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Momentum Compaction, Phase-Slip Factor and Gamma Transition in a Synchrotron

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1 Introduction

A synchrotron is the workhorse in charged particle acceleration and is applied for charged particles acceleration and ion acceleration to the highest energies [1, p. 77]. During the acceleration of a charged particle, each particle experiences a longitudinal force opposite in sign to it's "displacement from the central particle" like in a harmonic oscillator. This motion is similar also to the transverse motion of the particle in the accelerator (betatron oscillations) and this is the reason that the particle stays inside the bucket during the acceleration. In this technical note, we discuss the concepts like, transition energy, momentum compaction and phase slip factor associated with a synchrotron accelerator, and we use a simple example to present the physical meaning of the "transition energy". We present a simple model of a circular accelerator having straight sections and bends only to provide an explanation and the physical meaning of transition energy.

2 Path Length and Momentum Compaction

The knowledge of momentum compaction and phase-slip factor play a very important role in designing a circular accelerator. A relativistic particle beam in a circular accelerator must have a transition point. This "transition point" is a relativistic effect and we provide the definitions of the momentum compaction factor in a bend magnet. The momentum compaction factor is due to the different amounts of time it takes for the high and low momentum particles to go through the bend magnets relative to the straight sections [2, 3].

Following the textbook [1], we derive the formulas for momentum compaction, and transition gamma in a synchrotron. We consider linear beam dynamics and neglect higher order correction to the path length. In this approximation, only linear contribution to the path length comes from the curved sections of the beam transport line. The total path length in synchrotron is therefore defined by

$$L = \int (1 + \kappa x) dz. \tag{1}$$

The first term in the above integral is the original ideal path length of the beam line or the design circumference L_0 for $\delta = 0$, i.e., $L_0 = \int dz$. The second term in the above integral gives the deviation from the design circumference. The term κ is the dipole field strength proportional to the inverse of the dipole bending radius ρ. The transverse co-ordinate x is defined by, $x = D(z)\delta$, where $D(z)$ is the dispersion function along the ring and $\delta = \Delta p/p$ is the momentum spread. Now, Eq. (1) can be re-write as

$$\begin{aligned} L &= L_0 + \delta \int \kappa(z) D(z) dz, \\ \Delta L &= L - L_0 = \delta \int \kappa(z) D(z) dz. \end{aligned} \tag{2}$$

Particles with different momenta in a synchrotron will follow closed orbit with different lengths. The variation of the path length with momentum is determined by the momentum compaction factor, denoted by α_c and defined by

$$\alpha_c = \frac{\Delta L / L_0}{\delta} \quad \text{with } \delta = \Delta p/p. \tag{3}$$

Combining Eq. (2) and Eq. (3), Eq. (3) can be re-write as

$$\alpha_c = \frac{1}{L_0} \int_0^L \kappa(z) D(z) dz = \left\langle \frac{D(z)}{\rho} \right\rangle. \tag{4}$$

The above expression is derived under the approximation that the path length variation is determined only by the dispersion function in the bending magnets and the path length depends only on the energy of the particles.

For the given path length L in a synchrotron, the travel time is defined by

$$\tau = \frac{L}{\beta c}, \tag{5}$$

where $\beta = v/c$ is the relativistic velocity of a given particle, and c is the velocity of light in a vacuum. Taking logarithmic differentiation of Eq. (5) yields

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L_0} - \frac{\Delta\beta}{\beta}. \tag{6}$$

In Eq. (6), the first term $\Delta L/L_0$ is related to momentum compaction given by Eq. (3), i.e. $\alpha_c\delta = \Delta L/L_0$. The second term can be derived in the following way.

For a given momentum p, relativistic velocity β, and the rest mass energy E_0 of a given particle, the following relationships hold true:

$$\begin{aligned} cp &= \frac{E_0}{\sqrt{\frac{1}{\beta^2} - 1}}, \\ cp &= \frac{E_0\beta}{\sqrt{1-\beta^2}} = \gamma\beta E_0, \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \\ cp &= \beta E, \quad E = \gamma E_0. \end{aligned} \tag{7}$$

Further, taking logarithmic differentiation of $cp = \beta E$ yields

$$\begin{aligned} \frac{dc}{c} + \frac{dp}{p} &= \frac{d\beta}{\beta} + \frac{dE}{E}, \\ \frac{dp}{p} &= \frac{d\beta}{\beta} + \frac{dE}{E}, \quad \frac{dc}{c} = 0. \\ \to \frac{d\beta}{\beta} &= \frac{dp}{p} - \frac{dE}{E}. \end{aligned} \tag{8}$$

The total energy of a relativistic particle is the sum of its rest mass energy and the kinetic energy. Hence, the total energy E can be expressed as

$$E = (p^2c^2 + m_0^2c^4)^{1/2}, \tag{9}$$

where m_0 is the rest mass of the given particle. Differentiating above Eq. (9)

with respect to p, we get

$$\begin{aligned}\frac{dE}{dp} &= \frac{d}{dp}([p^2c^2 + m_0^2c^4]^{\frac{1}{2}}) \\ &= \frac{1}{2}[p^2c^2 + m_0^2c^4]^{-\frac{1}{2}}2pc^2 \\ &= \frac{pc^2}{E} = \frac{E}{p}\left(\frac{cp}{E}\right)^2 \\ &= \frac{E}{p}\beta^2, \quad cp = \beta E \\ &\rightarrow \frac{dE}{E} = \beta^2\frac{dp}{p}.\end{aligned} \tag{10}$$

Combining Eqs. (10), (8), and (6), we can re-write Eq. (6) as

$$\begin{aligned}\frac{\Delta\tau}{\tau} &= \alpha_c\frac{dp}{p} - \frac{dp}{p} + \beta^2\frac{dp}{p} \\ &= \alpha_c\frac{dp}{p} - (1-\beta^2)\frac{dp}{p} \\ &= \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{dp}{p} = -\left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p} = -\eta_c\frac{dp}{p}.\end{aligned} \tag{11}$$

The term $\eta_c = (1/\gamma^2 - \alpha_c) = (1/\gamma^2 - 1/\gamma_t^2)$ is called phase-slip factor in a synchrotron. The energy $\gamma_t = 1/\sqrt{\alpha_c}$ is the transition energy and it depends on lattice parameters of the circular machine.

Above the transition energy, $\gamma > \gamma_t$ and $\eta_c < 0$. This gives $\Delta\tau/\tau > 0$. It means a particle with a higher energy needs a longer time for one revolution than a particle with a lower energy. This is because the dispersion function causes particles with a higher energy to follow an equilibrium orbit with a larger average radius compared to the radius of the ideal orbit [1]. Similarly, below the transition energy, $\gamma < \gamma_t$ and $\eta_c > 0$. This gives $\Delta\tau/\tau < 0$. It means a particles with a higher energy needs a shorter time for one revolution than that of the synchronous particle. Figure 1 schematically presents this concept.

At $\gamma = \gamma_t$, $\eta_c = 0$, and the revolution period is independent of the particle momentum. All particles at different momenta travel thoroughly around the machine with equal revolution frequencies. This is the isochronous condition [4].

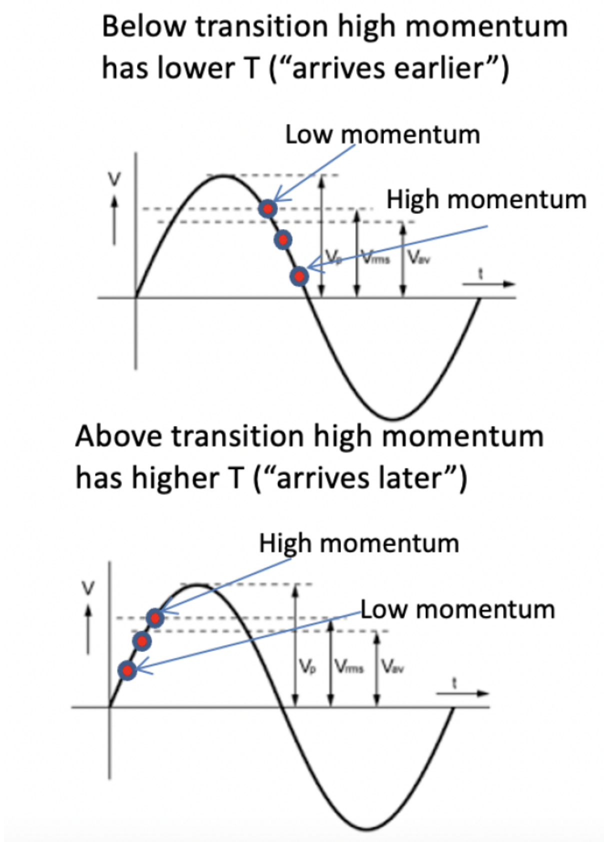

Figure 1: Schematic drawing of an rf wave, and the rf phase angle for a synchronous, a higher, and a lower momentum particles.

3 Transition Gamma

This chapter provides the physical meaning of the transition gamma. We use a simple example to explain the physical meaning of transition gamma. We consider a toy model of a synchrotron having straight sections and arc sections. The linear optics is designed using basic “FODO” (focusing and defocusing set of quadrupoles) optics. Based on this simple model, our goal will be to explain the physical meaning of gamma transition under the condition $dT/d\gamma = 0$. The transition energy is defined by

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}, \tag{12}$$

and it can be calculated by taking the ratio between the beam energy at transition and the rest mass energy of the particle. The value of the transition energy or transition gamma is independent of the particle mass and depends only on the machine optics and geometry [5]. For a regular lattice in a circular accelerator, the value of γ_t is approximately equal to the horizontal tune Q_x, i.e. $\gamma_t \approx Q_x$ [1]. The transition energy which is also known as gamma transition plays an important role in phase-stability of the bunched beam acceleration. From Eq. (11), it is clear that crossing transition changes the sign of the phase-slip factor. In general, all electron synchrotrons operate above transition gamma, whereas many proton and hadron synchrotrons must pass through transition as the beam is accelerated.

There are many experiments performed to understand and calculate the momentum compaction factor and transition energy in a ring accelerators [6, 7]. In this section, we consider a simple model of a circular accelerator with straight sections and only bending dipole magnets and derive the condition for the transition gamma.

As shown in Figure 2, let us consider a circular accelerator with the regular straight sections each of length s and the bending arcs with a bending radius ρ. The time spent by a charged particle along the straight sections and along the bending regions are different. The total time it takes for a particle to go along the circular accelerator is defined by

$$T = 4(T_m + T_s) = \frac{2\pi m}{qB}\gamma + \frac{4s}{c}\frac{\gamma}{\sqrt{\gamma^2 - 1}}. \tag{13}$$

The factor '4' comes for 4 straight sections and 4 bending sections along the circular accelerator. B is the dipole magnetic field, q is the charge, and m is the mass of the charged particle. The details on the derivation of time period is given in Appendix A.

Further, we make a T versus γ plot and transition energy value γ_t is obtained for $dT/d\gamma = 0$. From Eq. (13), $dT/d\gamma$ can be calculated as

$$\frac{dT}{d\gamma} = \frac{2\pi m}{qB} + \frac{4s}{c}\frac{d}{d\gamma}\left(\frac{\gamma}{\sqrt{\gamma^2 - 1}}\right). \tag{14}$$

Taking the second term on the right hand side of above equation and differ-

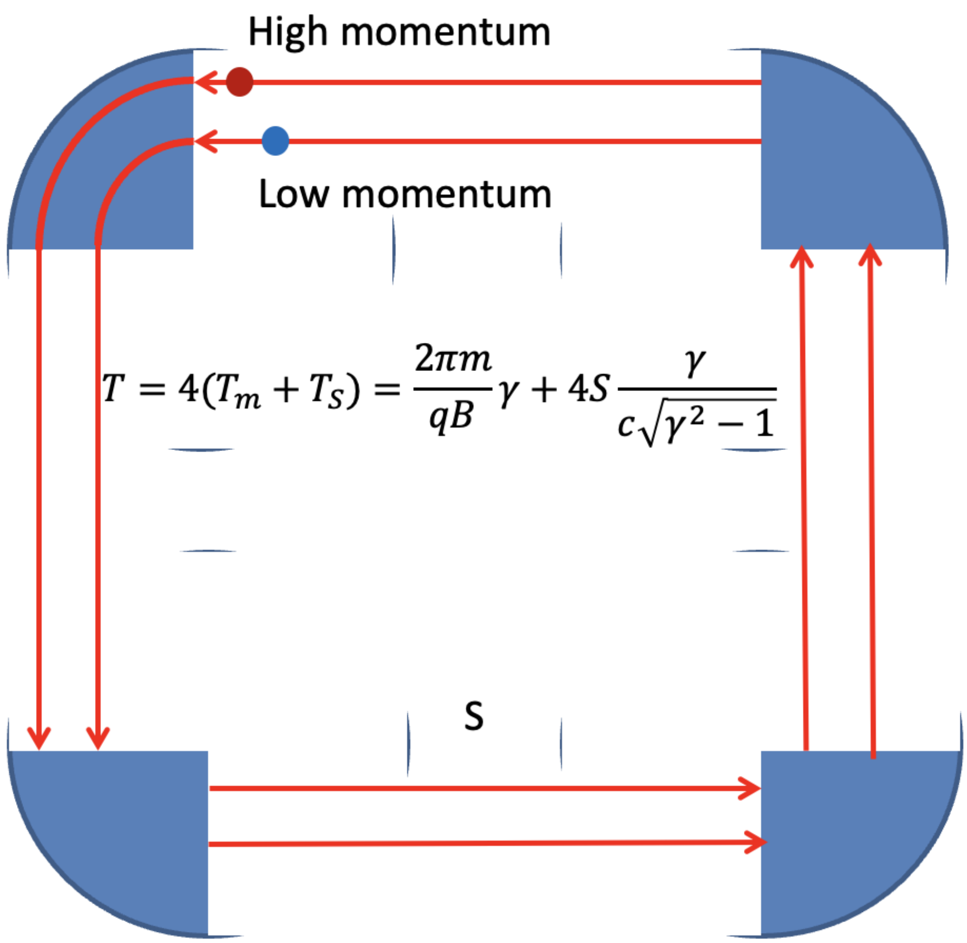

Figure 2: Schematic drawing of a circular accelerator showing straight sections and bends.

entiating by parts, we get

$$
\begin{aligned}
\frac{d}{d\gamma}\left(\frac{\gamma}{\sqrt{\gamma^2-1}}\right) &= \gamma\frac{d}{d\gamma}\left(\frac{1}{\sqrt{\gamma^2-1}}\right) + \frac{1}{\sqrt{\gamma^2-1}}\frac{d\gamma}{d\gamma} \\
&= \frac{-\gamma^2}{(\gamma^2-1)\sqrt{\gamma^2-1}} + \frac{1}{\sqrt{\gamma^2-1}} \qquad (15) \\
&= -\frac{1}{\sqrt{\gamma^2-1}}.
\end{aligned}
$$

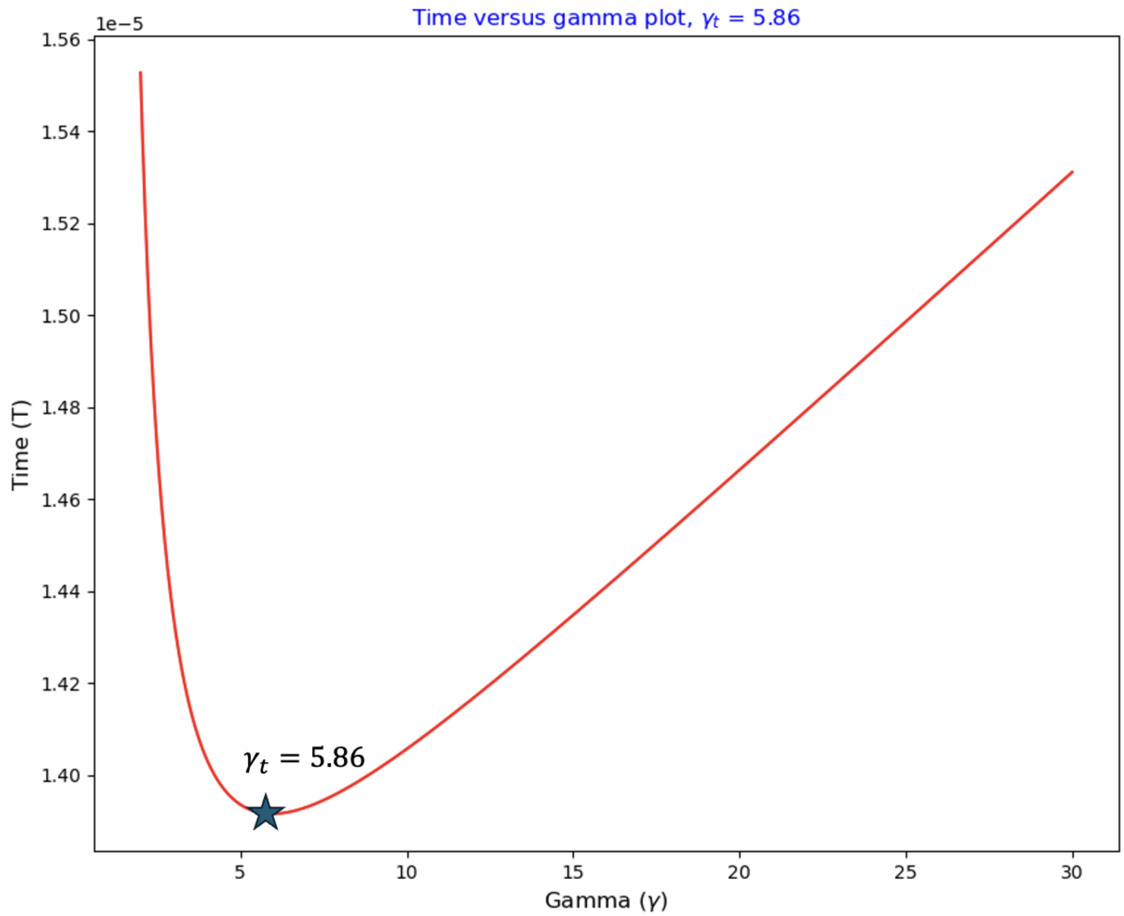

Figure 3: T versus γ plot with $\gamma_t = 5.86$.

Substituting the value from Eq. (15), Eq. (14) takes the form

$$\frac{dT}{d\gamma} = \frac{2\pi m}{qB} - \frac{4s}{c}\frac{1}{(\gamma^2 - 1)^{3/2}}. \tag{16}$$

Now, γ_t value is obtained with $dT/d\gamma\Big|_{\gamma=\gamma_t} = 0$, which gives the transition gamma value

$$\gamma_t = \left[1 + \left(\frac{2sqB}{\pi mc}\right)^{2/3}\right]^{1/2}. \tag{17}$$

We consider a simple case to calculate γ_t. Let us consider a ring which has a circumference of about 3.8 km. We consider proton beam with electronic charge $q = 1.6\times10^{-19}$C, mass $= 1.67\times10^{-27}$kg, $s = 1000.0$ m, and $B = 1.0$ T. This gives $\gamma_t = 5.86$.

4 Discussion

Figure 3 shows time (T) versus energy (γ) plot with the value of transition gamma (γ_t) 5.86. At high speeds approaching the speed of light the particle's total energy increases proportionally to γ, and the speed of the particle β, increases slowly to towards the value of the speed of light, as dictated by the theory of relativity. The shape of curve in Figure 3 is derived using equations of relativity as applied to the simple accelerator model in Figure 2. The minimum of the this curve in in Figure 3 where dT/d(gamma)=0 is the transition energy.

Furthermore, equation of longitudinal phase oscillation relative to synchronous particle can be written as [1]

$$\ddot{\varphi} + \frac{h\omega_s^2\eta_{tr}qV}{2\pi\beta^2U_s}(\sin\phi_s - \sin\phi) = 0. \tag{18}$$

For small amplitude oscillation,

$$\begin{aligned} \sin\phi &= \sin(\phi_s + \varphi) \\ &\approx \varphi\cos\phi_s + \sin\phi_s. \end{aligned} \tag{19}$$

Hence for the given small amplitude of oscillation, Eq. (18) can be written as

$$\ddot{\varphi} + \Omega_s^2\varphi = 0, \tag{20}$$

where $\Omega_s = \omega_s\sqrt{\frac{h\eta_{tr}\cos\phi_s}{2\pi\beta^2\gamma}\frac{qV}{mc^2}}$ is angular synchrotron oscillation frequency. Further, synchrotron tune Q_s can be defined as

$$Q_s = \frac{\Omega_s}{\omega_s} = \sqrt{\frac{h\eta_{tr}\cos\phi_s}{2\pi\beta^2\gamma}\frac{qV}{mc^2}}. \tag{21}$$

The following conclusions can be made:

- For oscillation about the synchronous phase, $\eta_{tr} \cos \phi_s$ must remain positive (or at least have the same sign as qV). Otherwise, oscillation frequency becomes imaginary.

- As η_{tr} flips sign when the beam accelerates through transition, the synchronous phase must shift to maintain longitudinal stability. This means ϕ_s becomes $\pi - \phi_s$, which is another side of sinusoidal curve (phase focusing above transition).

- At transition $\eta_{tr} = 0$. This gives $\Omega_s = 0$ and consequently $Q_s = 0$. Synchrotron frequency drops to zero at transition and the longitudinal phase space almost got *frozen* around gamma transition.

- From Eq. (20) and for $\Omega_s = 0$, $\ddot{\varphi} = 0$. This means $\dot{\varphi} = constant$.

A Time Period Along the Circular Accelerator with Straight Sections and Bends Only

The total time period is given by

$$T = 4(T_m + T_s), \tag{22}$$

where T_m and T_s are the time spent by a particle in bending arc and in a straight section of a circular accelerator respectively. The time spent in a straight section is

$$T_s = \frac{s}{c\beta}, \tag{23}$$

where $\beta = \sqrt{1 - 1/\gamma^2}$ is the relativistic velocity and γ is the relativistic energy factor. Substituting the value of β in Eq. 23, the time spent by a particle in a straight section becomes

$$T_s = \frac{s}{c}\frac{\gamma}{\sqrt{\gamma^2 - 1}}. \tag{24}$$

In the arc section of a circular accelerator, the centrifugal force balances Lorentz's force, i.e.

$$\frac{\gamma m v^2}{\rho} = qvB \tag{25}$$

where q is the charge, B is the magnetic field in the dipole magnet and ρ is the bending radius of the dipole magnet. Solving the above relationship, the bending radius can be expressed as

$$\rho = \frac{\gamma m \beta c}{qB} \tag{26}$$

Finally, time spent by a particle in the bending arc is given by

$$T_m = \frac{1}{4}\frac{2\pi\rho}{\beta c} = \frac{\pi m}{2qB}\gamma. \tag{27}$$

The total time period is then

$$T = 4(T_m + T_s) = \frac{2\pi m\gamma}{qB} + \frac{4s}{c}\frac{\gamma}{\sqrt{\gamma^2 - 1}}. \tag{28}$$

B Python Script to Calculate Transition Gamma

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

#defining some constants and parameters
c = 3e8 #velocity of light
s = 1000.0 #length of the straight section of the synchrotron
q = 1.6e-19 #charge of the particle
B = 1.0 # magnetic field in Tesla
m = 1.6726e-27 # mass of the proton in kg
#creating vectors T and gamma
g = np.linspace(2,30,1000) #gamma energy
array = np.sqrt((g*g)-1)
T = (2*np.pi*m*g/(B*q)) + 4*s*g/(c*(array))

#plot T versus gamma
fig = plt.figure(figsize = (10,8))
plt.plot(g,T)
plt.xlabel("Gamma ($\gamma$)",fontsize=12)
plt.ylabel("Time (T)",fontsize=12)
#plt.xticks(0,15)
plt.xlim(0,60,10)
plt.xticks(np.arange(0, 60, 10),fontsize=12)
plt.yticks(np.arange(2.7e-8, 3.10e-8, 0.05e-8),fontsize=12)
plt.savefig('transition.png')

a = 2*s*q*B/(np.pi*m*c)
b = pow(a,0.66)
gamma_t = pow((1+b),0.5)
gamma_t = 5.8598747346172555
```

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