A Damping Ring for the Rapid Cycling Synchrotron

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Abstract
The Electron-Ion Collider (EIC) [1] electron accelerator chain consists of a 400 MeV electron Linear Accelerator (LINAC), the Rapid Cycling Synchrotron (RCS), and the Electron Storage Ring (ESR). The LINAC injects into the RCS, the RCS ramps to a top energy of 18 GeV and the electron beam is extracted to the ESR. To minimize the momentum spread of the 400 MeV injected beam and serve as a 28 nC electron accumulator ring, a damping ring is proposed. This ring may also serve as an intermediate energy booster for the RCS. The damping ring is not within the current baseline of the EIC project.
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# 1 Introduction

Traditional accumulator rings such as the Positive Intensity Accumulator (PIA) [2] [3] are designed to be compact and have an intermediate energy. The electron accumulation methods may be from individual bunch injection and bunch merging, or from injection schemes that are used for top-up injection [4]. The 400 MeV EIC LINAC delivers 7 nC per bunch. The bunch charge need for the ESR at 10 GeV and 5 GeV design energies is 28 nC. At 18 GeV the charge per bunch requirement is 11 nC. The lattice design was performed using the Bmad toolkit [5].

Due to the large energy spread, it may be necessary to damp the electron beam before injection into the RCS. At the exit end of the 400 MeV LINAC the transverse emittance is 51 nm and energy spread of $5.56 \times 10^{-3}$. Table 1 lists key damping ring (DR) lattice parameters. The RCS ramp frequency of 1 Hz requires that the damping times must be on the order of milliseconds. The 400 MeV LINAC will inject directly into the DR.

In the development of the RCS damping ring, the Chasman-Green double bend achromat [6, 7] arc lattice design was explored. It was found that the double bend achromat design of the accelerator footprint was too large. The multi-bend achromat designs were then explored [8] because of their short damping times and small accelerator footprint.

<table>
<thead>
<tr>
<th>Table 1: Table of general parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Energy</strong></td>
</tr>
<tr>
<td><strong>Circumference</strong></td>
</tr>
<tr>
<td><strong>Horizontal Tune</strong></td>
</tr>
<tr>
<td><strong>Vertical Tune</strong></td>
</tr>
<tr>
<td><strong>Synchrotron Tune</strong></td>
</tr>
<tr>
<td><strong>Horizontal Natural Chromaticity</strong></td>
</tr>
<tr>
<td><strong>Vertical Natural Chromaticity</strong></td>
</tr>
<tr>
<td><strong>Horizontal Damping Time</strong></td>
</tr>
<tr>
<td><strong>Vertical Damping Time</strong></td>
</tr>
<tr>
<td><strong>Longitudinal Damping Time</strong></td>
</tr>
<tr>
<td><strong>Natural Horizontal Emittance</strong></td>
</tr>
<tr>
<td><strong>Natural Bunch Length</strong></td>
</tr>
<tr>
<td><strong>Natural Energy Spread</strong></td>
</tr>
<tr>
<td><strong>Energy Loss per Turn</strong></td>
</tr>
<tr>
<td><strong>Revolution Frequency</strong></td>
</tr>
<tr>
<td><strong>Harmonic Number</strong></td>
</tr>
<tr>
<td><strong>RF Voltage</strong></td>
</tr>
<tr>
<td><strong>RF Frequency</strong></td>
</tr>
<tr>
<td><strong>Momentum Compaction</strong></td>
</tr>
</tbody>
</table>

## 1.1 A Bit of Theory

At this point, it is helpful to establish a coordinate system and the Hamiltonian representation of the system. Using the Lorentz force,

$$F_{e/m} = e[E + v \times B]$$

(1)

, and the subsequent Lagrangian,

$$L = -m_0 c^2 \sqrt{1 - \beta^2} + eA \cdot v - e\phi$$

(2)
where $\phi$ is the electric potential, $A$ is the magnetic potential, $q_i$ is the position coordinate, and $p_i$ is the momentum with its momenta, the Hamiltonian of a particle under electromagnetic forces can be written as:

$$H(q, p, t) = e\phi + c\sqrt{(p - eA)^2 + m_0^2c^2}$$  \hspace{1cm} (3)

The equation of motion are:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$  \hspace{1cm} (4)

We will now perform a canonical transformation to the Frenet-Serret coordinate system [9], which is shown in Fig. 2. We have that

![Frenet-Serret coordinate system](image)

Figure 2: Frenet-Serret coordinate system where the reference particle $z$ coordinate is zero by construction. $\hat{z}$ is tangent, $\hat{x}$ is normal, and $\hat{y}$ is binormal to the curvature. $r_0$ is the reference curve and $s$ is the position along the reference orbit.

$$r(x, y, s) = r_0(s) + x\hat{x} + y\hat{y}$$  \hspace{1cm} (5)
The triorthogonal planes of the Dreibein have the relations:

$$\begin{align*}
\frac{dr_0}{ds} &= \hat{z} \\
\frac{d\hat{z}}{ds} &= \kappa \hat{x} \\
\frac{d\hat{x}}{ds} &= \tau \hat{y} - \kappa \hat{z} \\
\frac{d\hat{y}}{ds} &= -\tau \hat{x}
\end{align*}$$

(6)

where \( \kappa = 1/\rho \) and \( \rho \) is the radius of curvature and \( \tau \) is the torsion of the curve. If the reference curve is in a plane, \( \tau = 0 \). Using the Hamiltonian equations of motion, Equ. 4, the momentum along the direction of the reference orbit

$$p_s = \mathbf{p} \cdot \hat{z}(1 - \kappa x)$$

(7)

and the magnetic vector potential

$$A_s = \mathbf{A} \cdot \hat{z}(1 - \kappa x)$$

(8)

. Neglecting static electric field, \( \phi = 0 \) and the Hamiltonian is [10, 11]:

$$H_s = -eA_s - (1 - \kappa x) \sqrt{\frac{H^2 - m_0^2 c^4}{c^2} - (p_x - eA_x)^2 + (p_y - eA_y)^2}$$

(9)

The total momentum is

$$P = \mathbf{p} \cdot \mathbf{p} = \frac{H^2 - m_0^2 c^4}{c^2} = m_0^2 c^2 (\gamma^2 - 1) = m_0^2 c^2 \beta^2 \gamma^2 = P_0 + \Delta P$$

(10)

With a simple transformation,

$$\bar{q} = q, \bar{s} = s, \bar{p} = p/P_0, \bar{H} = H_s/P_0$$

(11)

where we let \( P_0 \) be the momentum of the reference particle.

$$H = -\frac{eA_s}{P_0} - (1 - \kappa x) \sqrt{\frac{P}{P_0} - (\bar{p}_x - \frac{eA_x}{P_0})^2 - (\bar{p}_y - \frac{eA_y}{P_0})^2}$$

(12)

The vector potential, \( A_s \), is written as a series:

$$A_s = \sum_n A_n (x + iy)^n$$

(13)

The real components describe the normal field and the complex components describe the skew field. Since the magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} \frac{eA_s}{P_0} = -\frac{B_y x^2}{2p^2 B_0} - \frac{1}{B_0 \rho} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^{n-1} B_y}{\partial x^{n-1}} |_{x=0, y=0} (x + iy)^n$$

(14)

The rigidity \( B_0 \rho \) is a term that is produced from the total momentum and the charge of the particle species, \( P_0/q_{\text{charge}} \). This means that \( B_0 \) is the bending field of the reference orbit at a
given curvature. We now have a description of the Hamiltonian which includes the contribution from the magnetic field written as a Taylor series

\[ H = \frac{B_y}{2\rho^2B_0} + \frac{1}{B_0\rho} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^{n-1}B_y}{\partial x^{n-1}}|_{x=0,y=0}(x + iy)^n \]

\[-(1 - \kappa x) \sqrt{\frac{P}{P_0} - \left(\frac{eA_x}{F_0}\right)^2 - \left(\frac{eA_y}{F_0}\right)^2} \]

The values from Tab. 1 are derived directly from the lattice using the *synchrotron radiation integrals* [12, 13].

\[ I_1 = \int_0^CG\eta ds \]
\[ I_2 = \int_0^CG^2ds \]
\[ I_3 = \int_0^C|G^3|ds \]
\[ I_{4_{a,b}} = \int_0^C(G^2 + 2K_{x,y})G\eta ds \]
\[ I_{5_{a,b}} = \int_0^C|G|^3\mathcal{H}_{a,b}ds \]

Where

\[ G(s) = \frac{r''(s) \cdot r_0(s)}{|r_0(s)|} \]

is the geometric strength, \( K_{x,y} \) is the field gradient normalized to the rigidity, and \( \eta_{a,b} \) is the mode dispersion of this uncoupled periodic system. The primes indicate the differential with respect to \( s \). The “dispersion invariant”, \( \mathcal{H} \), is

\[ \mathcal{H}_{a,b} = \gamma_{a,b}\eta_{a,b}^2 + 2\alpha_{a,b}\eta'_{a,b} + \beta_{a,b}\eta'^2_{a,b} \]

The \( \alpha \) and \( \beta \) in Equ. 18 are the Twiss parameters [14, 15, 16].

### 2 Floor Plan and Design

The accelerator footprint is approximately \( 13 \times 3.5 \) m\(^2\). The circumference of the lattice is 28.64 m with straight sections composed of 3 FODO cells of length 2.75 m. The filling factor of the arc, \( f_f \), defined by

\[ \frac{L_{bend}}{L_{arc}} \]

is 0.6, where \( L_{bend} \) is the total length of the bending magnets and \( L_{arc} \) is the length of the arc. The path length through the five bend achromat arc is 5.03 m. Figure 3 illustrates the floor plan of the RCS damping ring.

The arcs of the damping ring are composed of the 5 combined function dipole magnets with 3 of the dipoles in the arc of the length 0.84 m and the other two half length, 0.42 m. The bend angle is 45°. The quadrupole lengths are .2 m and the sextupole lengths are 0.05 m. There are a
total of 10 dipoles, 26 quadrupoles and 16 sextupoles in the RCS damping ring. The dipole lengths are defined in the lattice file as:

$$L_{\text{dipole}} = R_{\text{arc}} \cdot \sin(\theta_{\text{bend}}) - (2 \cdot L_{\text{quadrupole}})$$  \hspace{1cm} (20)

where $R_{\text{arc}}$ is 1.75 m, $\theta_{\text{bend}}$ is $2\pi/n_{\text{dipole}}$. The number of dipoles, $n_{\text{dipole}}$, is 8. The two half dipoles are considered one dipole in the calculations. The beam rigidity at 400 MeV is 1.33 T m. The length of the quadrupoles and sextupoles are based on the magnet design of the Compact Storage ring for Actinic Mask Inspection (COSAMI) [17]. An 80 cm RF cavity is placed in the straight section on the opposite side of the injection into the damping ring. Table 2 lists the magnets with there lengths and strengths. The RCS damping ring does not require any superconducting systems. The lattice file, in Bmad format, is located in Appendix A.

3 Optics

The lattice combines two separate beam lines, the arc and the straight section. Figure 4 plots the $\beta$-functions (both horizontal (black) and vertical (red)) and the lower plot is of the dispersion. Since there are no vertical bends, elements that excite coupling, and the lattice is ideal, the vertical dispersion (red) is zero. The tunes of the lattice $\nu_{x,y}$, are 2.832, 1.562, respectively. The fractional tunes are far enough away from the fifth order resonance horizontally and the strong half integer resonance vertically. The resonance condition can be written as:

$$pQ_x + qQ_y + rQ_s = n$$  \hspace{1cm} (21)

where $p$, $q$, $r$, and $n$ are integers. If $p$ and $q$ have the same sign, the resonance is known as a \textit{sum resonance}. If the signs are different, a \textit{difference resonance}. The highest order of resonance that is
Figure 4: The top plot is of the $\beta_{x,y}$ Twiss functions. The bottom plot is of the $\eta_{x,y}$ dispersion functions.
Table 2: Magnet types used in the RCS damping ring

<table>
<thead>
<tr>
<th>Element Name</th>
<th>Element Type</th>
<th>Length (m)</th>
<th>B field (T)</th>
<th>B(_1) gradient (Tm(^{-1}))</th>
<th>B(_2) gradient/b(_2) multipole (Tm(^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>edip</td>
<td>Sbend</td>
<td>0.419</td>
<td>-1.2513</td>
<td>4.393</td>
<td>(-2.1328 \times 10^{-3})</td>
</tr>
<tr>
<td>mdip</td>
<td>Sbend</td>
<td>0.837</td>
<td>-1.2513</td>
<td>3.383</td>
<td>(-2.1328 \times 10^{-3})</td>
</tr>
<tr>
<td>efq</td>
<td>Quadrupole</td>
<td>0.200</td>
<td>-4.088</td>
<td></td>
<td></td>
</tr>
<tr>
<td>edq</td>
<td>Quadrupole</td>
<td>0.200</td>
<td>0.013</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mfq</td>
<td>Quadrupole</td>
<td>0.100</td>
<td>-16.809</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sqd</td>
<td>Quadrupole</td>
<td>0.200</td>
<td>4.004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>asfq</td>
<td>Sextupole</td>
<td>0.050</td>
<td>-379.429</td>
<td>-379.429</td>
<td></td>
</tr>
<tr>
<td>asf2</td>
<td>Sextupole</td>
<td>0.050</td>
<td>-185.344</td>
<td>-185.344</td>
<td></td>
</tr>
<tr>
<td>ssf</td>
<td>Sextupole</td>
<td>0.050</td>
<td>-134.877</td>
<td>-134.877</td>
<td></td>
</tr>
<tr>
<td>ssd</td>
<td>Sextupole</td>
<td>0.050</td>
<td>11.895</td>
<td></td>
<td></td>
</tr>
<tr>
<td>asf3</td>
<td>Sextupole</td>
<td>0.050</td>
<td>-212.524</td>
<td>-212.524</td>
<td></td>
</tr>
<tr>
<td>asf4</td>
<td>Sextupole</td>
<td>0.050</td>
<td>-471.262</td>
<td>-471.262</td>
<td></td>
</tr>
</tbody>
</table>

considered is |n| \leq 5. The values for \(Q_x\), \(Q_y\), and \(Q_s\) are obtained by solving quotient of the eigenvalues \((\lambda_{1,3,5})\) and \(2\pi\) of the DR \(6 \times 6\) transfer matrix, \(M_{DR}\). Using the Lie Algebra method [18] the \(6 \times 6\) symplectic matrix can be described as the *Lie transformation* of integral of the Hamiltonian between \(s_i\) and \(s_f\):

\[
M_{DR} = e^{-\int_{s_i}^{s_f} H(x,s) ds} 
\]

where \(s\) are along the particles trajectory and

\[
e^{\cdot f} : \equiv \sum_{k=0}^{\infty} f :^k / k! 
\]

A few helpful relations are,

\[
: f : \equiv \frac{\partial f}{\partial q} \frac{\partial }{\partial p} - \frac{\partial f}{\partial p} \frac{\partial }{\partial q} 
\]

\[
: f : g \equiv \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} = [f,g] 
\]

\[
: f :^2 g = [f,[f,g]] 
\]

\[
e^{\cdot f} g = g + [f,g] + [f,[f,g]]/2! + ... 
\]

where \(f(z)\) is any function of \(q,p\). The six dimensional phase space coordinates are written as \(x = (x, p_x, y, p_y, z, p_z)\). And since \(s_i\) and \(s_f\) are two arbitrary points along \(s\), the beginning and end of the lattice are chosen to give the *one-turn-map*. The condition for symplecticity is as follows,

\[
J = M_{DR} J M_{DR}^T 
\]

where \(J\) is the \(6 \times 6\) matrix

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]
The Hamiltonian equations are written as the Lie derivative generated by $H$ \cite{19},

$$\dot{x} = [x, H] = - : H : x$$  \hspace{1cm} (27)

and if the Hamiltonian is s position (time) independent, then

$$x(s) = M(s)x^i = e^{-t:H::x^i}$$  \hspace{1cm} (28)

otherwise (s position/time dependent case)

$$\frac{d}{ds}M(s) = -(s) : H :$$  \hspace{1cm} (29)

A more general description of the transfer matrix may be obtained from \cite{20}.

$$M_{DR} = \begin{pmatrix} A & b & D_a \\ a & B & D_b \\ p_1 & p_2 & C \end{pmatrix}$$  \hspace{1cm} (30)

Here the elements of the matrix are $2 \times 2$ sub-matrices.

- The $A$ and $B$ matrices indicate the focusing of the lattice
- The lower case $a$ and $b$ indicate coupling
- $D_a$ and $D_b$ indicate dispersion and time of flight dependence on transverse initial conditions, $D_b$ specifically coupled dispersion
- $C$ shows changes in path length

The third order sum resonance is close, shown in Fig. 5 the chromatic footprint, but the fractional tune does not cross. Here we will define $\delta = p_z/P_0$ as the momentum spread of the beam. The chromaticities from the natural to third order are shown in Tab. 3. Due to the designed sextupole field of the dipoles, the vertical natural chromaticity is positive.

3.1 The Arc

The maximum $\beta$-functions in the arc is $3.88$ m, horizontal and $2.71$ m vertical. The constraint on the dispersion at the entrance and exit of the arc demands that the horizontal phase advance, $\phi_a$, be $2\pi$ through the arc.

The $M_{arc}$ matrix is the concatenated matrix of the sector bend (beam enters and exits perpendicular to the face of the dipole) combined-function dipoles, quadrupoles, sextupoles, and drifts. A dipole map, in the absence of coupling, is written as:

$$M_d = \begin{pmatrix} \cos \theta & \sin \theta/g & 0 & 0 & 0 & (1 - \cos \theta)/g \\ -g \sin \theta & \cos \theta & 0 & 0 & 0 & \sin \theta \\ 0 & 0 & 1 & \theta/g & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin \theta & -(1 - \cos \theta)/g & 0 & 0 & 1 & L_d/(\gamma\beta)^2 - (\theta - \sin \theta)/g \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (31)
Figure 5: The chromatic footprint of the damping ring with a $\delta \pm 1.5\%$ over a tune space of 0.8 to 0.9 horizontally and 0.5 to 0.6 vertically. The numbers within the parentheses of the tune diagram are read as $p, q, r,$ and $n$. 

Resonance line restrictions:

$[pQ_x + qQ_y + rQ_z = n]$

$|p| + |q| + |r| \leq 5$

$|r| \leq 1$

$Q_z = .012$
In the transfer map Equ. 31, \( g = 1/\rho \) is the geometric strength of the dipole (\( \rho \) is the bend radius), \( \theta \) is the bend angle, and \( L_d \) is the length of the dipole. To account for the quadrupolar field of the combined-function magnet, an additional map must be included [21, 22]:

\[
\mathcal{M}_{x,cf} = \begin{pmatrix}
K_S q \sin K_x + C_y \cos K_x & -\frac{[K_y] - [K_x]}{2 \sqrt{K_y} K_x} \sinh K_y \left( \cos K_x - \frac{[K_y] + [K_x]}{[K_x] - [K_y]} \right) + \frac{1}{\sqrt{K_x}} \cosh \frac{\sqrt{K_y}}{2} \sin K_x \\
\frac{[K_y] + [K_x]}{2 \sqrt{K_y} K_x} \sinh K_y \left( \cos K_x - \frac{[K_y] + [K_x]}{[K_x] - [K_y]} \right) + \frac{1}{\sqrt{K_x}} \cosh \frac{\sqrt{K_y}}{2} \sin K_x & K_S q \sin K_x + \cos K_y \cos K_x
\end{pmatrix}
\]

Equation 32

\[
\mathcal{M}_{y,cf} = \begin{pmatrix}
\frac{\sqrt{K_y} L_d}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{K_y} L_d}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{K_y} L_d}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{K_y} L_d}{2} & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Equation 33

where \( K_{x,y} = \frac{\sqrt{K_{x,y} L_d}}{2}, S_{x,y} = \sinh \frac{\sqrt{K_{x,y} L_d}}{2}, C_{x,y} = \cosh \frac{\sqrt{K_{x,y} L_d}}{2}, \) and \( K = \frac{[K_y] - [K_x]}{2 \sqrt{K_x} \sqrt{K_y}}. \) The complete transfer map of the combined-function magnet becomes:

\[
\mathcal{M}_{cf} = M_d + \begin{pmatrix}
\mathcal{M}_{x,cf} & 0 & 0 \\
0 & \mathcal{M}_{y,cf} & 0 \\
0 & 0 & M_z
\end{pmatrix}
\]

Equation 34

These combined function magnets increase the damping partition, a more complete description of the radiation integrals for bending magnets with quadrupolar field gradients can be found [23]. \( M_z \) is the description of longitudinal rotation

\[
M_z = \begin{pmatrix}
1 & L_d/(\gamma \beta)^2 \\
0 & 1
\end{pmatrix}
\]

Equation 35

The \( K_{x,y} \) value is the normalized strength of the quadrupole which is the field gradient, \( \partial B_{y,x}/\partial B_{x,y} \), of the quadrupole normalized to the rigidity, \( B_0 \gamma = p/q_{\text{charge}}. \) The quadrupole transfer map is:

\[
\mathcal{M}_q = \begin{pmatrix}
\frac{\cos \sqrt{K_x} L_q}{\sqrt{K_x} L_q} & \frac{\sin \sqrt{K_x} L_q}{\sqrt{K_x} L_q} & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{K_x} \sin \sqrt{K_x} L_q}{\sqrt{K_x} L_q} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\cosh \sqrt{K_y} L_q}{\sqrt{K_y} L_q} & \frac{\sinh \sqrt{K_y} L_q}{\sqrt{K_y} L_q} & 0 & 0 \\
0 & 0 & \sqrt{K_y} \sinh \sqrt{K_y} L_q & \cosh \sqrt{K_y} L_q & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Equation 36

\( L_q \) is the effective length of the quadrupole. Since we assume that the beam is on axis, there will be no contribution from the sextupoles on the linear optics. We treat the sextupoles as a drift with the drift transfer map:

\[
\mathcal{M}_{\text{drift}} = \begin{pmatrix}
1 & L & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & L & 0 & 0 \\
0 & 0 & 0 & 1 & L/(\gamma \beta)^2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Equation 37
Figure 6: The top plot is of the $\beta_{x,y}$ functions. The middle plot is of the $\eta_{x,y}$ functions. The bottom plot is the lattice layout where the black box: dipole, blue box: quadrupole, and the green box: sextupole.
From these maps we can now write the complete map of the arc as:

\[ \mathcal{M}_{\text{arc}} = \mathcal{M}_{1-31} \]  

(38)

. The \( \mathcal{M}_{\text{arc}} \) is the concatenation of 31 elements.

### 3.2 The Straight Section

Figure 7 shows the optics of the straight section. It is composed of simple FODO cells with two families of harmonic sextupoles that do not affect the linear chromaticity and are used to control the resonance driving terms. One of the straight sections will contain the RF system for the DR. The other straight will contain the injection system from the 400 MeV LINAC.

The beam will be injection from the LINAC through the spin rotator [24] and into the DR. The LINAC will inject a total of 4 bunches of 7 nC into the DR. Many different methods [25, 26] are available for injection into the DR. The injection method most favorable for the DR is longitudinal injection, where a septum and a short pulsed kicker places the slightly higher momentum beam onto a closed orbit of higher momentum. The beam then damps to the equilibrium orbit. The benefit is a closed orbit bump is unnecessary.

#### 3.2.1 Longitudinal Dynamics

If we consider the longitudinal electric field from the radiofrequency cavities, we will need to revisit Equ. 3. The longitudinal electric field is described by [27, 28], however to remain consistent and keeping \( s \) dependence of the Hamiltonian, we write the electric field along the \( s \) coordinate as [29]:

\[
E_s = -\frac{\partial A_s}{\partial t} = \sum_i V_i \delta_p(s - s_i) \sin(\omega_{rf} t + \phi_{0i})
\]

(39)

where number of cavities is \( i \), with a voltage gain of \( V_i \) located at \( s = s_i \). The angular frequency is \( \omega_{rf} = h\omega_{\text{rev}} \), \( \phi_{0i} \) is the initial phase, and \( \delta_p \) is a periodic delta function with the circumference being the period. The RF cavity harmonic, \( h \), and the revolution frequency, \( \omega_{\text{rev}} \) are listed in Tab. 1. The vector potential can is:

\[
A_s = \sum_i \frac{V_i}{\omega_{rf}} \delta_p(s - s_i) \cos(\omega_{rf} t + \phi_{0i})
\]

(40)

From this point with the help of a canonical transformation, the synchrotron Hamiltonian for electrons is [30, 31],

\[
H_{\text{syn}} = -\frac{1}{2} \left( \frac{\eta_e}{\rho} - \frac{1}{\gamma^2} \right) \delta^2 - \sum_i \frac{eV_i}{\hbar\beta^2 E} \delta_p(\theta - \theta_i)(\cos(\phi + \phi_{0i}) + \phi \sin(\phi_{0i}))
\]

(41)

where the \( \beta c \) is the speed, \( \gamma \) is the Lorentz factor, and \( E \) is the energy of the particle. Here \( \phi = \omega_{rf} t \) is the RF phase relative to \( \phi_s \). The total Hamiltonian becomes the sum of \( H \) (Equ. 15), and \( H_{\text{syn}} \).
Figure 7: The top plot is of the $\beta_{x,y}$ Twiss functions. The middle plot is of the $\eta_{x,y}$ dispersion functions. The bottom plot is the lattice layout where the black box:dipole, blue box:quarupole, and the green box:sextupole. The diamond:red indicates the position of the RF cavity.
4 Dynamic Aperture

The dynamic aperture is the stable region within phase space after a large number of applications of the map, $\mathcal{M}$, onto a set of coordinates, $x$. The chromaticities, as mentioned in a previous section, is to shown in Tab. 3. The first, second, and third order chromaticities are:

<table>
<thead>
<tr>
<th>Chromatic Order</th>
<th>Horizontal</th>
<th>Vertical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural</td>
<td>-11.50</td>
<td>6.24</td>
</tr>
<tr>
<td>Linear</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>Second</td>
<td>37.79</td>
<td>-32.49</td>
</tr>
<tr>
<td>Third</td>
<td>-850.0</td>
<td>79.98</td>
</tr>
</tbody>
</table>

\[
\xi_{x,y}^{(1)} = \frac{\partial Q_{x,y}}{\partial \delta} \\
\xi_{x,y}^{(2)} = \frac{\partial Q_{x,y}^2}{\partial \delta^2} \\
\xi_{x,y}^{(3)} = \frac{\partial Q_{x,y}^3}{\partial \delta^3}
\]

(42)

If we have $\nu = |p, q|/|n|$ of Equ. 21, then we consider the particle motion to resonate on a $p^{th}$ or $q^{th}$ harmonic $n^{th}$ order (integer) resonance [32, 33]. The KAM-Theory describes the position of the stability border of a the non-integrable perturbed Hamiltonian system. The theorem states

![Figure 8: A scan of the horizontal and vertical tune over ±1.5% momentum deviation](image)

$q^{th}$ harmonic $n^{th}$ order (integer) resonance [32, 33]. The KAM-Theory describes the position of the stability border of a the non-integrable perturbed Hamiltonian system. The theorem states
that if the perturbation is small and the frequencies of the Hamiltonian are incommensurate, the motion is confined to an $N$-torus. The exception is the negligible set of initial conditions that result in meandering trajectories on the energy surface [34]. As the order increases, the resonance strength decreases. The separatrix may be defined here as the border trajectory between rotation and oscillation which separates motions of different type. If separatrices begin to touch each other, the resonances begin to overlap. The motion about overlapping resonances are unstable and are called stochastic oscillations/instability, see Appendix D.

Through normalizing the coordinates transformation into action-angle variable, $J, \phi$, found in Appendix B, the general n-D integrable system with a perturbation in which the Hamiltonian is a function of its angle has the form

$$H(J, \psi) = H_0(J) + \epsilon H_p(J, \psi)$$ \hspace{1cm} (43)

where $\epsilon H_p(J, \psi)$ describe the external perturbation.

4.1 Optimization

To describe the optimization of the dynamic aperture, we use the resonance base that is defined the Appendix C to define the resonance driving terms (RDT). We define the Lie generator as:

$$h_{ijklm} \equiv A_{ijklm} e^{i\phi_{ijklm}}$$ \hspace{1cm} (44)

where $A_{ijklm}$ and $\phi_{ijklm}$ are the amplitude and phase of the RDT. The first order driving terms that drive linear chromaticity are:

$$h_{11001} = \frac{1}{4} \sum_{i=1}^{N} [(k_1 L)_i - 2(k_2 L)_i \eta^{(1)}_{xi}] \beta_{xi} + \mathcal{O}(\delta^2)$$ \hspace{1cm} (45)

$$h_{00111} = -\frac{1}{4} \sum_{i=1}^{N} [(k_1 L)_i - 2(k_2 L)_i \eta^{(1)}_{yi}] \beta_{yi} + \mathcal{O}(\delta^2)$$

where $k_1$ and $k_2$ are the normalized strengths of the quadrupole and sextupole, respectively. $L$ is the length of the slice of the lattice at that moment.

$$h_{20001} = \frac{1}{8} \sum_{i=1}^{N} [(k_1 L)_i - 2(k_2 L)_i \eta^{(1)}_{xi}] \beta_{xi} e^{2i\phi_{xi}} + \mathcal{O}(\delta^2)$$

$$h_{00201} = -\frac{1}{8} \sum_{i=1}^{N} [(k_1 L)_i - 2(k_2 L)_i \eta^{(1)}_{yi}] \beta_{yi} e^{2i\phi_{yi}} + \mathcal{O}(\delta^2)$$ \hspace{1cm} (46)

$$h_{10002} = \frac{1}{8} \sum_{i=1}^{N} [(k_1 L)_i - 2(k_2 L)_i \eta^{(1)}_{xi}] \beta_{xi} \sqrt{\beta_{yi}} e^{2i\phi_{xi}} + \mathcal{O}(\delta^2)$$

We also have that complex conjugates $h_{20001} = h^*_{02001}$, $h_{00201} = h^*_{00201}$, and $h_{10002} = h^*_{10002}$. The synchrobetatron resonances are driven by $h_{20001}$ and $h_{00201}$. The second order dispersion is driven by $h_{10002}$. Equations 45 and 46 are known as chromatic terms due to their dependence on dispersion. The first order geometric terms that are independent of the dispersion and with frequencies $p = 1$, $p = 3$, and $p = 1, q = \pm 2$ drive the betatron modes are listed in Equation 97 of [35]. A more comprehensive list of driving terms can be found in [37]. Table 4 lists all the driving
The goal of the optimization is to minimize the absolute value of the first and second order terms and the real component of the terms $h_{22000}$, $h_{11110}$, and $h_{00220}$ which drive the amplitude dependent tune shift.

<table>
<thead>
<tr>
<th>Term</th>
<th>Resonance Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{20001}$</td>
<td>4.293213915535</td>
</tr>
<tr>
<td>$h_{00201}$</td>
<td>0.619985917889</td>
</tr>
<tr>
<td>$h_{10002}$</td>
<td>1.67164484610</td>
</tr>
<tr>
<td>$h_{21000}$</td>
<td>3.853470780730</td>
</tr>
<tr>
<td>$h_{30000}$</td>
<td>4.530626940065</td>
</tr>
<tr>
<td>$h_{10110}$</td>
<td>2.482651709279</td>
</tr>
<tr>
<td>$h_{10020}$</td>
<td>4.200984095308</td>
</tr>
<tr>
<td>$h_{10200}$</td>
<td>5.641184353058</td>
</tr>
<tr>
<td>$h_{22000}$</td>
<td>200.2780397596</td>
</tr>
<tr>
<td>$h_{11110}$</td>
<td>214.4308899946</td>
</tr>
<tr>
<td>$h_{00220}$</td>
<td>2.735127566092</td>
</tr>
<tr>
<td>$h_{11020}$</td>
<td>239.5874528930</td>
</tr>
<tr>
<td>$h_{31000}$</td>
<td>200.9172734928</td>
</tr>
<tr>
<td>$h_{20110}$</td>
<td>141.8863601545</td>
</tr>
<tr>
<td>$h_{11200}$</td>
<td>71.60543118530</td>
</tr>
<tr>
<td>$h_{20020}$</td>
<td>111.6841084502</td>
</tr>
<tr>
<td>$h_{20200}$</td>
<td>42.65041090370</td>
</tr>
<tr>
<td>$h_{00310}$</td>
<td>47.28542658925</td>
</tr>
<tr>
<td>$h_{00400}$</td>
<td>33.10986649867</td>
</tr>
</tbody>
</table>

In addition to the RDTs, the chromatic perturbations described in Appendix E must be corrected. These $W$ functions show the $\beta_{x,y}$-beating that occurs due to the momentum spread of the beam. By minimizing the absolute value of the functions, the momentum aperture of the lattice improves which in turn allows a dynamic aperture to be found at higher momenta. Figure 9 shows the $W$ functions for each element of the DR.

The dynamic aperture in physical space after optimization is shown in Fig. 10. The beam sizes, $\sigma_x$ and $\sigma_y$, at the location of observation are 0.462 mm and and 0.327 mm, respectively. The solid black line of the plot indicated the aperture of the beam pipe, 32.9 mm inner diameter. The dynamic aperture is scanned by the process of increasing the orbital amplitude of the tracked particle for 282497 turns which is the turns within two damping periods. Synchrotron oscillations, radiation damping, and stochastic radiation fluctuations are included within the tracking. A total of 30 evenly spaced phase angles were taken over a 180° span. The total number of sextupole families in the DR is six, where each arc has two sextupole families.

## 5 Radiation Damping

The DR is designed to minimize the momentum spread of the beam before injection into the RCS. The energy loss per turn, $U_0$ is

$$U_0 = \frac{2r_cE_0^4}{3(mc^2)^3}I_2$$

(47)
Figure 9: The W-function for the DR. Black: horizontal and red: vertical.
where $r_c$ is the classical electron radius, $mc^2$ is the rest mass energy, and $E_0$ is the nominal stored energy. The equilibrium emittances and energy spread are calculated using Equ. 16:

$$
\epsilon_x = \frac{C_q}{I_2 - I_{4a}} \gamma_0 I_{5a}
$$

$$
\sigma_p = \frac{\sigma_E}{E_0} = \gamma_0 \sqrt{\frac{C_q I_3}{2I_2 + I_{4z}}}
$$

(48)

and $C_q = 3.832 \times 10^{-13}$. The transverse and longitudinal damping for the electrons over a ±40 MeV energy range can be summarized in Tab. 5. The damping times are found using

$$
\tau_x = \frac{2pc}{J_x E_f \omega_{rev}}
$$

$$
\tau_y = \frac{2pc}{J_y E_f \omega_{rev}}
$$

$$
\tau_z = \frac{2pc}{J_z E_f \omega_{rev}}
$$

(50)

with $J_x$, $J_y$, and $J_z$ are the partition numbers that show the distribution of damping in the three degrees of freedom of system and $E_f$ is the final energy after one turn. The inverse of these
damping time are known as the damping coefficients, $\alpha_i$. The partition numbers are

$$J_x = 1 - \frac{I_{4a}}{I_2}, \quad J_x = 1 - \frac{I_{4b}}{I_2}, \quad J_z = 2 + \frac{I_{4z}}{I_2}$$

(51)

For tracking, a fifth order map of the DR is generated. The projected beam sizes and momentum spread from a tracked bunch at 400 MeV after four damping periods are shown in Fig. 11 and Fig. 12. It is clear that within the first damping period the beam sizes and momentum spread are reduced. The stochastic radiation fluctuations, which prevents the vertical emittance of the beam from becoming zero [38], is calculated while tracking 1000 particles. Over a 180 ms, twenty damping periods, is shown in Fig. 13 and Fig. 14. The tracking results show emittance blowup with the first three damping periods which can be prevented by avoiding the fifth order sum resonance. From the tracking results, over 87% of the beam survives for the twenty damping periods. Further study is needed to include the intensity dependent collective effects such as intrabeam scattering [39]. The RCS has a dynamic range of 45 which at a 400 MeV injection

Table 5: The damping times of the DR with respect to varying energy

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$\gamma$</th>
<th>Turns</th>
<th>$\tau_x$ (ms)</th>
<th>$\tau_y$ (ms)</th>
<th>$\tau_z$ (ms)</th>
<th>$\epsilon_x$ nm</th>
</tr>
</thead>
<tbody>
<tr>
<td>364</td>
<td>712.33023082</td>
<td>124958.16</td>
<td>18.4984</td>
<td>47.7527</td>
<td>114.0908</td>
<td>34.3787</td>
</tr>
<tr>
<td>368</td>
<td>720.15803555</td>
<td>120927.74</td>
<td>17.9018</td>
<td>46.2124</td>
<td>110.4087</td>
<td>35.1385</td>
</tr>
<tr>
<td>372</td>
<td>727.98584028</td>
<td>117068.78</td>
<td>17.3306</td>
<td>44.7377</td>
<td>106.8833</td>
<td>35.9066</td>
</tr>
<tr>
<td>376</td>
<td>735.81364502</td>
<td>113372.28</td>
<td>16.7834</td>
<td>43.3251</td>
<td>103.5065</td>
<td>36.683</td>
</tr>
<tr>
<td>380</td>
<td>743.64144975</td>
<td>109829.79</td>
<td>16.259</td>
<td>41.9713</td>
<td>100.2705</td>
<td>37.4677</td>
</tr>
<tr>
<td>384</td>
<td>751.46925449</td>
<td>106433.35</td>
<td>15.7562</td>
<td>40.6733</td>
<td>97.168</td>
<td>38.2607</td>
</tr>
<tr>
<td>388</td>
<td>759.29705922</td>
<td>103175.53</td>
<td>15.274</td>
<td>39.4283</td>
<td>94.1922</td>
<td>39.062</td>
</tr>
<tr>
<td>392</td>
<td>767.12486396</td>
<td>100049.32</td>
<td>14.8112</td>
<td>38.2336</td>
<td>91.3367</td>
<td>39.8716</td>
</tr>
<tr>
<td>396</td>
<td>774.95266869</td>
<td>97048.15</td>
<td>14.3669</td>
<td>37.0867</td>
<td>88.5955</td>
<td>40.6895</td>
</tr>
<tr>
<td>400</td>
<td>782.78047342</td>
<td>94165.82</td>
<td>13.9403</td>
<td>35.9852</td>
<td>85.9629</td>
<td>41.5158</td>
</tr>
<tr>
<td>404</td>
<td>790.60827816</td>
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<td>13.5303</td>
<td>34.9269</td>
<td>83.4335</td>
<td>42.3503</td>
</tr>
<tr>
<td>408</td>
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<td>33.9097</td>
<td>81.0025</td>
<td>43.1931</td>
</tr>
<tr>
<td>412</td>
<td>806.26388763</td>
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<td>12.7574</td>
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<td>78.665</td>
<td>44.0442</td>
</tr>
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<td>814.09169236</td>
<td>83713.39</td>
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<td>31.9908</td>
<td>76.4166</td>
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<td>420</td>
<td>821.91949709</td>
<td>81344.36</td>
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<td>31.0854</td>
<td>74.253</td>
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<td>11.7047</td>
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<td>72.1704</td>
<td>46.6475</td>
</tr>
<tr>
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<td>29.3747</td>
<td>70.165</td>
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<td>432</td>
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<td>74752.4</td>
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<tr>
<td>436</td>
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<td>27.7872</td>
<td>66.3716</td>
<td>49.3254</td>
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<tr>
<td>440</td>
<td>861.0582077</td>
<td>70748.8</td>
<td>10.4738</td>
<td>27.0363</td>
<td>64.5772</td>
<td>50.2347</td>
</tr>
</tbody>
</table>

energy experiences additional magnetic field errors due to the field at 400 MeV not being well defined. The damping ring, theory, can also serve as a booster to the RCS to energies up to 1 GeV. This will require that the DR in its current configuration to utilize superconducting dipole magnets. Appendix F gives a discussion on the feasibility of the DR being used as a booster to the RCS.
Figure 11: The projected transverse beam sizes from a tracked beam four damping periods. In the tracking model, radiation fluctuations are included. The initial beam sizes at the observation point, where $\beta_{x,y} = (4.18, 2.09)$ m, are $\sigma_{\beta_{x,y}} = (0.46, 0.33)$ mm. The beam sizes after one damping period $\sigma_{h1,v1} = (0.44, 0.25)$ mm, two damping periods $\sigma_{h2,v2} = (0.44, 0.20)$ mm, three damping periods $\sigma_{h3,v3} = (0.43, 0.16)$ mm and four damping periods $\sigma_{h4,v4} = (0.42, 0.12)$ mm.
Figure 12: The projected bunch length and energy spread from a tracked beam four damping periods. In the tracking model, radiation fluctuations are included. The initial bunch length is $\sigma_{zi} = 54$ mm and the energy spread, $\sigma_e = \beta^2 \delta$, is initially $\sigma_{e1} = 5.56 \times 10^{-3}$. The bunch length after one damping period is $\sigma_{z1} = 91$ mm, after two $\sigma_{z2} = 82$ mm, after three $\sigma_{z3} = 72$ mm, and after four $\sigma_{z4} = 69$ mm. The energy spread after one damping period is $\sigma_{e1} = 3.1 \times 10^{-3}$, after two $\sigma_{e2} = 2.7 \times 10^{-3}$, after three $\sigma_{e3} = 2.5 \times 10^{-3}$ and after four $\sigma_{e4} = 2.2 \times 10^{-3}$. 

![Damping Ring $\sigma_{z,e}$ vs Turn](image)
Figure 13: The projected transverse beam sizes from a tracked beam twenty damping periods. In the tracking model, radiation fluctuations are included. The initial beam sizes at the observation point, where $\beta_{x,y} = (4.18, 2.09)$ m, are $\sigma_{h_i,v_i} = (0.46, 0.33)$ mm. The final beam sizes are $\sigma_{h_f,v_f} = (0.41, 2.4 \times 10^{-3})$ mm.
Figure 14: The projected bunch length and energy spread from a tracked beam twenty damping periods. In the tracking model, radiation fluctuations are included. The initial bunch length is $\sigma_{zi} = 54\,\text{mm}$ and the final $\sigma_{zi} = 25\,\text{mm}$. The energy spread, $\sigma_e = \beta^2 \delta$, is initially $\sigma_{ei} = 5.56 \times 10^{-3}$. The final energy spread is $\sigma_{ef} = 0.81 \times 10^{-3}$. 

![Figure 14](image-url)
6 Conclusion

The RCS damping ring is presented. The damping times at 400 MeV, $\tau_x = 13.9403 \text{ ms}$, $\tau_y = 35.9852 \text{ ms}$, and $\tau_z = 85.9629 \text{ ms}$ were found. The projected emittances after tracking a single damping period are $\epsilon_x = 46.9 \text{ nm}$, $\epsilon_y = 3.1 \text{ nm}$, and $\epsilon_z = 280.65 \mu\text{m}$. The difference in the natural emittance calculated by the synchrotron radiation integrals, and the tracked beam is due to the use of combined function magnets used in the lattice. The horizontal emittance and bunch length are within 13% of the natural emittance and bunch length. The energy spread has a percent difference of 74% which is due to inclusion of the RF system. However, after 4 damping periods, the horizontal emittance is $\epsilon_x = 42.0 \text{ nm}$ which is a less than 1% difference in the emittance calculated by the radiation integrals. The energy spread after 7 damping periods has a percent difference of less than 2%.

7 Acknowledgments

We thank the RCS injector group for the initial inspiration on designing the DR. We thank C. Nieves-Rosada for his extensive work with blender to produce the 3D model of Fig. 1. We thank S. Peggs for his wonderful discussion on the DR.
A Lattice File (damping_ring.bmad: dr-indi-arc.var)

parameter[lattice] = "RCS damping ring"
parameter[geometry] = closed

parameter[e_tot] = .4e9
parameter[particle] = Electron
parameter[n_part] = 1.74762254084901367E+11 ! full current

bmad_com[max_aperture_limit] = 0.04/2
bmad_com[rel_tol_tracking] = 1E-3
bmad_com[abs_tol_tracking] = 1E-6
bmad_com[taylor_order] = 3
bmad_com[auto_bookkeeper] = F
bmad_com[csr_and_space_charge_on] = f
bmad_com[spin_tracking_on] = T
bmad_com[radiation_damping_on] = t
bmad_com[radiation_fluctuations_on] = t
bmad_com[absolute_time_tracking] = T

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

!--Constants

deg_to_rad = pi/180
rigidity=1.334255292039262

!~~cell parameters
arc_radius = 1.75
ndipole=8
bend_ang=((2*pi)/ndipole)
quadrature= .2
sex_length= 0.05!.2
drift_length=quad_length*1.1

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

dipole_length=(arc_radius*abs(sin(bend_ang)))-(2*quad_length)
focal_length = dipole_length/sqrt(2)

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
cell_length = 2*dipole_length+2*quad_length+4*drift_length

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

beginning[beta_a] = 9
beginning[beta_b] = 2

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

! cfms

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

dp_edge:dipole,L=dipole_length/2,ang=bend_ang/2!,k2=-17.1/rigidity
midd_dip:dipole,L=dipole_length,ang=bend_ang!,k2=-17.1/rigidity

edge_fquad:quadrupole,l=quad_length/2
edge_dquad:quadrupole,l=quad_length/2

mid_fquad:quadrupole,l=quad_length/2
mid_dquad:quadrupole,l=quad_length/2

arc_sf1:sextupole,l=sex_length,superimpose,ref = beginning, offset = 2.442822-drift_length/6,
    ref_origin = beginning, ele_origin = center
arc_sf2:sextupole,l=sex_length,superimpose,ref = beginning, offset = 3.700259-drift_length/6,
    ref_origin = beginning, ele_origin = center
arc_sf3:sextupole,l=sex_length,superimpose,ref = beginning, offset = 16.763653-drift_length/6,
ref_origin = beginning, ele_origin = center
arc_sf4:sextupole,l=sex_length,superimpose,ref = beginning, offset = 18.021090-drift_length/6,
ref_origin = beginning, ele_origin = center

!arc_sex:overlay={arc_sf1[k2]:focus,arc_sf2[k2]:focus,arc_sf3[k2]:focus,arc_sf4[k2]:focus},
var = {focus}, focus = 174.69941881
dip_sex:overlay={edge_dip:b2,midd_dip:b2}, var = {b2}, b2 = -12.8161380375

!dip_mid:marker,superimpose,ref = midd_dip, offset = 0,ref_origin =center, ele_origin = center

fba_drifti:drift,L=drift_length/3
fba_drifto:drift,L=drift_length/2
tba_beg:marker
tba_end:marker
str2arc:marker
tba_arc:line=(str2arc,edge_fquad,fba_drifto,2*edge_dquad,fba_drifto,
tba_beg,edge_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
midd_dip,fba_drifti,mid_fquad,fba_drifti,mid_fquad,fba_drifti,
edge_dip,tba_end,fba_drifto,2*edge_dquad,fba_drifto,edge_fquad,
str2arc)
tba_arc_closed:line=(2*tba_arc)
use,tba_arc

EDGE_FQUAD[K1] = 3.09273207594915E+00
MID_FQUAD[K1] = 1.24911958085089E+01
EDGE_DQUAD[K1] = -1.75424308161356E-03
EDGE_DIP[K1] = -3.22121444392064E+00
MIDD_DIP[K1] = -2.48408319508227E+00

! DIP_SEX[FOCUS] = -1.78885887599440E+01
! ARC_SEX[FOCUS] = 2.35968889193645E+02

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
cavity:rfcavity,l=0.8,voltage=450e3,harmon=10,superimpose,ref = 129, offset = 0,
ref_origin = center, ele_origin = center

zero_point:fiducial, origin_ele = 133

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

!match lines

!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

!Straight Section

L_cell=1.1*cell_length

str_drift:drift,L=((L_cell/2)-2*quad_length-2*sex_length)
str_drift_chic:drift,L=((L_cell/2)-quad_length)
str_qf:quadrupole,L=quad_length/2,k1=edge_fquad[k1]
str_qd:quadrupole,L=quad_length/2
str_sf:sextupole,L=sex_length
str_sd:sextupole,L=sex_length
str_foc:marker
str_def:marker
chic_beg:marker
chic_end:marker

!str:line=(str_qd,str_def,str_drift,str_qd,str_def,str_drift,str_qd,str_def,str_drift,EDGE_FQUAD,str_foc)
str:line=(EDGE_FQUAD,str_sf,str_drift,str_qd,str_def,str_drift,EDGE_FQUAD,str_foc)
use,str

EDGE_FQUAD[K1] = 3.06357595872161870E+000
MID_FQUAD[K1] = 1.25983648150456258E+001
EDGE_DQUAD[K1] = -9.40230260185378162E-003
STR_QD[K1] = -3.00106109855128445E+000
EDGE_DIP[K1] = -3.29212182468477943E+000
MIDD_DIP[K1] = -2.53523369616839434E+000
DIP_SEX[B2] = -2.24920670515815985E-003
ARC_SF1[K2] = 2.51685641178211199E+002
ARC_SF2[K2] = 1.97541452872327511E+002
ARC_SF3[K2] = 2.26908132981568599E+002
ARC_SF4[K2] = 2.89036261261280401E+002
STR_SF[K2] = 8.44737477223389561E+001
STR_SD[K2] = -8.471426757558283E+000

ring:line=(tba_arc,3*str,tba_arc,3*str)
use,ring

*[aperture_type] = elliptical
*[aperture_at] = both_ends
*[x_limit] = 0.04/2
*[y_limit] = 0.04/2

sbend::*[ds_step] = 5e-2
sbend::*[r0_mag] = 30e-3/2

quadrupole::*[ds_step] = 5e-2
quadrupole::*[r0_mag] = 30e-3/2

sextupole::*[ds_step] = 5e-2
sextupole::*[r0_mag] = 30e-3/2

*[scale_multiploes] = T

B Coordinate Transformation action-angle

We can write a simplified version of the Hamiltonian,

$$ H = \frac{p^2}{2} + \frac{K(s)}{2} x^2 $$

where we describe the reference particle only allowing only horizontal motion. We also describe transverse field ($A_x = A_y = 0$)only with no skew field. The periodic function $K(s)$, ($K(s) = \frac{1}{B_y} \frac{\partial B_x}{\partial s}$) equals $K(s + C)$ where $C$ is the circumference $2\pi R$. The momentum $p$, is in the normal plane and $p = \frac{dx}{ds}$. The common periodic solution of the Hamiltonian for the motion, $x$ of the particle is:

$$ x(s) = \sqrt{\epsilon_x} \beta_x(s) \cos (\phi(s) + \phi_0) $$
where we define the phase advance \( \phi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')} \). If we switch to the action-angle variables, \( J \) and \( \psi \), then the Hamiltonian can now be derived by:

\[
\psi = \int_0^s \frac{ds'}{\beta_x(s')} + \phi_0
\]

\[
F_1(x, \psi, s) = \frac{x^2}{2\beta} \left( \tan \psi - \frac{\beta'}{2} \right)
\]

where \( F_1(x, \psi, s) \) is the generating function. We define \( p = \frac{\partial F_1}{\partial x} \) and \( J = \frac{\partial F_1}{\partial \psi} \). We can write a new Hamiltonian as,

\[
\mathcal{H} = H + \frac{\partial F_1}{\partial s}
\]

We have \( p = x' \),

\[
x' = -\frac{x}{\beta} \left( \tan \psi - \frac{\beta'}{2} \right)
\]

\[
\tan^2 \psi = \left[ x' - \left( \frac{\beta' x}{2\beta} \right) \right]^2
\]

\[
J = \frac{\partial F_1}{\partial \psi} = \frac{x^2}{2\beta} \sec^2 \psi = \frac{x^2}{2\beta} \left[ 1 + \tan^2 \psi \right]
\]

\[
= \frac{x^2}{2\beta} \left[ 1 + \left( \left( x' - \frac{\beta' x}{2\beta} \right) \right)^2 \right]
\]

The Courant and Snyder invariant, \( J \), of the particle motion can be expressed as the emittance, \( \epsilon \), by

\[
\epsilon = 2J = \frac{1}{\beta} \left[ x^2 + \left( \beta x' - \frac{\beta' x}{2} \right)^2 \right]
\]

The action of the particle is:

\[
J = \frac{2}{\epsilon} = \frac{1}{2\beta} \left[ x^2 + \left( \beta x' - \frac{\beta' x}{2} \right)^2 \right]
\]

We have \( \mathcal{H}(J, \psi) = \frac{J}{\beta(s)} \), where \( s \) is still an independent. From the Hamiltonian equations, \( \frac{d\psi}{ds} = \frac{\partial \mathcal{H}}{\partial J} = \frac{1}{\beta(s)} \).

Remember, \( \int_0^s \frac{ds}{\beta_x(s)} \) is the phase advance and

\[
x = \sqrt{2\beta J} \cos \psi,
\]

\[
x' = \sqrt{2\beta J} \left[ \sin \psi - \frac{\beta' \cos \psi}{2} \right]
\]

We again rewrite the Hamiltonian,

\[
F_2(\psi, J_1) = J_1 \left[ \frac{2\pi
s}{C} - \int_0^s \frac{ds'}{\beta_x(s')} \right] + \psi J_1
\]

\[
\psi = \frac{\partial F_2}{\partial J_1} = \psi + \frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta_x(s')}
\]

with \( J_1 = \frac{\partial F_2}{\partial \psi} = J \) and

\[
H_1 = \frac{2\pi\nu}{C} J_1
\]
C Resonance Basis

*Normal Forms* are symplectic matrices that become “pure” rotations that have the dynamics of a given transfer matrix $\mathcal{M}$.

$$\mathcal{R} = A \mathcal{M} A^{-1}$$  (62)

where $A$ is normalizing map $\mathcal{M}$. $\mathcal{R}$ is the normal form of $\mathcal{M}$. The resonance basis can be generated by \[40\]:

$$f_2 = \sum_{k=1}^{N} \frac{\phi_k}{2} \left[ x_k^2 + (\xi_k - \bar{\xi}_k)p_k^2 \right] = \sum_{k=1}^{N} f_2^k$$

$$\mathcal{R} = e^{i f_2}$$

$$[f_2, h_k^\pm] = \frac{\partial f_2}{\partial x} \frac{\partial h_k^\pm}{\partial p} - \frac{\partial f_2}{\partial p} \frac{\partial h_k^\pm}{\partial x}$$

$$= \mp (i \xi_k + \bar{\xi}_k) \phi_k h_k^\pm = \mp \lambda h_k^\pm$$

$$h_k^\pm = x_k \pm (i \xi_k + \bar{\xi}_k) p_k$$

$$= \sqrt{2J_k} e^{i\phi_k} = \sqrt{2J_k} \cos \phi_k \pm i \sqrt{2J_k} \sin \phi_k$$

where $x_k$ and $p_k$ are the position and momentum variables for a given plane $x$, $y$, and $z$. For stable systems, $\xi_n = 1$ and $\bar{\xi}_n = 0$. The linear map generator, $h_2 = f_2$, is a *effective Hamiltonian* of the total Lie map where

$$M = A^{-1} TA$$  (64)

The $T$ matrix is the linear transformation

$$T = \begin{pmatrix}
T_{11} & T_{12} & 0 & 0 & 0 & \eta(s) - T_{12}T_{11}\eta(0) - T_{12}T_{12}\eta(0) \\
T_{21} & T_{22} & 0 & 0 & 0 & \eta(s) - T_{22}T_{21}\eta(0) - T_{22}T_{22}\eta(0) \\
0 & 0 & T_{33} & T_{34} & 0 & 0 \\
0 & 0 & T_{43} & T_{44} & 0 & 0 \\
A & B & 0 & 0 & 1 & -C\alpha_c + A\eta \sin \mu_x \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$  (65)

where $\alpha_c$ is the momentum compaction and $\mu(x)$ is the phase advance for the closed periodic lattice. The elements are defined as

$$T_{11} = \cos \mu_x + \alpha_x \sin \mu_x$$

$$T_{12} = \beta_x \sin \mu_x$$

$$T_{21} = -\gamma_x \sin \mu_x$$

$$T_{22} = \cos \mu_x - \alpha_x \sin \mu_x$$

$$T_{33} = \cos \mu_y + \alpha_y \sin \mu_y$$

$$T_{34} = \beta_y \sin \mu_y$$

$$T_{43} = -\gamma_y \sin \mu_y$$

$$T_{44} = \cos \mu_y - \alpha_y \sin \mu_y$$

$$A = \eta' - \eta T_{11} + \eta T_{21}$$

$$B = -\eta - \eta T_{22} + \eta T_{12}$$

and $A\eta = \gamma_x \eta^2 + 2\alpha_x \eta\eta' + \beta_x \eta'^2$. From here, the $M$ matrix can be simplified as

$$M = \begin{pmatrix}
\cos \mu_x & \sin \mu_x & 0 & 0 & 0 & 0 \\
-\sin \mu_x & \cos \mu_x & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \mu_y & \sin \mu_y & 0 & 0 \\
0 & 0 & -\sin \mu_y & \cos \mu_y & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\oint ds\eta(s)/\rho(s)\alpha_c \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$  (67)
The action-angle transformation allow us to write $h_s$ as

$$h_2 = -\mu_x J_x - \mu_y J_y - \frac{1}{2} \alpha_c \delta^2$$ (68)

The eigenmodes of $h_2$ are

$$|ijkl,m> = J_x^{(i+j)/2} J_y^{(k+l)/2} e^{(i-j)\phi_x} e^{(k-l)\phi_y} \delta^m$$ (69)

here we have that

$$h_2 : |ijkl,m> = H_i [(i-j)\mu_x + (k-l)\mu_y] |ijkl,m>$$ (70)

We have now that $|ijkl,m>$ are the resonance basis. The eigenmode expansion of the effective Hamiltonian to the $n^{th}$ generator can be written as:

$$h_n = \sum_{ijklm \geq 0} A_{ijklm}^n |ijkl,m>$$ (71)

where $i + j + k + l + m = n$ the resonance order.

D Definition of Chirikov Criterion

Let us start with the definition of an integrable Hamiltonian with action $J$ and constants $\alpha$, $\beta$, and $m$,

$$H_0 = \alpha J + \beta J^m$$ (72)

The Hamiltonian equations are

$$\frac{dJ}{dt} = -\frac{\partial H_0}{\partial \phi} = 0$$

$$\frac{d\phi}{dt} = \alpha + \beta m J^{m-1} \equiv \omega(J)$$ (73)

The solution are of $J$ = invariant and $\phi = \omega(J)t + \phi_0$. The perturbed Hamiltonian may be written as [41]:

$$H = H_0 + \epsilon H_\lambda = H_0 + \epsilon \cos \phi \sum_{l=-\infty}^{\infty} \delta(t - l)$$ (74)

where $l$ is an integer and $\epsilon$ is the perturbation width or size. Let us rewrite $\sum_{l=-\infty}^{\infty} \delta(t - l)$ as $\lambda + 2 \sum_{m=\lambda}^{\infty} \cos 2\pi mt$. The perturbation period is defined as $\omega_0 = 2\pi$. The Hamiltonian can now be written as

$$H = \alpha J + \beta J^m + \frac{\epsilon \omega_0}{2\pi} \left( \cos(\phi) + \sum_{m=\lambda}^{\infty} (\cos(\phi + m\omega_0 t) + \cos(\phi - m\omega_0 t)) \right)$$ (75)

For small $\epsilon$, we find $\cos \omega(J)t \pm m\omega_0 t$. When the function,

$$m\omega(J) = \pm \omega_0$$ (76)

a resonance is generated. The system is dominated by a single resonance if the action is at the resonance condition and the perturbation term is small. The Hamiltonian about the resonance is:

$$H_r = \alpha J + \beta J^m + \frac{\epsilon \omega_0}{2\pi} \cos(\phi - m\omega_0 t)$$ (77)

The Hamiltonian equations of motion are:

$$\frac{dJ}{dt} = -\frac{\partial H_r}{\partial \phi} \frac{\epsilon \omega_0}{2\pi} \sin(\phi - m\omega_0 t)$$

$$\frac{d\phi}{dt} = \frac{\partial H_r}{\partial J} = \alpha + \beta m J^{m-1} = \omega(J)$$ (78)
We have that the action, $J$, can be expressed as,

$$J = J_R + \delta J$$

(79)

and $\omega(J_R) = m\omega_0$. We can approximate the difference in the resonances as,

$$d(\phi - m\omega_0 t) = \omega(J) - \omega(J_R)$$

$$= \omega(J) - m\omega_0$$

$$\simeq \Delta J \frac{d\omega(J_R)}{dJ} \equiv \Delta J \omega'(J_R)$$

(80)

We let $\psi = \phi - m\omega_0 t$, and write the Hamiltonian equations of motion as

$$\frac{d\psi}{dt} = \Delta J \omega'(J_R)$$

$$\frac{d\Delta J}{dt} = \frac{\epsilon\omega_0}{2\pi} \sin \psi$$

(81)

The resonance width can be described as

$$\delta J = 4 \sqrt{\frac{\epsilon\omega_0}{2\pi \omega'(J_R)}}$$

(82)

A plausible condition for the occurrence of the stochastic instability seems to be the approach of resonances down to the distance of the order of a resonance size [42]. The reason why the motion of the system is known as stochastic, is due to the unstable motion of the system as though a random force was applied.

From the distance between resonances, $\Delta \omega = \omega_0$, and the width of resonances, $\delta \omega \simeq \frac{d\omega}{dt} \delta J = 4 \sqrt{\frac{\epsilon\omega_0 \omega'(J_R)}{2\pi}}$, we can define the Chirikov Criterion

$$s = \frac{\delta \omega}{\Delta \omega} = 4 \sqrt{\frac{\epsilon\omega'(J_R)}{2\pi \omega_0}} \geq 1$$

(83)

where we define $s$ as the stochasticity parameter.

**E  W-function Derivation**

If we allow $\delta = \frac{\Delta p}{p}$

$$\beta_\delta = \beta(\delta), \alpha_\delta = \alpha(\delta), \phi_\delta = \phi(\delta)$$

$$\beta_0 = \beta(0), \alpha_0 = \alpha(0), \phi_0 = \phi(0)$$

(84)

where $\alpha$ and $\beta$ are the Twiss parameters and $\phi$ the phase advance. We define the difference in the $\beta$ function for the particle with momentum error, $\delta p$, and the particle on momentum, $p$, as $\Delta \beta$. Analogously, we define define $\Delta \phi$ as the difference particle with momentum error and the on momentum particle.

$$\Delta \beta = \beta_\delta - \beta_0, \beta = \sqrt{\beta_\delta \beta_0}$$

$$\Delta \phi = \phi_\delta - \phi_0, \phi = \frac{\phi_\delta + \phi_0}{2}$$

(85)

We can define the chromatic $\beta$-beat, $B$ and chromatic change in the $\beta$ function slope with respect to the position, $A$, in the magnetic lattice as [43]:

$$B = \frac{\beta_\delta - \beta_0}{\sqrt{\beta_\delta \beta_0}} = \frac{\Delta \beta}{\beta}$$

$$A = \frac{\alpha_\delta \beta_0 - \alpha_0 \beta_\delta}{\sqrt{\beta_\delta \beta_0}}$$

(86)
Through achromatic regions $A^2 + B^2$ is invariant, and if we both variables to $\delta$ we have:

$$B = \lim_{\delta \to 0} \frac{\beta_i - \beta_0}{\sqrt{\beta_i \beta_0}} \times \frac{1}{\delta}$$

$$A = \lim_{\delta \to 0} \frac{\alpha_i \beta_0 - \alpha_0 \beta_i}{\sqrt{\beta_i \beta_0}} \times \frac{1}{\delta}$$

$$W = \sqrt{A^2 + B^2}$$

(F) Damping Ring as RCS Booster

The RCS dipole field at injection is unstable due to the main field and multipole field becoming unpredictable at $6.3369 \times 10^{-3}$. To correct this issue, developing the damping ring to ramp to 1 GeV has been proposed. The dipole strength of the DR at 1 GeV is $3.13 \text{T}$ which enters the superconducting regime. A table of the key parameter changes with respect to energy are shown is Tab. 6.

Table 6: Key parameters of the Damping Ring with energies greater than 450 MeV.

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$U_0$ (KeV)</th>
<th>Damping Time (ms)</th>
<th>$\alpha_x$ (1/s)</th>
<th>$\alpha_y$ (1/s)</th>
<th>$\alpha_z$ (1/s)</th>
<th>$\epsilon_x$ nm</th>
<th>$\delta$ $\times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>2.12</td>
<td>9.00</td>
<td>$6.9 \times 10^{-6}$</td>
<td>$2.7 \times 10^{-6}$</td>
<td>$1.1 \times 10^{-6}$</td>
<td>41.52</td>
<td>7.25</td>
</tr>
<tr>
<td>500</td>
<td>5.19</td>
<td>4.61</td>
<td>$1.3 \times 10^{-5}$</td>
<td>$5.2 \times 10^{-6}$</td>
<td>$2.2 \times 10^{-6}$</td>
<td>64.87</td>
<td>9.07</td>
</tr>
<tr>
<td>600</td>
<td>10.75</td>
<td>2.67</td>
<td>$2.3 \times 10^{-5}$</td>
<td>$9.0 \times 10^{-6}$</td>
<td>$3.8 \times 10^{-6}$</td>
<td>93.41</td>
<td>10.88</td>
</tr>
<tr>
<td>700</td>
<td>19.92</td>
<td>1.68</td>
<td>$3.7 \times 10^{-5}$</td>
<td>$1.4 \times 10^{-5}$</td>
<td>$6.0 \times 10^{-6}$</td>
<td>127.15</td>
<td>12.69</td>
</tr>
<tr>
<td>800</td>
<td>33.98</td>
<td>1.12</td>
<td>$5.5 \times 10^{-5}$</td>
<td>$2.1 \times 10^{-5}$</td>
<td>$8.9 \times 10^{-6}$</td>
<td>166.07</td>
<td>14.50</td>
</tr>
<tr>
<td>900</td>
<td>54.43</td>
<td>0.79</td>
<td>$7.8 \times 10^{-5}$</td>
<td>$3.0 \times 10^{-5}$</td>
<td>$1.3 \times 10^{-5}$</td>
<td>210.18</td>
<td>16.32</td>
</tr>
<tr>
<td>1000</td>
<td>82.96</td>
<td>0.58</td>
<td>$1.1 \times 10^{-4}$</td>
<td>$4.1 \times 10^{-5}$</td>
<td>$1.7 \times 10^{-5}$</td>
<td>259.47</td>
<td>18.13</td>
</tr>
</tbody>
</table>
References


A. Dragt and C. P. C. f. T. P. University of Maryland, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*. University of Maryland, Center for Theoretical Physics, Department of Physics, 1997.


C.-X. Wang, “Explicit formulas for 2nd-order driving terms due to sextupoles and chromatic effects of quadrupoles,” 4 2012.


