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EXACT MAPS TO SECOND ORDER FOR SELECTED ELEMENTS IN MAD-X VARIABLES

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EXACT MAPS TO SECOND ORDER FOR SELECTED ELEMENTS IN MAD-X VARIABLES

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Abstract

This report presents calculations for exact maps to second order about an arbitrary orbit for certain elements: drifts, solenoids, and rotations about a transverse axis. These expressions were used in recent updates to MAD-X, and thus the phase space variables used are those of MAD-X. Formulas are given here for the final orbit and its first and second derivatives with respect to the incoming phase space coordinates for a drift, solenoid, and coordinate system rotations about transverse axes. I do not claim these results to be new: these expressions are well-known, and are presented here only for reference. This document does not collect all maps that could be expressed exactly, in particular dipoles are omitted.

INTRODUCTION

Exact expressions for the maps through certain elements are well known; see, for instance, Ref. [1]. For the `TWISS` command in MAD-X, the map about the orbit and its first and second derivatives are needed in general. MAD version 8 would preform computations using third order maps about the zero phase space vector [2], and that model was originally carried over to MAD-X. However, when a particle follows a trajectory that is far off axis thorough a magnet (or a drift), this could introduce unnecessary errors into the calculation. Systems with such off-axis design orbits often require coordinate transformations to be implemented; thus I include expressions for the nontrivial forms of those as well (rotations about a transverse axis). For implementation of the exact maps in MAD-X, expressions for the map and its first two derivatvies are needed; this report gives those expressions, sufficient background to understand their calculation, and forms to improve numerical properties in a couple cases. The only cases included are the drift, the solenoid, and rotations about a transverse axis. I did not include expressions for a dipole, since the model in MAD-X handles cases where there would be no exact expression (a dipole with a quadrupole component, for instance). Other cases cannnot be expressed exactly (a quadrupole, for instance); though with some approximations many expressions could be found, I do not propose to derive them here.

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VARIABLES AND HAMILTONIAN

The transverse variables are X, $P_x = p_x/p_0$, Y, and $P_y = p_y/p_0$, where p_x and p_y are the physical momenta and p_0 is an arbitrary reference momentum in the same physical dimensions. The Hamiltonian, with s as the independent variable is scaled by p_0. The longitudinal coordinate is $T = -c[t - t_0(s)]$, and its conjugate momentum is $P_t = (E - E_0)/(p_0 c)$. E_0 is the total energy corresponding to p_0, and similarly the relativistic parameters β_0 and γ_0 used below will also correspond to p_0.

Hamilton's equations of motion in these variables are

$$\frac{dX}{ds} = \frac{\partial H}{\partial P_x} \qquad \frac{dP_x}{ds} = -\frac{\partial H}{\partial X} \tag{1}$$

$$\frac{dY}{ds} = \frac{\partial H}{\partial P_y} \qquad \frac{dP_y}{ds} = -\frac{\partial H}{\partial Y} \tag{2}$$

$$\frac{dT}{ds} = \frac{\partial H}{\partial P_t} \qquad \frac{dP_t}{ds} = -\frac{\partial H}{\partial T} \tag{3}$$

DRIFT

In these variables, the Hamiltonian for a drift is

$$-\sqrt{1 + 2\beta_0^{-1}P_t + P_t^2 - P_x^2 - P_y^2} = -P_s \tag{4}$$

Thus the map for a drift of length L is

$$X_1 = X_0 + \frac{P_{x0}}{P_{s0}}L \qquad P_{x1} = P_{x0} \tag{5}$$

$$Y_1 = Y_0 + \frac{P_{y0}}{P_{s0}}L \qquad P_{y1} = P_{y0} \tag{6}$$

$$T_1 = T_0 + \frac{L}{\beta_0} - \frac{\beta_0^{-1} + P_{t0}}{P_{s0}}L \qquad P_{y1} = P_{y0} \tag{7}$$

For better numerical precision, the time equation can be written as

$$T_1 = T_0 + \frac{\gamma_0^{-2}(2\beta_0^{-1}P_{t0} + P_{t0}^2) - P_{x0}^2 - P_{y0}^2}{\beta_0^2 P_{s0}(\beta_0^{-1}P_{s0} + \beta_0^{-1} + P_t)}L \tag{8}$$

The nontrivial first derivatives of this map are given by

$$\frac{\partial X_1}{\partial P_{x0}} = \left(\frac{1}{P_{s0}} + \frac{P_{x0}^2}{P_{s0}^3}\right)L \tag{9}$$

$$\frac{\partial X_1}{\partial P_{y0}} = \frac{P_{x0}P_{y0}}{P_{s0}^3}L \tag{10}$$

$$\frac{\partial X_1}{\partial P_{t0}} = -\frac{P_{x0}(\beta_0^{-1} + P_{t0})}{P_{s0}^3}L \tag{11}$$

$$\frac{\partial Y_1}{\partial P_{y0}} = \left(\frac{1}{P_{s0}} + \frac{P_{y0}^2}{P_{s0}^3}\right)L \tag{12}$$

$$\frac{\partial Y_1}{\partial P_{t0}} = -\frac{P_{y0}(\beta_0^{-1}+P_{t0})}{P_{s0}^3}L \tag{13}$$

$$\frac{\partial T_1}{\partial P_{t0}} = -\frac{L}{P_{s0}} + \frac{(\beta_0^{-1}+P_{t0})^2}{P_{s0}^3}L \tag{14}$$

and the symmetry conditions

$$\frac{\partial Y_1}{\partial P_{x0}} = \frac{\partial X_1}{\partial P_{y0}} \qquad \frac{\partial T_1}{\partial P_{x0}} = \frac{\partial X_1}{\partial P_{t0}} \qquad \frac{\partial T_1}{\partial P_{y0}} = \frac{\partial Y_1}{\partial P_{t0}} \tag{15}$$

Some of these can be rewritten for improved numerical properties

$$\frac{\partial X_1}{\partial P_{x0}} = \frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2-P_{y0}^2}{P_{s0}^3}L \tag{16}$$

$$\frac{\partial Y_1}{\partial P_{y0}} = \frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2-P_{x0}^2}{P_{s0}^3}L \tag{17}$$

$$\frac{\partial T_1}{\partial P_{t0}} = \frac{\beta_0^{-2}\gamma_0^{-2}+P_{x0}^2+P_{y0}^2}{P_{s0}^3}L \tag{18}$$

The second derivatives are given by

$$\frac{\partial^2 X_1}{\partial P_{x0}^2} = 3\frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2-P_{y0}^2}{P_{s0}^5}P_{x0}L \tag{19}$$

$$\frac{\partial^2 X_1}{\partial P_{x0}P_{y0}} = \left(\frac{1}{P_{s0}^3}+3\frac{P_{x0}^2}{P_{s0}^5}\right)P_{y0}L \tag{20}$$

$$\frac{\partial^2 X_1}{\partial P_{x0}P_{t0}} = -\left(\frac{1}{P_{s0}^3}+3\frac{P_{x0}^2}{P_{s0}^5}\right)(\beta_0^{-1}+P_{t0})L \tag{21}$$

$$\frac{\partial^2 X_1}{\partial P_{y0}^2} = \left(\frac{1}{P_{s0}^3}+3\frac{P_{y0}^2}{P_{s0}^5}\right)P_{x0}L \tag{22}$$

$$\frac{\partial^2 X_1}{\partial P_{y0}P_{t0}} = -3\frac{P_{x0}P_{y0}(\beta_0^{-1}+P_{t0})}{P_{s0}^5}L \tag{23}$$

$$\frac{\partial^2 X_1}{\partial P_{t0}^2} = \left(3\frac{(\beta_0^{-1}+P_{t0})^2}{P_{s0}^5}-\frac{1}{P_{s0}^3}\right)P_{x0}L \tag{24}$$

$$\frac{\partial^2 Y_1}{\partial P_{y0}^2} = 3\frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2-P_{x0}^2}{P_{s0}^5}P_{y0}L \tag{25}$$

$$\frac{\partial^2 Y_1}{\partial P_{y0}P_{t0}} = -\left(\frac{1}{P_{s0}^3}+3\frac{P_{y0}^2}{P_{s0}^5}\right)(\beta_0^{-1}+P_{t0})L \tag{26}$$

$$\frac{\partial^2 Y_1}{\partial P_{t0}^2} = \left(3\frac{(\beta_0^{-1}+P_{t0})^2}{P_{s0}^5}-\frac{1}{P_{s0}^3}\right)P_{y0}L \tag{27}$$

$$\frac{\partial^2 T_1}{\partial P_{t0}^2} = -3\frac{\beta_0^{-2}\gamma_0^{-2}+P_{x0}^2+P_{y0}^2}{P_{s0}^5}(\beta_0^{-1}+P_{t0})L \tag{28}$$

along with the symmetry conditions

$$\frac{\partial^2 Y_1}{\partial P_{x0}^2} = \frac{\partial^2 X_1}{\partial P_{x0}P_{y0}} \tag{29}$$

$$\frac{\partial^2 Y_1}{\partial P_{x0}P_{y0}} = \frac{\partial^2 X_1}{\partial P_{y0}^2} \tag{30}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}P_{y0}} = \frac{\partial^2 Y_1}{\partial P_{x0}P_{t0}} = \frac{\partial^2 X_1}{\partial P_{y0}P_{t0}} \tag{31}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}^2} = \frac{\partial^2 X_1}{\partial P_{x0}P_{t0}} \tag{32}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}P_{t0}} = \frac{\partial^2 X_1}{\partial P_{t0}^2} \tag{33}$$

$$\frac{\partial^2 T_1}{\partial P_{y0}^2} = \frac{\partial^2 Y_1}{\partial P_{y0}P_{t0}} \tag{34}$$

$$\frac{\partial^2 T_1}{\partial P_{y0}P_{t0}} = \frac{\partial^2 Y_1}{\partial P_{t0}^2} \tag{35}$$

For improved numerical properties, some sub-expressions can be rewritten:

$$\frac{1}{P_{s0}^3}+3\frac{P_{x0}^2}{P_{s0}^5} = \frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2+2P_{x0}^2-P_{y0}^2}{P_{x0}^5} \tag{36}$$

$$\frac{1}{P_{s0}^3}+3\frac{P_{y0}^2}{P_{s0}^5} = \frac{1+2\beta_0^{-1}P_{t0}+P_{t0}^2-P_{x0}^2+2P_{y0}^2}{P_{x0}^5} \tag{37}$$

$$3\frac{(\beta_0^{-1}+P_{t0})^2}{P_{s0}^5}-\frac{1}{P_{s0}^3} = \frac{2(\beta_0^{-1}+P_{t0})^2+\beta_0^{-2}\gamma_0^{-2}+P_{x0}^2+P_{y0}^2}{P_{s0}^5} \tag{38}$$

ROTATION ABOUT THE Y AXIS

Say the coordinate plane is rotated by an angle θ_y about the y axis. Then the momenta are transformed by

$$P_{x1} = P_{x0}\cos\theta_y - P_{s0}\sin\theta_y \tag{39}$$

$$P_{s1} = P_{x0}\sin\theta_y + P_{s0}\cos\theta_y \tag{40}$$

P_y and P_t are invariant so they are not subscripted with a 0 or 1. For a particle not at $X_0 = 0$, the particle must be then transported to the new plane. This transport results in

$$X_1 = X_0\frac{P_{s0}}{P_{s1}} \tag{41}$$

$$Y_1 = Y_0 - X_0\sin\theta_y\frac{P_y}{P_{s1}} \tag{42}$$

$$T_1 = T_0 + X_0\sin\theta_y\frac{\beta_0^{-1}+P_t}{P_{s1}} \tag{43}$$

For simplifying the calculations of the derivatives of the map, it is helpful to derive the map from a mixed-variable generating function. That generating function is

$$\begin{aligned} &G(X_0,P_{x1},P_y,P_t)\\ &\quad= X_0P_{x0}(P_{x1},P_y,P_t)+Y_0P_y+T_0P_t\\ &\quad= X_0[P_{x1}\cos\theta_y+P_s(P_{x1},P_y,P_t)\sin\theta_y]\\ &\qquad+Y_0P_y+T_0P_t \end{aligned} \tag{44}$$

and the coordinate components of the map can be written as

$$X_1 = \frac{\partial G}{\partial P_{x1}} = X_0 \frac{\partial P_{x0}}{\partial P_{x1}} \tag{45}$$

$$Y_1 = Y_0 + \frac{\partial G}{\partial P_y} = Y_0 + X_0 \frac{\partial P_{x0}}{\partial P_y} \tag{46}$$

$$T_1 = T_0 + \frac{\partial G}{\partial P_t} = X_0 \frac{\partial P_{x0}}{\partial P_t} \tag{47}$$

For compactness, I introduce a notation for derivatives,

$$\frac{\partial^n P_{x0}}{\partial P_{x1}^p \partial P_y^q \partial P_t^r} = D_{0x\cdots xy\cdots yt\cdots t} \tag{48}$$

$$\frac{\partial^n P_{x1}}{\partial P_{x0}^p \partial P_y^q \partial P_t^r} = D_{1x\cdots xy\cdots yt\cdots t} \tag{49}$$

where the number of copies of x, y, and t in the subscript of G are p, q, and r respectively.

The first derivatives required for the map are

$$\frac{\partial P_{x0}}{\partial P_{x1}} = D_{0x} = \frac{P_{s0}}{P_{s1}} \tag{50}$$

$$\frac{\partial P_{x0}}{\partial P_y} = D_{0y} = -\sin\theta \frac{P_y}{P_{s1}} \tag{51}$$

$$\frac{\partial P_{x0}}{\partial P_t} = D_{0t} = \sin\theta \frac{\beta_0^{-1} + P_t}{P_{s1}} \tag{52}$$

As for the derivatives of this map, a few intermediate calculations are helpful:

$$P_m^2 = P_x^2 + P_s^2 = 1 + 2\beta_0^{-1} P_t + P_t^2 - P_y^2 \tag{53}$$

$$\frac{\partial P_{s0}}{\partial P_{x0}} = -\frac{P_{x0}}{P_{s0}} \tag{54}$$

$$\frac{\partial P_{s0}}{\partial P_y} = -\frac{P_y}{P_{s0}} \tag{55}$$

$$\frac{\partial P_{s0}}{\partial P_t} = \frac{\beta_0^{-1} + P_t}{P_{s0}} \tag{56}$$

$$\frac{\partial P_{s1}}{\partial P_{x0}} = -\frac{P_{x1}}{P_{s0}} \tag{57}$$

$$\frac{\partial P_{s1}}{\partial P_y} = -\frac{P_y}{P_{s0}} \cos\theta_y \tag{58}$$

$$\frac{\partial P_{s1}}{\partial P_t} = \frac{\beta_0^{-1} + P_t}{P_{s0}} \cos\theta_y \tag{59}$$

From the generating function representation, we have

$$\frac{\partial X_1}{\partial X_0} = D_{0x} \tag{60}$$

$$\frac{\partial X_1}{\partial P_{x0}} = X_0 D_{0xx} D_{1x} \tag{61}$$

$$\frac{\partial X_1}{\partial P_y} = X_0 (D_{0xy} + D_{0xx} D_{1y}) \tag{62}$$

$$\frac{\partial X_1}{\partial P_t} = X_0 (D_{0xt} + D_{0xx} D_{1t}) \tag{63}$$

$$\frac{\partial Y_1}{\partial X_0} = D_{0y} \tag{64}$$

$$\frac{\partial Y_1}{\partial P_{x0}} = X_0 D_{0xy} D_{1x} \tag{65}$$

$$\frac{\partial Y_1}{\partial P_y} = X_0 (D_{0yy} + D_{0xy} D_{1y}) \tag{66}$$

$$\frac{\partial Y_1}{\partial P_t} = X_0 (D_{0yt} + D_{0xy} D_{1t}) \tag{67}$$

$$\frac{\partial T_1}{\partial X_0} = D_{0t} \tag{68}$$

$$\frac{\partial T_1}{\partial P_{x0}} = X_0 D_{0xt} D_{1x} \tag{69}$$

$$\frac{\partial T_1}{\partial P_y} = X_0 (D_{0yt} + D_{0xt} D_{1y}) \tag{70}$$

$$\frac{\partial T_1}{\partial P_t} = X_0 (D_{0tt} + D_{0xt} D_{1t}) \tag{71}$$

Note in these computations that when a partial derivative is taken of P_{x0} it is treated as a function of P_{x1}, P_y, and P_t, while if a partial derivative is taken of P_{x1}, it is treated as a function of P_{x0}, P_y, and P_t. The second derivatives of P_{x0} are

$$D_{0xx} = -\sin\theta_y \frac{1 + 2\beta_0^{-1} P_t + P_t^2 - P_y^2}{P_{s1}^3} \tag{72}$$

$$D_{0xy} = -\sin\theta_y \frac{P_{x1} P_y}{P_{s1}^3} \tag{73}$$

$$D_{0xt} = \sin\theta_y \frac{P_{x1}(\beta_0^{-1} + P_t)}{P_{s1}^3} \tag{74}$$

$$D_{0yy} = -\sin\theta_y \frac{1 + 2\beta_0^{-1} P_t + P_t^2 - P_{x1}^2}{P_{s1}^3} \tag{75}$$

$$D_{0yt} = \sin\theta_y \frac{P_y(\beta_0^{-1} + P_t)}{P_{s1}^3} \tag{76}$$

$$D_{0tt} = -\sin\theta_y \frac{\beta_0^{-2}\gamma_0^{-2} + P_{x1}^2 + P_y^2}{P_{s1}^3} \tag{77}$$

To compute the first derivatives of the map, first compute the first derivatives of P_{x1} which are needed for the generating function derivatives:

$$\frac{\partial P_{x1}}{\partial P_{x0}} = D_{1x} = \frac{P_{s1}}{P_{s0}} \tag{78}$$

$$\frac{\partial P_{x1}}{\partial P_y} = D_{1y} = \sin\theta_y \frac{P_y}{P_{s0}} \tag{79}$$

$$\frac{\partial P_{x1}}{\partial P_t} = D_{1t} = -\sin\theta_y \frac{\beta_0^{-1} + P_t}{P_{s0}} \tag{80}$$

Then the first derivatives of the map become

$$\frac{\partial X_1}{\partial X_0} = \frac{P_{s0}}{P_{s1}} \tag{81}$$

$$\frac{\partial X_1}{\partial P_{x0}} = -X_0 \sin\theta_y \frac{1 + 2\beta_0^{-1} P_t + P_t^2 - P_y^2}{P_{s0} P_{s1}^2} \tag{82}$$

$$\frac{\partial X_1}{\partial P_y} = -X_0 \sin\theta_y \frac{P_{x0} P_y}{P_{s0} P_{s1}^2} \tag{83}$$

$$\frac{\partial X_1}{\partial P_t} = X_0 \sin\theta_y \frac{P_{x0}(\beta_0^{-1} + P_t)}{P_{s0}P_{s1}^2} \tag{84}$$

$$\frac{\partial Y_1}{\partial X_0} = -\sin\theta_y \frac{P_y}{P_{s1}} \tag{85}$$

$$\frac{\partial Y_1}{\partial P_{x0}} = -X_0 \sin\theta_y \frac{P_{x1}P_y}{P_{s0}P_{s1}^2} \tag{86}$$

$$\frac{\partial Y_1}{\partial P_y} = X_0 \sin\theta_y \frac{P_{x0}P_{x1} - (1 + 2\beta_0^{-1}P_t + P_t^2)\cos\theta_y}{P_{s0}P_{s1}^2} \tag{87}$$

$$\frac{\partial Y_1}{\partial P_t} = X_0 \sin\theta_y \cos\theta_y \frac{P_y(\beta_0^{-1} + P_t)}{P_{s0}P_{s1}^2} \tag{88}$$

$$\frac{\partial T_1}{\partial X_0} = \sin\theta_y \frac{\beta_0^{-1} + P_t}{P_{s1}} \tag{89}$$

$$\frac{\partial T_1}{\partial P_{x0}} = X_0 \sin\theta_y \frac{P_{x1}(\beta_0^{-1} + P_t)}{P_{s0}P_{s1}^2} \tag{90}$$

$$\frac{\partial T_1}{\partial P_y} = X_0 \sin\theta_y \cos\theta_y \frac{P_y(\beta_0^{-1} + P_t)}{P_{s0}P_{s1}^2} \tag{91}$$

$$\frac{\partial T_1}{\partial P_t} = -X_0 \sin\theta_y \frac{P_{x0}P_{x1} + (\beta_0^{-2}\gamma_0^{-2} + P_y^2)\cos\theta_y}{P_{s0}P_{s1}^2} \tag{92}$$

For computing the second derivatives, use

$$\frac{\partial^2 X_1}{\partial P_{x0}^2} = X_0(D_{0xxx}D_{1x}^2 + D_{0xx}D_{1xx}) \tag{93}$$

$$\frac{\partial^2 X_1}{\partial P_{x0}\partial P_y} = X_0(D_{0xxy}D_{1x} + D_{0xxx}D_{1x}D_{1y} + D_{0xx}D_{1xy}) \tag{94}$$

$$\frac{\partial^2 X_1}{\partial P_{x0}\partial P_t} = X_0(D_{0xxt}D_{1x} + D_{0xxx}D_{1x}D_{1t} + D_{0xx}D_{1xt}) \tag{95}$$

$$\frac{\partial^2 X_1}{\partial P_y^2} = X_0(D_{0xyy} + 2D_{0xxy}D_{1y} + D_{0xxx}D_{1y}^2 + D_{0xx}D_{1yy}) \tag{96}$$

$$\frac{\partial^2 X_1}{\partial P_y\partial P_t} = X_0(D_{0xyt} + D_{0xxy}D_{1t} + D_{0xxt}D_{1y} + D_{0xxx}D_{1y}D_{1t} + D_{0xx}D_{1yt}) \tag{97}$$

$$\frac{\partial^2 X_1}{\partial P_t^2} = X_0(D_{0xtt} + 2D_{0xxt}D_{1t} + D_{0xxx}D_{1t}^2 + D_{0xx}D_{1tt}) \tag{98}$$

$$\frac{\partial^2 Y_1}{\partial P_{x0}^2} = X_0(D_{0xxy}D_{1x}^2 + D_{0xy}D_{1xx}) \tag{99}$$

$$\frac{\partial^2 Y_1}{\partial P_{x0}\partial P_y} = X_0(D_{0xyy}D_{1x} + D_{0xxy}D_{1x}D_{1y} + D_{0xy}D_{1xy}) \tag{100}$$

$$\frac{\partial^2 Y_1}{\partial P_{x0}\partial P_t} = X_0(D_{0xyt}D_{1x} + D_{0xxy}D_{1x}D_{1t} + D_{0xy}D_{1xt}) \tag{101}$$

$$\frac{\partial^2 Y_1}{\partial P_y^2} = X_0(D_{0yyy} + 2D_{0xyy}D_{1y} + D_{0xxy}D_{1y}^2 + D_{0xy}D_{1yy}) \tag{102}$$

$$\frac{\partial^2 Y_1}{\partial P_y\partial P_t} = X_0(D_{0yyt} + D_{0xyy}D_{1t} + D_{0xyt}D_{1y} + D_{0xxy}D_{1y}D_{1t} + D_{0xy}D_{1yt}) \tag{103}$$

$$\frac{\partial^2 Y_1}{\partial P_t^2} = X_0(D_{0ytt} + 2D_{0xyt}D_{1t} + D_{0xxy}D_{1t}^2 + D_{0xy}D_{1tt}) \tag{104}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}^2} = X_0(D_{0xxt}D_{1x}^2 + D_{0xt}D_{1xx}) \tag{105}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}\partial P_y} = X_0(D_{0xyt}D_{1x} + D_{0xxt}D_{1x}D_{1y} + D_{0xt}D_{1xy}) \tag{106}$$

$$\frac{\partial^2 T_1}{\partial P_{x0}\partial P_t} = X_0(D_{0xtt}D_{1x} + D_{0xxt}D_{1x}D_{1t} + D_{0xt}D_{1xt}) \tag{107}$$

$$\frac{\partial^2 T_1}{\partial P_y^2} = X_0(D_{0yyt} + 2D_{0xyt}D_{1y} + D_{0xxt}D_{1y}^2 + D_{0xt}D_{1yy}) \tag{108}$$

$$\frac{\partial^2 T_1}{\partial P_y\partial P_t} = X_0(D_{0ytt} + D_{0xyt}D_{1t} + D_{0xtt}D_{1y} + D_{0xxt}D_{1y}D_{1t} + D_{0xt}D_{1yt}) \tag{109}$$

$$\frac{\partial^2 T_1}{\partial P_t^2} = X_0(D_{0ttt} + 2D_{0xtt}D_{1t} + D_{0xxt}D_{1t}^2 + D_{0xt}D_{1tt}) \tag{110}$$

The third derivatives of P_{x0} are needed:

$$D_{0xxx} = -3P_{x1}\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_y^2}{P_{s1}^5} \tag{111}$$

$$D_{0xxy} = -P_y\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_y^2 + 2P_{x1}^2}{P_{s1}^5} \tag{112}$$

$$D_{0xxt} = (\beta_0^{-1} + P_t)\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_y^2 + 2P_{x1}^2}{P_{s1}^5} \tag{113}$$

$$D_{0xyy} = -P_{x1}\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_{x1}^2 + 2P_y^2}{P_{s1}^5} \tag{114}$$

$$D_{0xyt} = 3\sin\theta_y \frac{P_{x1}P_y(\beta_0^{-1} + P_t)}{P_{s1}^5} \tag{115}$$

$$D_{0xtt} = -P_{x1}\sin\theta_y \frac{2(\beta_0^{-1}+P_t)^2 + \beta_0^{-2}\gamma_0^{-2} + P_{x1}^2 + P_y^2}{P_{s1}^5} \tag{116}$$

$$D_{0yyy} = -3P_y\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_{x1}^2}{P_{s1}^5} \tag{117}$$

$$D_{0yyt} = (\beta_0^{-1}+P_t)\sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_{x1}^2 + 2P_y^2}{P_{s1}^5} \tag{118}$$

$$D_{0ytt} = -P_y\sin\theta_y \frac{2(\beta_0^{-1}+P_t)^2 + \beta_0^{-2}\gamma_0^{-2} + P_{x1}^2 + P_y^2}{P_{s1}^5} \tag{119}$$

$$D_{0ttt} = 3(\beta_0^{-1}+P_t)\sin\theta_y \frac{\beta_0^{-2}\gamma_0^{-2} + P_{x1}^2 + P_y^2}{P_{s1}^3} \tag{120}$$

and the second derivatives of P_{x1} are

$$D_{1xx} = \sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_y^2}{P_{s0}^3} \tag{121}$$

$$D_{1xy} = \sin\theta_y \frac{P_{x0}P_y}{P_{s0}^3} \tag{122}$$

$$D_{1xt} = -\sin\theta_y \frac{P_{x0}(\beta_0^{-1}+P_t)}{P_{s0}^3} \tag{123}$$

$$D_{1yy} = \sin\theta_y \frac{1 + 2\beta_0^{-1}P_t + P_t^2 - P_{x0}^2}{P_{s0}^3} \tag{124}$$

$$D_{1yt} = -\sin\theta_y \frac{P_y(\beta_0^{-1}+P_t)}{P_{s0}^3} \tag{125}$$

$$D_{1tt} = \sin\theta_y \frac{\beta_0^{-2}\gamma_0^{-2} + P_{x0}^2 + P_y^2}{P_{s0}^3} \tag{126}$$

SOLENOID

The Hamiltonian for a uniform solenoid is

$$-\sqrt{\begin{aligned}1 + 2\beta_0^{-1}P_t + P_t^2 - (P_x + k_sY/2)^2\\ - (P_y - k_sX/2)^2\end{aligned}} = -P_s \tag{127}$$

The map for this can be solved exactly. All calculations will be done using canonical momenta, which are the same as the kinetic momenta outside the solenoid, but differ from the kinetic momenta inside the solenoid. On passing from outside to inside the solenoid, the canonical momenta are unchnaged, but the kinetic momenta have a discrete change depending on the position. P_s is invariant within the solenoid, but changes from inside to outside the solenoid. In the following equations, when P_s appears, it refers to its value inside the solneoid.

The map is straightforward to compute:

$$\begin{bmatrix} X_1 \\ P_{x1} \\ Y_1 \\ P_{y1} \end{bmatrix} = M_{S0}\left(\frac{k_s}{2}, \frac{k_sL}{2P_s}\right) \begin{bmatrix} X_0 \\ P_{x0} \\ Y_0 \\ P_{y0} \end{bmatrix} \tag{128}$$

$$T_1 = T_0 + \frac{L}{\beta_0} - \frac{\beta_0^{-1}+P_t}{P_s}L \tag{129}$$

where

$$M_{S0}(\kappa,\phi) = \begin{bmatrix} \cos^2\phi & \kappa^{-1}\cos\phi\sin\phi & \cos\phi\sin\phi & \kappa^{-1}\sin^2\phi \\ -\kappa\cos\phi\sin\phi & \cos^2\phi & -\kappa\sin^2\phi & \cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\kappa^{-1}\sin^2\phi & \cos^2\phi & \kappa^{-1}\cos\phi\sin\phi \\ \kappa\sin^2\phi & -\cos\phi\sin\phi & -\kappa\cos\phi\sin\phi & \cos^2\phi \end{bmatrix} \tag{130}$$

and L the length of the solenoid. For numerical precision,

$$T_1 = T_0 + \frac{\gamma_0^{-2}(2\beta_0^{-1}P_t + P_t^2) - P_\perp^2}{\beta_0^2 P_s(\beta_0^{-1}P_s + \beta_0^{-1} + P_t)}L \tag{131}$$

with

$$P_\perp^2 = (P_x + k_sY/2)^2 + (P_y - k_sX/2)^2 \tag{132}$$

which is invariant within the solenoid.

For derivatives, it will be convenient index in the transverse variables, so we define a transverse phase space vector $\boldsymbol{Z} = (X, P_x, Y, P_y)$. Thus, for instance,

$$\frac{\partial P_s}{\partial \boldsymbol{Z}} = -\frac{1}{P_s}\Pi\left(\frac{k_s}{2}\right)\boldsymbol{Z} \tag{133}$$

with

$$\Pi(\kappa) = \begin{bmatrix} \kappa^2 & 0 & 0 & -\kappa \\ 0 & 1 & \kappa & 0 \\ 0 & \kappa & \kappa^2 & 0 \\ -\kappa & 0 & 0 & 1 \end{bmatrix} \tag{134}$$

Further defining

$$M_{S1}(\kappa,\psi) = \begin{bmatrix} -\sin\psi & \kappa^{-1}\cos\psi & \cos\psi & \kappa^{-1}\sin\psi \\ -\kappa\cos\psi & -\sin\psi & -\kappa\sin\psi & \cos\psi \\ -\cos\psi & -\kappa^{-1}\sin\psi & -\sin\psi & \kappa^{-1}\cos\psi \\ \kappa\sin\psi & -\cos\psi & -\kappa\cos\psi & -\sin\psi \end{bmatrix} \tag{135}$$

we have

$$\frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_0} = M_{S0}\left(\frac{k_s}{2}, \frac{k_s L}{2P_s}\right) + \mu\pi^T \tag{136}$$

$$\frac{\partial \mathbf{Z}_1}{\partial P_t} = -(\beta_0^{-1} + P_t)\boldsymbol{\mu} \tag{137}$$

$$\frac{\partial T_1}{\partial \mathbf{Z}_0} = -L\frac{\beta_0^{-1} + P_t}{P_s^3}\boldsymbol{\pi} \tag{138}$$

$$\frac{\partial T_1}{\partial P_t} = \frac{\beta_0^{-2}\gamma_0^{-2} + P_\perp^2}{P_s^3}L \tag{139}$$

with

$$\boldsymbol{\pi} = \Pi\left(\frac{k_s}{2}\right)\mathbf{Z}_0 \tag{140}$$

$$\boldsymbol{\mu} = \frac{k_s L}{2P_s^3}M_{S1}\left(\frac{k_s}{2}, \frac{k_s L}{P_s}\right)\mathbf{Z}_0 \tag{141}$$

Note the difference between the second arguments of M_{S0} and M_{S1}.

For the second derivatives,

$$\frac{\partial^2 Z_{1i}}{\partial Z_{0j}\partial Z_{0k}} = \frac{k_s L}{2P_s^3}(M_{S1;ij}\pi_k + M_{S1;ik}\pi_j) + \mu_i\Pi_{jk} + \tau_i\pi_j\pi_k \tag{142}$$

$$\frac{\partial^2 \mathbf{Z}_1}{\partial \mathbf{Z}_0\partial P_t} = -(\beta_0^{-1} + P_t)\left(\boldsymbol{\tau}\boldsymbol{\pi}^T + \frac{k_s L}{2P_s^3}M_{S1}\right) \tag{143}$$

$$\frac{\partial^2 \mathbf{Z}_1}{\partial P_t^2} = (\beta_0^{-1} + P_t)^2\boldsymbol{\tau} - \boldsymbol{\mu} \tag{144}$$

$$\frac{\partial^2 T_1}{\partial \mathbf{Z}_0\partial \mathbf{Z}_0} = -3L\frac{\beta_0^{-1} + P_t}{P_s^5}\boldsymbol{\pi}\boldsymbol{\pi}^T - L\frac{\beta_0^{-1} + P_t}{P_s^3}\Pi \tag{145}$$

$$\frac{\partial^2 T_1}{\partial \mathbf{Z}_0\partial P_t} = L\frac{3(\beta_0^{-1} + P_t)^2 - P_s^2}{P_s^5}\boldsymbol{\pi} \tag{146}$$

$$\frac{\partial^2 T_1}{\partial P_t^2} = -3\frac{(\beta_0^{-1} + P_t)(\beta_0^{-2}\gamma_0^{-2} + P_\perp^2)}{P_s^5}L \tag{147}$$

with

$$\boldsymbol{\tau} = \frac{3}{P_s^2}\boldsymbol{\mu} - \frac{1}{2}\left(\frac{k_s L}{P_s^3}\right)^2 M_{S2}\left(\frac{k_s}{2}, \frac{k_s L}{P_s}\right)\mathbf{Z}_0 \tag{148}$$

$$M_{S2}(\kappa,\psi) = \left[\begin{matrix} \cos\psi & \kappa^{-1}\sin\psi \\ -\kappa\sin\psi & \cos\psi \\ -\sin\psi & \kappa^{-1}\cos\psi \\ -\kappa\cos\psi & -\sin\psi \end{matrix}\right. \\ \left.\begin{matrix} \sin\psi & -\kappa^{-1}\cos\psi \\ \kappa\cos\psi & \sin\psi \\ \cos\psi & \kappa^{-1}\sin\psi \\ -\kappa\sin\psi & \cos\psi \end{matrix}\right] \tag{149}$$

REFERENCES

[1] É. Forest, *Beam Dynamics, A New Attitude and Framework*. CRC Press, 1998.

[2] F. C. Iselin, *The MAD program, Physical methods manual*, 1994, `https://mad.web.cern.ch/mad/`