

# Integrals of Polynomial Functions over Spheres and Balls

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## 1 Definitions and Problem

Let  $B_n = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$  be the unit  $n$ -dimensional ball and  $S_{n-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$  be the unit  $(n-1)$ -dimensional sphere. Note that this sphere is the surface of the ball:  $S_{n-1} = \partial B_n$ .

We want to evaluate the integral

$$I_{a_1 \dots a_n}^{B_n} = \int_{B_n} \left( \prod_{i=1}^n x_i^{a_i} \right) d^n \mathbf{x}$$

of a general monomial term over the  $n$ -dimensional ball. It will be helpful to define a similar integral

$$I_{a_1 \dots a_n}^{S_{n-1}} = \int_{S_{n-1}} \left( \prod_{i=1}^n x_i^{a_i} \right) d^{n-1} \mathbf{x}$$

over the surface of the sphere. Finally, an integral over all space but weighted by an  $n$ -dimensional unit Gaussian distribution will also be useful:

$$I_{a_1 \dots a_n}^{g_n} = \int_{\mathbb{R}^n} \left( \prod_{i=1}^n x_i^{a_i} \right) \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}|\mathbf{x}|^2} d^n \mathbf{x}.$$

## 2 Separability of Gaussian Integral

The Gaussian integrand can be written as a product

$$I_{a_1 \dots a_n}^{g_n} = \int_{\mathbb{R}^n} \left( \prod_{i=1}^n x_i^{a_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2} \right) d^n \mathbf{x}.$$

Since each term in the product only depends on  $x_i$ , this is a separable integral that is the product of one-dimensional integrals:

$$I_{a_1 \dots a_n}^{g_n} = \prod_{i=1}^n \int_{\mathbb{R}} x_i^{a_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2} dx_i = \prod_{i=1}^n I_{a_i}^{g_1}.$$

### 3 Values of One-Dimensional Gaussian Integral $I_a^{g_1}$

The one-dimensional Gaussian integrals can be evaluated by noting

$$\frac{d}{dx} \left[ x^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right] = (ax^{a-1} - x^{a+1}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and therefore by integrating both sides over  $\mathbb{R}$ ,

$$0 = aI_{a-1}^{g_1} - I_{a+1}^{g_1}.$$

The recurrence of values can be started by noting  $I_0^{g_1} = 1$  because the Gaussian is normalised and  $I_1^{g_1} = 0$  because it is the integral of an odd function. Writing the recurrence as  $I_{a+2}^{g_1} = (a+1)I_a^{g_1}$  makes it clear that  $I_a^{g_1} = 0$  for  $a$  odd. For even values,

$$I_2^{g_1} = 1, \quad I_4^{g_1} = 3, \quad I_6^{g_1} = 3 \times 5, \quad I_8^{g_1} = 3 \times 5 \times 7, \quad \dots,$$

giving the general formula

$$I_{2a}^{g_1} = \prod_{b=1}^a (2b-1) = (2a-1)!! = \frac{(2a)!}{2^a a!}.$$

#### 3.1 Integrals on the Half Real Line $I_a^{h_1}$

The following calculations will also need 1D Gaussian integrals on the half real line defined by

$$I_a^{h_1} = \int_0^\infty x^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

Starting as before but integrating over  $[0, \infty)$  gives

$$\left[ x^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right]_{x=0}^\infty = aI_{a-1}^{h_1} - I_{a+1}^{h_1}.$$

The left hand side is equal to  $\frac{-1}{\sqrt{2\pi}}$  when  $a = 0$  and zero otherwise. The recurrence can still be rewritten  $I_{a+2}^{h_1} = (a+1)I_a^{h_1}$  valid for  $a \geq 0$ . The original  $a = 0$  case gives

$$\frac{-1}{\sqrt{2\pi}} = -I_1^{h_1} \quad \Rightarrow \quad I_1^{h_1} = \frac{1}{\sqrt{2\pi}}.$$

For even functions we have  $I_{2a}^{h_1} = \frac{1}{2}I_{2a}^{g_1}$  and in particular  $I_0^{h_1} = \frac{1}{2}$ . The recurrence gives the general formulae

$$I_{2a}^{h_1} = \frac{1}{2}(2a-1)!! \quad \text{and} \quad I_{2a+1}^{h_1} = \frac{1}{\sqrt{2\pi}} \prod_{b=1}^a 2b = \frac{1}{\sqrt{2\pi}}(2a)!! = \frac{1}{\sqrt{2\pi}}2^a a!.$$

### 4 Relation of Gaussian to Spherical Integral

The Gaussian integrand can also be split into parts that depend on radius  $r = |\mathbf{x}|$  and parts that do not:

$$I_{a_1 \dots a_n}^{g_n} = \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}r^2} r^{\sum a_i} \left( \prod_{i=1}^n \left( \frac{x_i}{r} \right)^{a_i} \right) d^n \mathbf{x}.$$

Defining  $\mathbf{x} = r\mathbf{u}$  so that  $\mathbf{u} \in S_{n-1}$  gives  $d^n \mathbf{x} = r^{n-1} dr d^{n-1} \mathbf{u}$  and

$$\begin{aligned} I_{a_1 \dots a_n}^{g_n} &= \int_0^\infty \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}r^2} r^{\sum a_i} r^{n-1} dr \int_{S_{n-1}} \left( \prod_{i=1}^n u_i^{a_i} \right) d^{n-1} \mathbf{u} \\ &= \frac{1}{\sqrt{2\pi}^{n-1}} I_{n-1+\sum a_i}^{h_1} I_{a_1 \dots a_n}^{S_{n-1}}. \end{aligned}$$

## 5 Evaluating the Spherical Integral

By the formula in the previous section,

$$I_{a_1 \dots a_n}^{S_{n-1}} = \frac{\sqrt{2\pi}^{n-1} I_{a_1 \dots a_n}^{g_n}}{I_{n-1+\sum a_i}^{h_1}}.$$

The separability of the Gaussian integrals allows this to be written with only  $I_a^{g_1}$  and  $I_a^{h_1}$  terms:

$$I_{a_1 \dots a_n}^{S_{n-1}} = \sqrt{2\pi}^{n-1} \frac{\prod_{i=1}^n I_{a_i}^{g_1}}{I_{n-1+\sum a_i}^{h_1}}.$$

## 6 Evaluating the Ball Integral

Integrating over a smaller sphere of radius  $r$  will introduce a factor of  $r^{n-1}$  from the change in surface area and a factor of  $r^{\sum a_i}$  from the change in the monomial term itself. Thus,

$$I_{a_1 \dots a_n}^{B_n} = \int_0^1 r^{n-1+\sum a_i} I_{a_1 \dots a_n}^{S_{n-1}} dr = \frac{1}{n + \sum a_i} I_{a_1 \dots a_n}^{S_{n-1}}.$$

Using the formula from the previous section, this can be written in terms of  $I_a^{g_1}$  and  $I_a^{h_1}$ :

$$I_{a_1 \dots a_n}^{B_n} = \frac{\sqrt{2\pi}^{n-1}}{n + \sum a_i} \frac{\prod_{i=1}^n I_{a_i}^{g_1}}{I_{n-1+\sum a_i}^{h_1}}.$$