# Estimates of the Recombination Rate for the Strong Hadron Cooling System in the EIC 

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#### Abstract

Monitoring the recombination process of the ions as they go through the cooling section serves as a crucial tool for aligning their energy with that of the electrons. In this note, we calculate the recombination rate for the Strong Hadron Cooling system based on the MBEC.

\section*{Introduction}

The strong hadron cooling system for EIC consists of the modulator, the microbunching amplifier and the kicker section. In the modulator and the kicker section, the electrons are comoving with the protons. If the relative velocity of an electron with respect to a proton is small enough, it can be captured by the proton and the resulting neutral particle, i.e. a hydrogen atom, will deviate from the designed trajectory and get lost around the cooling section. Since the probability of a proton capturing an electron depends on the relative velocity between them, one can align the energy of the two beams based on the number of hydrogen atoms detected by a recombination monitor. In this note, we estimate the rate at which the hydrogen atoms produced by the recombination process for the SHC in EIC.


## Equations

The recombination rate is calculated from the following equation

$$
\begin{equation*}
R=\gamma^{-2} N_{p} N_{\text {bunch }} \frac{\sigma_{e s}}{\sigma_{p s}} n_{e} \frac{L_{\text {cool }}}{L_{\text {ring }}} \alpha_{r}, \tag{1}
\end{equation*}
$$

where $\gamma$ is the relativistic energy factor of the electrons and the ions, $N_{p}$ is the number of protons in a bunch, $N_{\text {bunch }}$ is the number of the proton bunches in the storage ring, $\sigma_{e s}$ is the RMS bunch length of the electrons, $\sigma_{p s}$ is the R.M.S. bunch length of the protons, $n_{e}$ is the spatial density of the electrons in the lab frame, $L_{\text {cool }}$ is the length of the cooling section, $L_{\text {ring }}$ is the circumference of the proton storage ring,

$$
\begin{equation*}
\alpha_{r}=\frac{\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)\left|\vec{v}_{e}-\vec{v}_{i}\right| \sigma\left(\left|\vec{v}_{e}-\vec{v}_{i}\right|\right)}{\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)} \tag{2}
\end{equation*}
$$

is the recombination rate coefficient averaged over the velocity distribution of the electrons and that of the ions with the recombination cross section given by

$$
\begin{equation*}
\sigma(v)=A \frac{2 h v_{0}}{m_{e} v^{2}}\left[\ln \left(\sqrt{\frac{2 h v_{0}}{m_{e} v^{2}}}\right)+\gamma_{1}+\gamma_{2}\left(\frac{m_{e} v^{2}}{2 h v_{0}}\right)^{1 / 3}\right] \tag{3}
\end{equation*}
$$

where $A=2.11 \times 10^{-22} \mathrm{~cm}^{2}, \gamma_{1}=0.1402, \gamma_{2}=0.525, h v_{0}=Z^{2} \alpha^{2} m_{e}^{2} / 2=Z^{2} \times 13.6 \mathrm{eV}$, and $Z=1$ for protons. We assume that the transverse velocity distribution of both the electrons and that of the ions have the Gaussian profile, i.e.

$$
\begin{equation*}
f_{e}\left(v_{e}\right)=\frac{1}{2 \pi \beta_{e, x} \beta_{e, y}} \exp \left(-\frac{v_{e, x}^{2}}{2 \beta_{e, x}^{2}}-\frac{v_{e, y}^{2}}{2 \beta_{e, y}^{2}}\right) f_{e, z}\left(v_{e, z}\right) \tag{4}
\end{equation*}
$$

and the velocity distribution of the protons

$$
\begin{equation*}
f_{I}\left(v_{i}\right)=\frac{1}{2 \pi \beta_{i, x} \beta_{i, y}} \exp \left(-\frac{v_{i, x}^{2}}{2 \beta_{i, x}^{2}}-\frac{v_{i, y}^{2}}{2 \beta_{i, y}^{2}}\right) f_{i, z}\left(v_{i, z}\right) . \tag{5}
\end{equation*}
$$

Since the longitudinal velocity spreads of the electrons and the ions in the co-moving frame are significantly smaller than the transverse velocity spreads, we assume that the longitudinal velocity spread is zero for this analysis, i.e.

$$
\begin{equation*}
f_{e, z}\left(v_{e, z}\right)=\delta\left(v_{e, z}-v_{z 0}\right), \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{i, z}\left(v_{i, z}\right)=\delta\left(v_{i, z}\right) . \tag{7}
\end{equation*}
$$

Inserting eq. (3)-(7) into eq. (2) and carrying out the integration (APPENDIX B) yields

$$
\begin{gather*}
\alpha_{r}=A c \frac{h v_{0}}{m_{e} \beta_{x} \beta_{y}} \sqrt{\frac{2 h v_{0}}{m_{e} c^{2}}} \\
\int_{0}^{\infty} \frac{1}{\sqrt{y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}}}\left[-\frac{1}{2} \ln \left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)+\gamma_{1}+\gamma_{2}\left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)^{1 / 3}\right] \exp \left(-\frac{h v_{0}}{k T_{1}} y\right) I_{0}\left(\frac{h v_{0}}{k T_{1}} \eta \cdot y\right) d y^{\prime} \tag{8}
\end{gather*}
$$

where

$$
\begin{align*}
k T_{1} & \equiv \frac{2 m_{e} \beta_{x}^{2} \beta_{y}^{2}}{\beta_{x}^{2}+\beta_{y}^{2}}  \tag{9}\\
\eta & \equiv \frac{\beta_{x}^{2}-\beta_{y}^{2}}{\beta_{x}^{2}+\beta_{y}^{2}} \tag{10}
\end{align*}
$$

$$
\begin{equation*}
\beta_{x} \equiv \sqrt{\beta_{i, x}^{2}+\beta_{e, x}^{2}}, \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta_{y} \equiv \sqrt{\beta_{i, y}^{2}+\beta_{e, y}^{2}} . \tag{12}
\end{equation*}
$$

## Estimates of the recombination rate for the SHC in EIC

With the parameters listed in table 1 and expression given by eq. (8), the recombination rate coefficient is calculated and shown in fig. 1 (left). As a comparison, fig. 1 (right) shows the recombination rate coefficient calculated for the CeC experiment at RHIC with parameters assumed in table 2 (see APPENDIX A for the expression of calculating the recombination rate with round electron and ion beams). As shown in fig. 1, the recombination rate expected for the SHC in the EIC is about a factor of two lower than that in the CeC experiment. However, there are 1160 proton bunches in the EIC while there is only one hadron bunch in the CeC experiment, which may lead to significantly higher background noise for detecting the recombination signal in the SHC as compared to that of the CeC experiment.

| Proton beam |  | Electron beam |  |
| :---: | :---: | :---: | :---: |
| Energy, gamma | 293.09 | Energy, gamma | 293.09 |
| Horizontal emittance | 11.3 nm | Horizontal norm. emittance | $2.8 \mu \mathrm{~m}$ |
| Vertical emittance | 1 nm | Vertical norm. emittance | $2.8 \mu \mathrm{~m}$ |
| $\beta_{\mathrm{x}}$ at modulator | 17.09 m | $\beta_{\mathrm{x}}$ at modulator | 76.4 m |
| $\beta_{\mathrm{y}}$ at modulator | 17.08 m | $\beta_{\mathrm{y}}$ at modulator | 10.15 m |
| $\sigma_{v x} / c$ at modulator (beam frame) | 7.537E-3 | $\sigma_{v x} / c$ at modulator (beam frame) | $3.277 \mathrm{E}-3$ |
| $\sigma_{\mathrm{vy}} / \mathrm{c}$ at modulator (beam frame) | $2.243 \mathrm{E}-3$ | $\sigma_{\mathrm{vy}} / \mathrm{c}$ at modulator (beam frame) | $8.992 \mathrm{E}-3$ |
| Energy spread | 6.8E-4 | Energy spread | 1E-4 |
| $\sigma_{\mathrm{vz}} / \mathrm{c}$ at modulator (beam frame) | $6.8 \mathrm{E}-4$ | $\sigma_{\mathrm{vz}} / \mathrm{c}$ at modulator (beam frame) | 1E-4 |
| Population | 6.9 E 10 | Bunch charge | 1 nC |
| RMS bunch length | 6 cm | RMS bunch length | 7 mm |

Table 1: parameters used for the estimates of the recombination rate in the SHC of the EIC


Figure 1: recombination coefficient as calculated for the SHC in EIC (left) and that for the CeC experiment (right). The abscissa is the longitudinal average velocity of the electrons with respect to the average velocity of the ions and the ordinate is the recombination rate coefficient, $\alpha_{r}$, as defined in eq. (1).

| Au beam |  | Electron beam |  |
| :---: | :---: | :---: | :---: |
| Energy, gamma | 28.5 | Energy, gamma | 28.5 |
| Horizontal emittance | 87.7 nm | Horizontal norm. emittance | $3 \mu \mathrm{~m}$ |
| Vertical emittance | 87.7 nm | Vertical norm. emittance | $3 \mu \mathrm{~m}$ |
| $\beta_{x}$ at cooling section | 5 m | $\beta_{x}$ at cooling section | 2.3 m |
| $\beta_{y}$ at modulator | 5 m | $\beta_{\mathrm{y}}$ at modulator | 2.3 m |
| $\sigma_{v x} / \mathrm{c}$ at modulator (beam frame) | 4E-3 | $\sigma_{v x} / c$ at modulator (beam frame) | 6.1E-3 |
| $\sigma_{v y} / c$ at modulator (beam frame) | 4E-3 | $\sigma_{v y} / c$ at modulator (beam frame) | 6.1E-3 |
| Energy spread | 1.3E-3 | Energy spread | 5E-4 |
| $\sigma_{\mathrm{vz}} / \mathrm{c}$ at modulator (beam frame) | 1.3E-3 | $\sigma_{\mathrm{vz}} / \mathrm{c}$ at modulator (beam frame) | 5E-4 |
| Population | 8.4E8 | Bunch charge | 1.5 nC |
| RMS bunch length | 1.05 m | RMS bunch length | 3.6 mm |

Table 2: Parameters used for the estimate of the recombination rate in the CeC experiment.


Figure 2: (left) The recombination rate as estimated from eq. (1) for the SHC in the EIC with the assumption of 35 meters of the cooling section and 1160 proton bunches; (right) the recombination rate as estimated for the CeC experiment at RHIC with the assumption of 1 Au bunch and 14 meters of the cooling section.

| Au beam |  | Electron beam |  |
| :---: | :---: | :---: | :---: |
| Energy, gamma | 4.1 | Energy, gamma | 4.1 |
| Horizontal emittance | 385 nm | Horizontal norm. emittance | $2.5 \mu \mathrm{~m}$ |
| Vertical emittance | 385 nm | Vertical norm. emittance | $2.5 \mu \mathrm{~m}$ |
| $\beta_{x}$ at cooling section | 25 m | $\beta_{x}$ at cooling section | 30 m |
| $\beta_{y}$ at cooling section | 25 m | $\beta_{\mathrm{v}}$ at modulator | 30 m |
| $\sigma_{v x} / c$ at modulator (beam frame) | 5.2E-4 | $\sigma_{v x} / \mathrm{c}$ at modulator (beam frame) | 5.9E-4 |
| $\sigma_{v y} / \mathrm{c}$ at modulator (beam frame) | $5.2 \mathrm{E}-4$ | $\sigma_{v y} / \mathrm{c}$ at modulator (beam frame) | $5.9 \mathrm{E}-4$ |
| Energy spread | 5E-4 | Energy spread | 3E-4 |
| $\sigma_{\mathrm{v}} / \mathrm{c}$ at modulator (beam frame) | 5E-4 | $\sigma_{\mathrm{vz}} / \mathrm{c}$ at modulator (beam frame) | 3E-4 |
| Population | 5 E8 | Bunch charge (1 bunch, 3 nC for 30 bunches) | 0.1 nC |
| RMS bunch length | 3.3 m | RMS bunch length | 3.6 mm |

Table 3: Parameters used for the estimate of the recombination rate in the LEReC system.

Using eq. (A3) and the parameters of LEReC shown in table 3, we can also calculate the recombination rate for the Low Energy RHIC electron Cooling (LEReC). The recombination rate is shown in fig. 3.


Figure 3: The recombination rate estimated for the LEReC as calculated from eq. (A3) (left) and that taken from [1](right). The factor of 2 difference between the two estimate comes from the difference in the current used in the estimate. The left plot assumes the peak current at the bunch center and the right plot is generated with averaging the electrons' current over the longitudinal bunch profile.

## Summary

As shown in fig. 2, the recombination rate expected for the SHC in the EIC is about a factor of two lower than that in the CeC experiment. However, there are 1160 proton bunches in the EIC while there is only one hadron bunch in the CeC experiment, which may lead to significantly higher background noise for detecting the recombination signal in the SHC as compared to that of the CeC experiment. We also reproduced the results of the expected recombination rate for the LEReC system as estimated in C-A/AP/582 and confirmed that the estimated recombination rate for the SHC in the EIC will be $2^{\sim} 3$ orders of magnitudes lower than that in the LEReC.

## REFERENCES

[1] S. Zhang and M. Blaskiewicz, Recombination monitor, Report No. C-A/AP/582 (2017).
[2] G. Wang, arXiv: 2109.13980 (2021).

## APPENDIX A: Expression for calculating the recombination rate with cylindrically symmetric electron beam and ion beam

For round electron beam and ion beam with Gaussian transverse velocity distribution, i.e.

$$
\begin{gather*}
f_{e}\left(v_{e}\right)=\frac{1}{(2 \pi)^{3 / 2} \beta_{e, 1}^{2} \beta_{e, z}} \exp \left(-\frac{v_{e, x}^{2}+v_{e, y}^{2}}{2 \beta_{e, 1}^{2}}\right) \exp \left(-\frac{\left(v_{e, z}-v_{z 0}\right)^{2}}{2 \beta_{e, z}^{2}}\right),  \tag{A1}\\
f_{I}\left(v_{i}\right)=\frac{1}{(2 \pi)^{3 / 2} \beta_{i, 1}^{2} \beta_{i, 2}} \exp \left(-\frac{v_{i, x}^{2}+v_{i, y}^{2}}{2 \beta_{i, 1}^{2}}\right) \exp \left(-\frac{v_{i, 2}^{2}}{2 \beta_{i, z}^{2}}\right) \tag{A2}
\end{gather*}
$$

with $\beta_{e, x}=\beta_{e, y}=\beta_{e, \perp}$ and $\beta_{i, x}=\beta_{i, y}=\beta_{i, \perp}$, eq. (2) can be reduced to [2]

$$
\begin{align*}
& \alpha_{r}=\frac{A c h v_{0}}{k T_{e i} \eta} \sqrt{\frac{2 h v_{0}}{m_{e} c^{2}}} \exp \left(\frac{m_{e} v_{z 0}^{2}}{2 k T_{e i} \eta^{2}} \int_{0}^{\infty}\left(-\ln y+\gamma_{1}+\gamma_{2} y^{2 / 3}\right) \exp \left(-\frac{h v_{0}}{k T_{e i}} y^{2}\right),\right. \\
& \times\left\{\operatorname{Erf}\left[\eta \sqrt{\frac{h v_{0}}{2 k T_{z}}}\left(y+\sqrt{\frac{m_{e}}{2 h v_{0}}} \frac{v_{z 0}}{\eta^{2}}\right)\right]-E r f\left[\eta \sqrt{\frac{h v_{0}}{2 k T_{z}}}\left(\frac{v_{z 0}}{\eta^{2}} \sqrt{\frac{m_{e}}{2 h v_{0}}}-y\right)\right]\right\} d y \tag{A3}
\end{align*}
$$

where

$$
\begin{gather*}
T_{e i}=\frac{m_{e}}{2 k}\left(\beta_{e, x}^{2}+\beta_{e, y}^{2}+\beta_{i, x}^{2}+\beta_{i, y}^{2}\right)=\frac{m_{e}}{k}\left(\beta_{e, \perp}^{2}+\beta_{i, \perp}^{2}\right)  \tag{A4}\\
T_{z}=\frac{m_{e}}{2 k}\left(\beta_{e, z}^{2}+\beta_{i, z}^{2}\right) \tag{A5}
\end{gather*}
$$

and

$$
\begin{equation*}
\eta=\sqrt{1-\frac{2 T_{z}}{T_{e i}}} \tag{A6}
\end{equation*}
$$

## APPENDIX B: Derivation of the recombination rate for the longitudinally cold electron and ion

 beam without cylindrical symmetryFor the electrons and ions with the following distribution,

$$
\begin{equation*}
f_{e}\left(v_{e}\right)=\frac{1}{2 \pi \beta_{e, x} \beta_{e, y}} \exp \left(-\frac{v_{e, x}^{2}}{2 \beta_{e, x}^{2}}-\frac{v_{e, y}^{2}}{2 \beta_{e, y}^{2}}\right) \delta\left(v_{e, z}-v_{z 0}\right) \tag{B1}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{I}\left(v_{i}\right)=\frac{1}{2 \pi \beta_{i, x} \beta_{i, y}} \exp \left(-\frac{v_{i, x}^{2}}{2 \beta_{i, x}^{2}}-\frac{v_{i, y}^{2}}{2 \beta_{i, y}^{2}}\right) \delta\left(v_{i, z}\right), \tag{B2}
\end{equation*}
$$

the nominator of the R.H.S. of eq. (2) becomes

$$
\begin{gather*}
\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)\left|\vec{v}_{e}-\vec{v}_{i}\right| \sigma\left(\left|\vec{v}_{e}-\vec{v}_{i}\right|\right) \\
=\frac{c_{1}}{(2 \pi)^{2} \beta_{e, x} \beta_{e, y} \beta_{i, x} \beta_{i, y}} \int_{-\infty}^{\infty} d v_{x} d v_{y} \sqrt{v_{x}^{2}+v_{y}^{2}+v_{z 0}^{2}} \sigma\left(\sqrt{v_{x}^{2}+v_{y}^{2}+v_{z 0}^{2}}\right) \exp \left(-\frac{v_{x}^{2}}{2 \beta_{x}^{2}}-\frac{v_{y}^{2}}{2 \beta_{y}^{2}}\right)^{\prime} \tag{B3}
\end{gather*}
$$

where

$$
\begin{gather*}
c_{1} \equiv \int_{-\infty}^{\infty} \exp \left(-\mu_{x} \tilde{v}_{i, x}^{2}\right) \exp \left(-\mu_{y} \tilde{v}_{i, y}^{2}\right) d \tilde{v}_{i, x} d \tilde{v}_{i, y}=\frac{\pi}{\sqrt{\mu_{x} \mu_{y}}},  \tag{B4}\\
\mu_{x} \equiv \frac{1}{2 \beta_{e, x}^{2}}+\frac{1}{2 \beta_{i, x}^{2}},  \tag{B5}\\
\mu_{y} \tag{B6}
\end{gather*}>\frac{1}{2 \beta_{e, y}^{2}}+\frac{1}{2 \beta_{i, y}^{2}},
$$

and

$$
\begin{equation*}
\beta_{y} \equiv \sqrt{\beta_{i, y}^{2}+\beta_{e, y}^{2}} . \tag{B8}
\end{equation*}
$$

The denominator of the R.H.S. of eq. (2) is

$$
\begin{equation*}
\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)=\frac{c_{1}}{(2 \pi)^{2} \beta_{e, x} \beta_{e, y} \beta_{i, x} \beta_{i, y}} \int_{-\infty}^{\infty} d v_{x} d v_{y} \exp \left(-\frac{v_{x}^{2}}{2 \beta_{x}^{2}}-\frac{v_{y}^{2}}{2 \beta_{y}^{2}}\right)=\frac{c_{1} \beta_{x} \beta_{y}}{2 \pi \beta_{e, x} \beta_{e, y} \beta_{i, x} \beta_{i, y}} . \tag{B9}
\end{equation*}
$$

Inserting eq. (B3) and eq. (B9) into eq. (2) yields

$$
\begin{align*}
\alpha_{r} & =\frac{\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)\left|\vec{v}_{e}-\vec{v}_{i}\right| \sigma\left(\left|\vec{v}_{e}-\vec{v}_{i}\right|\right)}{\int_{-\infty}^{\infty} d^{3} v_{i} d^{3} v_{e} f_{e}\left(v_{e}\right) f_{I}\left(v_{i}\right)}  \tag{B10}\\
& =\frac{1}{2 \pi \beta_{x} \beta_{y}} \int_{-\infty}^{\infty} \sqrt{v_{x}^{2}+v_{y}^{2}+v_{z 0}^{2}} \sigma\left(\sqrt{v_{x}{ }^{2}+v_{y}^{2}+v_{z 0}^{2}}\right) \exp \left(-\frac{v_{x}{ }^{2}}{2 \beta_{x}^{2}}-\frac{v_{y}^{2}}{2 \beta_{y}^{2}}\right) d v_{x} d v_{y} .
\end{align*}
$$

Eq. (B10) can be integrated in the polar system as the following:

$$
\begin{align*}
\alpha_{r} & =\frac{1}{2 \pi \beta_{x} \beta_{y}} \int_{-\infty}^{\infty} \sqrt{v_{x}^{2}+v_{y}^{2}+v_{z 0}^{2}} \sigma\left(\sqrt{v_{x}^{2}+v_{y}^{2}+v_{z 0}^{2}}\right) \exp \left(-\frac{v_{x}{ }^{2}}{2 \beta_{x}^{2}}-\frac{v_{y}^{2}}{2 \beta_{y}^{2}}\right) d v_{x} d v_{y} \\
& =\frac{1}{2 \pi \beta_{x} \beta_{y}} \int_{0}^{\infty} \int_{0}^{2 \pi} \sqrt{v^{2}+v_{z 0}^{2}} \sigma\left(\sqrt{v^{2}+v_{z 0}^{2}}\right) \exp \left(-\frac{v^{2} \cos ^{2} \theta}{2 \beta_{x}^{2}}-\frac{v^{2} \sin ^{2} \theta}{2 \beta_{y}^{2}}\right) v d v d \theta,  \tag{B11}\\
& =\frac{1}{2 \beta_{x} \beta_{y}} \int_{0}^{\infty} \sqrt{x+v_{z 0}^{2}} \sigma\left(\sqrt{x+v_{z 0}^{2}}\right) \exp \left(-\frac{x}{4 \beta_{1}^{2}}\right) I_{0}\left(-\frac{x}{4 \xi}\right) d x
\end{align*}
$$

where

$$
\begin{equation*}
\beta_{1}^{2} \equiv \frac{\beta_{x}^{2} \beta_{y}^{2}}{\beta_{x}^{2}+\beta_{y}^{2}} \tag{B12}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi \equiv-\frac{\beta_{x}^{2} \beta_{y}^{2}}{\beta_{x}^{2}-\beta_{y}^{2}} \tag{B13}
\end{equation*}
$$

Inserting eq. (3) into eq. (B11) yields

$$
\begin{gather*}
\alpha_{r}=A c \frac{h v_{0}}{m_{e} \beta_{\chi} \beta_{y}} \sqrt{\frac{2 h v_{0}}{m_{e} c^{2}}} \\
\int_{0}^{\infty} \frac{1}{\sqrt{y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}}}\left[-\frac{1}{2} \ln \left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)+\gamma_{1}+\gamma_{2}\left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)^{1 / 3}\right] \exp \left(-\frac{h v_{0}}{2 m_{e} \beta_{1}^{2}} y\right) I_{0}\left(-\frac{h v_{0}}{2 m_{e} \xi} y\right) d y \tag{B14}
\end{gather*}
$$

where I used $y=\frac{m_{e} x}{2 h v_{0}}$. If I define

$$
\begin{equation*}
k T_{1}=2 m_{e} \beta_{1}^{2}=2 m_{e} \frac{\beta_{x}^{2} \beta_{y}^{2}}{\beta_{x}^{2}+\beta_{y}^{2}}, \tag{B15}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta=-\frac{\beta_{1}^{2}}{\xi}=\frac{\beta_{x}^{2}-\beta_{y}^{2}}{\beta_{x}^{2}+\beta_{y}^{2}}, \tag{B16}
\end{equation*}
$$

eq. (B14) becomes

$$
\begin{gather*}
\alpha_{r}=A c \frac{h v_{0}}{m_{e} \beta_{x} \beta_{y}} \sqrt{\frac{2 h v_{0}}{m_{e} c^{2}}} \\
\int_{0}^{\infty} \frac{1}{\sqrt{y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}}}\left[-\frac{1}{2} \ln \left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)+\gamma_{1}+\gamma_{2}\left(y+\frac{m_{e} v_{z 0}^{2}}{2 h v_{0}}\right)^{1 / 3}\right] \exp \left(-\frac{h v_{0}}{k T_{1}} y\right) I_{0}\left(\frac{h v_{0}}{k T_{1}} \eta \cdot y\right) d y \tag{B17}
\end{gather*}
$$


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