2. The Quasistatic Model

The quasistatic model is a commonly used approximation to solve the electromagnetic field of a moving charged particle beam in the beam physics community. One common approach to derive the quasistatic model is based on the Lorentz transformation; the physics problem is first formulated as electrostatic in the beam's frame, and then the full set of governing equations is transformed into the lab frame [2]. Despite its popularity in the community, what the quasistatic model actually approximates was seldom discussed. Thus, we first try to formalize the idea of a quasistatic model based on the formulation originally appeared in [11] and provide the necessary theoretical reasoning. The tool developed in this section will be a foundation for the discussion in the next section.

The free-space inhomogeneous wave equations for the electric scalar potential  $\phi$  and the magnetic vector potential  $\mathbf{A}$  can be expressed as [12]

$$\nabla^2 \phi(x, y, s, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \phi(x, y, s, t) = -\frac{\rho(x, y, s, t)}{\varepsilon_0}, \quad (2.1)$$

$$\nabla^2 \mathbf{A}(x, y, s, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(x, y, s, t) = -\mu_0 \mathbf{J}(x, y, s, t), \quad (2.2)$$

where  $\varepsilon_0$  is the vacuum permittivity,  $\mu_0$  is the vacuum permeability, and  $c_0$  is the speed of light in vacuum. Here, the charge density  $\rho$  and the current density  $\mathbf{J}$  need to satisfy the continuity equation

$$\nabla \cdot \mathbf{J}(x, y, s, t) + \frac{\partial}{\partial t} \rho(x, y, s, t) = 0, \quad (2.3)$$

and the electromagnetic potentials need to satisfy the Lorenz gauge condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c_0^2} \frac{\partial \phi}{\partial t} = 0. \quad (2.4)$$

The quasistatic approximation relies on the following two assumptions:

1. Each particle moves almost with the same velocity  $v_0$ , and the charge and current densities for the particle beam can be modeled as

$$\rho(x, y, s, t) = \rho(x, y, s - v_0 t) \quad \text{and} \quad \mathbf{J}(x, y, s, t) = v_0 \rho(x, y, s - v_0 t) \hat{\mathbf{s}}.$$

2. The propagation of the electromagnetic field finishes instantaneously. In this case, given the charge and current densities of the forms  $\rho(x, y, s - v_0 t)$  and  $\mathbf{J}(x, y, s - v_0 t)$ , the corresponding  $\phi$  and  $\mathbf{A}$  satisfying Eq. (2.1) and Eq. (2.2) can be expressed in the forms  $\phi(x, y, s - v_0 t)$  and  $\mathbf{A}(x, y, s - v_0 t)$ . The proof is demonstrated in [Appendix A](#).

Applying the quasistatic approximation, Eq. (2.1) and Eq. (2.2) become

$$\nabla_{\perp}^2 \phi(x, y, z) - \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \phi(x, y, z) = -\frac{\rho(x, y, z)}{\epsilon_0}, \quad (2.5)$$

$$\nabla_{\perp}^2 A_s(x, y, z) - \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} A_s(x, y, z) = -\mu_0 v_0 \rho(x, y, z). \quad (2.6)$$

Here, the operator  $\nabla_{\perp} := \partial^2/\partial x^2 + \partial^2/\partial y^2$  denotes the two-dimensional Laplacian, and the new variable  $z := s - v_0 t$  stands for the relative position to a reference particle with a trajectory  $v_0 t$ . After some algebraic manipulations, Eq. (2.5) and Eq. (2.6) reduce to

$$\nabla_{\perp}^2 (A_s - \frac{v_0}{c_0^2} \phi) + \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} (A_s - \frac{v_0}{c_0^2} \phi) = 0 \quad \implies \quad A_s = \frac{v_0}{c_0^2} \phi. \quad (2.7)$$

This simple linear relation between  $\phi$  and  $\mathbf{A}$  suggests that under the quasistatic approximation, we only need to solve the wave equation for  $\phi$ , and both electric and magnetic fields can be directly derived from  $\phi$ .

### 3. The Quasistatic Field of a Focusing Charged Particle Beam

Before finding the quasistatic field of a focusing particle beam, we first need to discuss an analytical expression of its charge and current densities. The charge density of a focusing particle beam moving in a constant velocity  $v_0$  can be modeled as

$$\rho(x, y, s, t) = \frac{1}{(2\pi)^{3/2}} \frac{Ne}{\sigma_x(s)\sigma_y(s)\sigma_z} \exp\left(-\frac{x^2}{2\sigma_x^2(s)} - \frac{y^2}{2\sigma_y^2(s)} - \frac{(s - v_0 t)^2}{2\sigma_z^2}\right), \quad (3.1)$$

where  $N$  is the number of particles and  $e$  is the charge of each single particle. Here, the horizontal and vertical beam sizes  $\sigma_x$  and  $\sigma_y$  are defined as

$$\sigma_i(s) := \sigma_i^* \cdot \left(1 + \frac{s^2}{\beta_i^*}\right)^{1/2} \quad i \in \{x, y\}$$

with  $\sigma_i^*$  the transverse beam size and  $\beta_i^*$  the beta function at  $s = 0$ . Throughout this article, we will simply write  $\sigma_x$  and  $\sigma_y$  without specifying the function argument. It can be checked that Eq. (3.1) satisfies  $\iiint \rho(x, y, s, t) dx dy ds = Ne$  at a given time  $t$ . As each particle moves almost in the velocity  $v_0$ , it is reasonable to model the longitudinal current density as

$$J_s(x, y, s, t) = v_0 \rho(x, y, s, t). \quad (3.2)$$

Because the transverse beam sizes change with  $s$  during the propagation, some transverse current densities ( $J_x$  and  $J_y$ ) exist to account for the change in the charge distribution. However, in some scenarios, we may neglect these transverse current densities by claiming  $|J_x|, |J_y| \ll |J_s|$ . The strength of the current density is proportional to the particle's velocity; and particularly, the particle's transverse velocity is proportional to the rate of change of the beam sizes

$$|J_i| \approx v_0 \frac{d\sigma_i}{ds} \quad i \in \{x, y\}.$$

Thus, putting together with  $|J_s| \approx v_0$ , we have the following estimation

$$\left| \frac{J_i}{J_s} \right| \approx \frac{\sigma_i^*}{\beta_i^*} \frac{\frac{s}{\beta_i^*}}{\sqrt{1 + \left(\frac{s}{\beta_i^*}\right)^2}} \leq \frac{\sigma_i^*}{\beta_i^*} = \sqrt{\frac{\epsilon_i}{\beta_i^*}} \quad i \in \{x, y\},$$

where  $\epsilon_i$  is the transverse emittance of the particle beam. The ratio  $\sqrt{\epsilon_i \beta_i^*}$  is the beam divergence and should be small for most particle colliders. In fact,  $J_x$  and  $J_y$  can be explicitly solved by substituting Eq. (3.1) into the continuity equation (Eq. (2.3))

$$\begin{aligned} & \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_s}{\partial \sigma_x} \frac{\partial \sigma_x}{\partial s} + \frac{\partial J_s}{\partial \sigma_y} \frac{\partial \sigma_y}{\partial s} + \frac{\partial J_s}{\partial (s - v_0 t)} = v_0 \cancel{\frac{\partial \rho}{\partial t}} \\ \implies & \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \left( \frac{x^2}{\sigma_x^2} - 1 \right) \frac{1}{\sigma_x} \frac{\partial \sigma_x}{\partial s} J_s + \left( \frac{y^2}{\sigma_y^2} - 1 \right) \frac{1}{\sigma_y} \frac{\partial \sigma_y}{\partial s} J_s = 0. \end{aligned}$$

Hence, the expression for the transverse current densities can be written as

$$J_x = \frac{x}{\sigma_x} \frac{\partial \sigma_x}{\partial s} J_s \quad \text{and} \quad J_y = \frac{y}{\sigma_y} \frac{\partial \sigma_y}{\partial s} J_s.$$

When expressed in the coordinate  $(x, y, z, s)$  with  $z := s - v_0 t$ , because the charge density still depends explicitly on  $s$ , the corresponding electromagnetic potential needs to satisfy the wave equations below

$$\nabla_{\perp}^2 \phi(x, y, z, s) + \frac{\partial}{\partial s^2} \phi(x, y, z, s) - \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} \phi(x, y, z, s) = -\frac{\rho(x, y, z, s)}{\epsilon_0}, \quad (3.3)$$

$$\nabla_{\perp}^2 A_s(x, y, z, s) + \frac{\partial}{\partial s^2} A_s(x, y, z, s) - \frac{1}{\gamma^2} \frac{\partial^2}{\partial z^2} A_s(x, y, z, s) = -\mu_0 v_0 \rho(x, y, z, s). \quad (3.4)$$

By assuming  $\phi$  and  $A_s$  slowly varying in  $s$  (*i.e.*,  $\partial^2 \phi / \partial s^2 \approx 0$ ), the simple linear relation between  $\phi$  and  $A_s$  found in Eq. (2.7) can also be derived

$$A_s(x, y, z, s) = \frac{v_0}{c_0^2} \phi(x, y, z, s). \quad (3.5)$$

Therefore, it suffices just to solve Eq. (3.3), and the solution can be found by one of the methods used in [1, 3]

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{Ne}{(8\pi)^{1/2}} \int_0^\infty \frac{\exp\left(-\frac{x^2}{2(\sigma_x^2+q)} - \frac{y^2}{2(\sigma_y^2+q)} - \frac{z^2}{2(\sigma_z^2+q/\gamma^2)}\right)}{(\sigma_x^2+q)^{1/2}(\sigma_y^2+q)^{1/2}(\sigma_z^2+q/\gamma^2)^{1/2}} dq. \quad (3.6)$$

Eq. (3.6) is almost the same as the solution for a rigid bunch except that  $\sigma_x$  and  $\sigma_y$  are now functions of  $s$ . Applying Eq. (3.5), the calculation of the electric field can be simplified:

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \implies E_x = -\frac{\partial\phi}{\partial x}, E_y = -\frac{\partial\phi}{\partial y}, \text{ and } E_s = -\overbrace{\frac{\partial\phi}{\partial s}}^{=:E_{ss}} - \underbrace{\frac{1}{\gamma^2} \frac{\partial\phi}{\partial z}}_{=:E_{sz}}, \quad (3.7)$$

where the term  $E_{ss}$  is caused by the beam focusing during the propagation, and the term  $E_{sz}$  is due to the density variation of particles in  $z$ -coordinate.

#### 4. The 2D Approximation

To derive the 2D approximation of Eq. (3.6), we introduce a change of variable  $q := \min(\sigma_x, \sigma_y)^2 \cdot w/(1-w)$ . For an elliptical beam with  $\sigma_y \leq \sigma_x$ , we choose  $q = \sigma_y^2 \cdot w/(1-w)$ , and Eq. (3.6) becomes

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{Ne}{(8\pi)^{1/2}} \frac{\sigma_y}{\sigma_x\sigma_z} \times \int_0^1 \frac{\exp\left(-\frac{(1-w)x^2}{2\sigma_x^2(1-w+Aw)} - \frac{(1-w)y^2}{2\sigma_y^2} - \frac{(1-w)\gamma^2 z^2}{2\gamma^2\sigma_z^2(1-w+\epsilon w)}\right)}{\underbrace{(1-w)^{1/2}(1-w+Aw)^{1/2}(1-w+\epsilon w)^{1/2}}_{=: \psi_\epsilon(w)}} dw \quad (4.1)$$

with  $\epsilon := \sigma_y^2/(\gamma^2\sigma_z^2)$  and  $A := \sigma_y^2/\sigma_x^2$ . Now, we want to calculate the electric field  $E_x$ ,  $E_y$ ,  $E_{ss}$  and  $E_{sz}$  in the limit  $\epsilon \rightarrow 0$ . In the calculation for  $E_x$  and  $E_y$ , we can interchange the limit and integration

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} E_x &= \text{const} \cdot \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{\partial}{\partial x} \psi_\epsilon(w) dw = \text{const} \cdot \int_0^1 \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial x} \psi_\epsilon(w) dw, \\ \lim_{\epsilon \rightarrow 0} E_y &= \text{const} \cdot \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{\partial}{\partial y} \psi_\epsilon(w) dw = \text{const} \cdot \int_0^1 \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial y} \psi_\epsilon(w) dw, \end{aligned}$$

because  $|\partial\psi_\epsilon(w)/\partial x|$  and  $|\partial\psi_\epsilon(w)/\partial y|$  are smaller than some integrable functions and the dominated convergence theorem can be utilized [13]. Therefore, we can further

write out

$$\begin{aligned}\lim_{\epsilon \rightarrow 0} E_x &= \text{const} \cdot \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \frac{x}{\sigma_x^2} \int_0^1 \frac{\exp\left(-\frac{(1-w)x^2}{2\sigma_x^2(1-w+Aw)} - \frac{(1-w)y^2}{2\sigma_y^2}\right)}{(1-w+Aw)^{3/2}} dw, \\ \lim_{\epsilon \rightarrow 0} E_y &= \text{const} \cdot \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \frac{y}{\sigma_y^2} \int_0^1 \frac{\exp\left(-\frac{(1-w)x^2}{2\sigma_x^2(1-w+Aw)} - \frac{(1-w)y^2}{2\sigma_y^2}\right)}{(1-w+Aw)^{1/2}} dw.\end{aligned}\quad (4.2)$$

Using a substitution of  $w = q/(q + \sigma_y^2)$  and some algebraic manipulation, Eqs. (4.2) can be expressed in the form

$$E_x^{2D} := \lim_{\epsilon \rightarrow 0} E_x = -\frac{\partial \phi^{2D}}{\partial x} \quad \text{and} \quad E_y^{2D} := \lim_{\epsilon \rightarrow 0} E_y = -\frac{\partial \phi^{2D}}{\partial y}\quad (4.3)$$

Here, we define a 2D scalar potential

$$\phi^{2D} := \frac{1}{4\pi\epsilon_0} \frac{e \cdot \rho_{\parallel}(z)}{2} \int_0^{\infty} \frac{\exp\left(-\frac{x^2}{2(\sigma_x^2+q)} - \frac{y^2}{2(\sigma_y^2+q)}\right)}{(\sigma_x^2+q)^{1/2}(\sigma_y^2+q)^{1/2}} dq,\quad (4.4)$$

with

$$\rho_{\parallel}(z) := \frac{1}{(2\pi)^{1/2}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)\quad (4.5)$$

the normalized longitudinal particle density. Actually, Eq. (4.4) is the solution of the two-dimensional Poisson equation

$$\nabla_{\perp}^2 \phi(x, y, z) = -\frac{\rho(x, y, z)}{\epsilon_0},\quad (4.6)$$

which is a simplification of Eq. (2.5) by neglecting the term  $\frac{1}{\gamma^2} \frac{\partial^2 \phi}{\partial s^2}$  and is a theoretical foundation of some beam-beam studies, *e.g.*, the strong-strong simulation model [11]. For the calculation of  $\lim_{\epsilon \rightarrow 0} E_{ss}$ , we can first observe from Eq. (3.6) that [10]

$$E_{ss} = -\frac{\partial \phi}{\partial s} = -\frac{\partial(\sigma_x^2)}{\partial s} \frac{\partial \phi}{\partial(\sigma_x^2)} - \frac{\partial(\sigma_y^2)}{\partial s} \frac{\partial \phi}{\partial(\sigma_y^2)} = -\frac{1}{2} \frac{\partial(\sigma_x^2)}{\partial s} \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{2} \frac{\partial(\sigma_y^2)}{\partial s} \frac{\partial^2 \phi}{\partial y^2}.\quad (4.7)$$

Hence, we can apply the same machinery in the derivation of Eqs. (4.3) and eventually get the result

$$E_{ss}^{2D} := \lim_{\epsilon \rightarrow 0} E_{ss} = -\frac{\partial \phi^{2D}}{\partial s}.\quad (4.8)$$

A set of close-form expressions of  $E_x^{2D}$ ,  $E_y^{2D}$  and  $E_{ss}^{2D}$  can be further derived [6, 8], and has been widely used in many beam-beam studies [6].

Before going through a detailed derivation of  $\lim_{\epsilon \rightarrow 0} E_{sz}$ , we first try to guesstimate the answer. The limit  $\epsilon = \sigma_y^2/(\gamma^2\sigma_z^2) \rightarrow 0$  implies a particle beam with a very large Lorentz factor or a very long bunch length. We know the electric field lines emanating from a charged particle get compressed in the transverse direction due to the Lorentz



contraction [12]. In the highly relativistic limit  $\gamma \rightarrow \infty$ , the longitudinal field of each single particle in a bunch approaches zero, and the same thing also holds for the field of the whole bunch as it is just the superposition of single-particle fields. To discuss the case of an extremely long bunch, we consider a particle beam with a Gaussian-distributed density  $\rho$  and a bunch length  $\sigma_z$ . Given an arbitrary location  $z = z_0$  and a length  $l > 0$ , we have  $\rho(z = z_0 - l) \approx \rho(z = z_0 + l)$  as  $\sigma_z \rightarrow \infty$ ; and hence, the longitudinal electric fields generated from  $\rho(z = z_0 - l)$  and  $\rho(z = z_0 + l)$  cancel at  $z = z_0$  for all  $l$ .

In deriving  $\lim_{\epsilon \rightarrow 0} E_{sz}$ , if limit and integration are arbitrarily interchanged, we may get a result

$$\lim_{\epsilon \rightarrow 0} E_{sz} = \text{const} \cdot \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{\partial}{\partial z} \psi_\epsilon(w) dw \quad (4.9)$$

$$\stackrel{?}{=} \text{const} \cdot \int_0^1 \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial z} \psi_\epsilon(w) dw \quad (4.10)$$

$$= \text{const} \cdot \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \frac{z}{\sigma_z^2} \int_0^1 \frac{\exp\left(-\frac{(1-w)x^2}{2\sigma_x^2(1-w+Aw)} - \frac{(1-w)y^2}{2\sigma_y^2}\right)}{(1-w+Aw)^{1/2}(1-w)} dw \quad (4.11)$$

$$= -\frac{1}{\gamma^2} \frac{\partial \phi^{2D}}{\partial z} \quad (\text{substitute with } w = q/(q + \sigma_y^2)). \quad (4.12)$$

However, Eq. (4.12) diverges because the integrand in Eq. (4.11) is bigger than a function  $f(w) := \exp(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2})/(1-w)$  on the interval  $[0, 1]$ , and the integral  $\int_0^1 f(w) dw$  diverges. This means that the result of Eq. (4.12) and our previous guesstimation contradict each other. This contradiction arises because we cannot find an integrable function that is bigger than  $|\partial \psi(w)_\epsilon / \partial z|$ ; and hence, we are not allowed to interchange integration and limit in Eq. (4.9). To find out the right answer, we first write down  $E_{sz}$  explicitly

$$E_{sz} = \frac{1}{4\pi\epsilon_0} \frac{Ne}{(8\pi)^{1/2}} \frac{\sigma_y}{\gamma^2 \sigma_x \sigma_z^2} \frac{z}{\sigma_z} \times \underbrace{\int_0^1 \frac{(1-w)^{1/2} \exp\left(-\frac{(1-w)x^2}{2\sigma_x^2(1-w+Aw)} - \frac{(1-w)y^2}{2\sigma_y^2} - \frac{(1-w)\gamma^2 z^2}{2\gamma^2 \sigma_z^2(1-w+\epsilon w)}\right)}{(1-w+Aw)^{1/2}(1-w+\epsilon w)^{3/2}} dw}_{=: I_\epsilon} \quad (4.13)$$

With a new variable  $u := (1-w)/\epsilon$  defined, an upper bound of  $I_\epsilon$  can be derived

$$I_\epsilon \leq \int_0^{1/\epsilon} \frac{u^{1/2}}{(A + (1-A)\epsilon u)^{1/2} \cdot (1 + (1-\epsilon)u)^{3/2}} dw \quad (4.14)$$

$$\leq \frac{1}{(1-A)^{1/2} \cdot \epsilon^{1/2}} \int_0^{1/\epsilon} \frac{1}{(1 + (1-\epsilon)u)^{3/2}} du \quad (4.15)$$

$$= \frac{2}{(1-A)^{1/2} \cdot \epsilon^{1/2}} \frac{1 - \epsilon^{1/2}}{1 - \epsilon}. \quad (4.16)$$

In deriving Eq. (4.15), we apply an inequality  $1/(A + (1-A)\epsilon u) \leq 1/((1-A)\epsilon u)$ , which is true for  $u > 0$  because  $\sigma_y/\sigma_x \leq 1$  was assumed at the beginning, and  $(1-A) \geq 0$ .

Combining Eq. (4.13) and Eq. (4.16), we conclude that the magnitude of  $E_{sz}$  is bounded by

$$|E_{sz}| \leq \frac{1}{4\pi\epsilon_0} \frac{Ne}{(8\pi)^{1/2}} \frac{2}{\sigma_x\sigma_y} \frac{|z|}{\sigma_z} \frac{1}{1 - \sigma_y^2/\sigma_x^2} \frac{\epsilon^{1/2}}{1 + \epsilon^{1/2}},$$

and  $|E_{sz}| \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Therefore, we have  $E_{sz}^{2D} := \lim_{\epsilon \rightarrow 0} E_{sz} = 0$ .

## Acknowledgement

This work is supported by the U.S. DOE under contract NO. DE-AC02-05CH11231.

## A. A Property on the Solution of the Inhomogeneous Wave Equation

**Lemma 1.** *Assume the electromagnetic wave follows instantaneous propagation; in other words, every signal arrives at the observer right after its generation from the source without any retardation. Given a time-dependent charge source of the forms  $\rho(x, y, s - v_0t)$  and  $\mathbf{J}(x, y, s - v_0t)$ , the solutions to the inhomogeneous wave equations for the scalar and vector potentials can be expressed in the forms  $\phi(x, y, s - v_0t)$  and  $\mathbf{A}(x, y, s - v_0t)$ .*

*Proof.* We only show the case for  $\phi$ , and the case for  $\mathbf{A}$  can be done via the same machinery. The retarded solution of the inhomogeneous wave equation for  $\phi$  is [12]

$$\phi(x, y, s, t) = \frac{1}{4\pi\epsilon_0} \iiint g(x - x', y - y', s - s') \cdot \rho(x', y', s', t') dx' dy' ds' \quad (\text{A.1})$$

with a Green's function

$$g(x - x', y - y', s - s') = 1/\sqrt{(x - x')^2 + (y - y')^2 + (s - s')^2}. \quad (\text{A.2})$$

Here,  $t'$  satisfies the retardation condition

$$t' = t - \sqrt{(x - x')^2 + (y - y')^2 + (s - s')^2}/c_0. \quad (\text{A.3})$$

Substituting the source term in Eq. (A.1) with  $\rho(x', y', s' - v_0t')$  and applying the assumption  $t' = t$ , we can prove the conclusion

$$\begin{aligned} \phi(x, y, s, t) &= \frac{1}{4\pi\epsilon_0} \iiint g(x - x', y - y', s - s') \cdot \rho(x', y', s' - v_0t) dx' dy' ds' \\ &= \frac{1}{4\pi\epsilon_0} \iiint g(x - x', y - y', s - v_0t - s'') \cdot \rho(x', y', s'') dx' dy' ds'' \quad (s'' := s' - v_0t) \\ &= \frac{1}{4\pi\epsilon_0} \iiint g(x - x', y - y', (s - v_0t) - s'') \cdot \rho(x', y', s'') dx' dy' ds'' \\ &=: f(x, y, s - v_0t). \end{aligned}$$

□

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