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Notes on the 2 to 1 bunch merge with application to polarized proton bunches in AGS

C. Gardner

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Collider Accelerator Department

Brookhaven National Laboratory

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Notes on the 2 to 1 bunch merge with application to polarized proton bunches in AGS

C.J. Gardner

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A simple and useful program to produce a 2 to 1 bunch merge is developed in these notes and applied to polarized proton bunches in AGS. The notes are summarized in Sections 1 through 4. The reader may refer to the additional sections as needed. Given the large number of pages and references to various sections, it is recommended that the document viewer be set up to increment and decrement the page number with single key strokes. This gives fast access to the sections and allows sequences of figures to be viewed as movies.

1 The merge setup

The 2 to 1 merge is accomplished by judicious use of RF harmonics h and $2h$ where h is a positive integer. We use subscripts I and F to denote parameters at the start and end of the merge. At the start of the merge we have harmonic h and $2h$ voltages

$$V_1 = V_{1I}, \quad V_2 = V_{2I} \tag{1}$$

and at the end

$$V_1 = V_{1F}, \quad V_2 = V_{2F}. \tag{2}$$

As explained in Section 12, the merge starts and ends with ratios

$$R_I = \frac{V_{1I}}{V_{2I}} = 0, \quad R_F = \frac{V_{1F}}{V_{2F}} = 2 \tag{3}$$

which gives

$$V_{1I} = 0, \quad V_{1F} = 2V_{2F}. \tag{4}$$

As explained in Section 13, it is desirable to also have

$$V_{1F} = \frac{1}{2} V_{2I} \tag{5}$$

which, along with the second of equations (4), gives

$$V_{2F} = \frac{1}{4} V_{2I}. \tag{6}$$

The end of merge harmonic h and $2h$ voltages, V_{1F} and V_{2F}, are then completely determined by the initial harmonic $2h$ voltage V_{2I}.

For simplicity and ease of setup, the RF voltages are programmed to vary linearly with time. Taking T to be the time it takes to go from the start to the end of the merge, we have, as shown in Section 18,

$$V_1 = \frac{V_{2I}}{2}\left(\frac{t}{T}\right), \quad V_2 = V_{2I}\left(1 - \frac{3}{4}\frac{t}{T}\right) \tag{7}$$

where

$$0 \leq t \leq T. \tag{8}$$

It is easily seen that these equations satisfy conditions (1) through (6).

After the end of the merge we want to bring the harmonic $2h$ voltage down to zero while keeping the harmonic h voltage fixed. To accomplish this we take

$$V_1 = \frac{1}{2} V_{2I}, \quad V_2 = V_{2I}\left(1 - \frac{3}{4}\frac{t}{T}\right) \tag{9}$$

for

$$T < t \leq \mathcal{T} \tag{10}$$

and require that $V_2 = 0$ when $t = \mathcal{T}$. This gives

$$\mathcal{T} = \frac{4}{3} T \tag{11}$$

which is the total time needed for the merge. We also have

$$\mathcal{T} - T = \frac{1}{3} T = \frac{1}{4} \mathcal{T} \tag{12}$$

which is the additional time needed to bring the harmonic $2h$ voltage down to zero. Thus three fourths of the total time is used to do the merge, while one fourth of the time is used to bring V_2 down to zero.

2 Quality of the merge

If the merge is done sufficiently slowly, the longitudinal emittance of the merged bunch will equal the sum of the emittances of the initial 2 bunches. In that case we say that the quality of the merge is good. If the merge is done too quickly, the merged bunch will become filamented and diluted with empty phase space, making its gross emittance greater than that of the 2 unmerged bunches. In that case we say that there has been emittance growth (even though the phase space area occupied by the bunch particles has not changed). If the gross emittance is significantly greater than the sum of the emittances of the unmerged bunches then we say that the quality of the merge is poor.

One can also look at the distribution of particles with respect to initial longitudinal oscillation amplitude. If the merge is done sufficiently slowly, the distribution will remain intact and the gross emittance of the merged bunch will be conserved. In some instances the gross emittance may be conserved even if there is some disruption of the distribution of particles with respect to oscillation amplitude in the merged bunch.

3 RF bucket evolution

The merge resulting from RF voltage program (7) begins with unmerged bunches sitting in stationary harmonic $2h$ buckets. Adjacent harmonic $2h$ buckets then evolve into a bucket with an inner and outer separatrix as the merge progresses. The inner separatrix consists of two lobes which initially contain the unmerged bunches. As the merge progresses the stable fixed points associated with the lobes move toward one another and the lobe area decreases. If the merge is done sufficiently slowly, the bunches will follow the fixed points. When the decreasing lobe area equals the bunch area, bunch particles are squeezed out of the separatrix and begin to mix in the area enclosed by the outer separatrix. At the end of the merge, the lobe area reaches zero and all bunch particles are contained inside the outer separatrix.

The evolution of the inner and outer separatrices and their parameters are illustrated in Sections 20 and 21. There one sees that the area enclosed by the inner separatrix decreases nearly linearly with time as the merge progresses. This is a consequence of the linearity of the RF voltages and the particular proportions given by equations (7). It means that bunch particles will be squeezed out of the inner separatrix lobes at a nearly constant rate if the merge is done sufficiently slowly. In Section 22 it is argued that for given initial bunch area and merge time T, the quality of the merge should be about the same for a wide range of initial harmonic $2h$ voltages V_{2I}. Examples for which this reasoning works are given in the next section.

4 Simulation results

Simulations of the merge were done in accordance with equations (7) through (11). They were carried out for polarized protons in AGS on the injection porch where $G\gamma = 4.5$ and on the flattop where $G\gamma = 45.5$. Sections 23 and 24 show the simulations done on the injection porch. Sections 25 through 28 show the simulations done on the flattop.

For the merge of Section 23 the initial harmonic 12 voltage was chosen so that the associated bucket area was just slightly larger than the area of the bunch matched to the bucket. This resulted in bunch particles being squeezed out of the inner separatrix for nearly the entire merge time T. The time T was chosen to give a merge that keeps the longitudinal oscillation amplitude layers intact and the gross emittance unchanged.

For the merge of Section 24 the unmerged bunch area and merge time T were kept the same as those used in Section 23, but the initial harmonic 12 voltage was increased by a factor of 25. This increases the associated synchrotron frequency by a factor of 5. It also gives an initial harmonic 12 bucket area 5 times larger, which results in bunch particles being squeezed out of the inner separatrix for a period of time that is 5 times shorter. The effect of this shorter time is counteracted by the factor of 5 increase in synchrotron frequency. The resulting merge keeps the longitudinal oscillation amplitude layers intact and the gross emittance unchanged, consistent with the reasoning of Section 22.

The aim of the simulations carried out on the AGS flattop at $G\gamma = 45.5$ was to determine the minimum time needed to produce a bunch that has not suffered too much emittance growth during the merge. The simulations also show that the quality of the merge remains the same over a wide range of initial harmonic $2h$ voltages. Total merge times $\mathcal{T}$ of 4000, 2000, 1000, and 500 ms were used in the merges of Sections 25, 26, 27, and 28 respectively.

For the 4000 ms merge of Section 25 the initial harmonic 12 voltage was first chosen so that the corresponding bucket area was just slightly larger than the area of the unmerged bunch matched to the bucket. This produced a merged bunch with some disruption of the longitudinal oscillation amplitude layers but with little growth of gross emittance as shown in Figure 74. The initial harmonic 12 voltage was then increased by a factor of 100, keeping the unmerged bunch area and merge time T the same. This produced a merged bunch of the same quality, which is easily seen by comparing Figures 79 and 82. This is consistent with the reasoning of Section 22.

For the 2000 ms merge of Section 26 the initial harmonic 12 voltage was again chosen so that the corresponding bucket area was just slightly larger than the area of the unmerged bunch matched to the bucket. The shorter merge time produced a merged bunch with more disruption and filamentation of the longitudinal oscillation amplitude layers. Comparing Figures 74 and 85 one also sees a larger gross emittance of the merged bunch.

For the 1000 ms merge of Section 27 the initial harmonic 12 voltage was first chosen so that the corresponding bucket area was just slightly larger than the area of the unmerged bunch matched to the bucket. The shorter merge time produced a merged bunch with further disruption and filamentation of the longitudinal oscillation amplitude layers as shown in Figure 98. Here one also sees that the merged bunch comes right up against the border of the harmonic 6 bucket. From this and the known bucket area one can infer an emittance growth factor of 1.3. This result is consistent with observations and measurements [1] of the actual merge that was set up on the AGS flattop during the FY2022 polarized proton run. The merge simulation was then repeated with increasing initial harmonic 12 voltages while keeping the unmerged bunch area and merge time the same. In all of the associated figures one sees that the quality of the merged bunch remains the same, which is again consistent with the reasoning of Section 22.

For the 500 ms merge of Section 28 the initial harmonic 12 voltage was again chosen so that the corresponding bucket area was just slightly larger than the area of the unmerged bunch matched to the bucket. The shorter merge time produced a merge in which the bunches remain separated with the oscillation amplitude layers of each bunch essentially intact as shown in Figure 112. Here one sees that some of the bunch particles are outside the harmonic 6 bucket, indicating further emittance growth with the merge time reduced from 1000 to 500 ms.

These simulation results show that 1000 ms is a reasonable compromise between merge quality and the need to keep the time on the flattop to a minimum.

5 The double-harmonic RF bucket

To simplify the discussion here and in subsequent sections, we work below transition.

The double-harmonic RF bucket is one in which harmonic numbers h and $2h$ are active. For the case in which two bunches are to be merged we have the "force" function [2]

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi \tag{13}$$

and associated "potential"

$$U(\phi) = A_1 \cos\phi - \frac{1}{2} A_2 \cos 2\phi \tag{14}$$

where the amplitudes A_1 and A_2 are either zero or positive. One starts with bunches sitting sitting in harmonic $2h$ buckets with

$$A_1 = 0, \quad A_2 > 0. \tag{15}$$

Amplitude A_1 is then raised from zero and A_2 is decreased toward zero. Eventually A_2 reaches zero, leaving A_1 as the only nonzero amplitude. If done properly, one ends up with merged bunches sitting in harmonic h buckets. According to the notation and conventions used in [2] we have

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h} \tag{16}$$

where V_1 and V_2 are the harmonic h and $2h$ voltages respectively. For amplitude

$$A_2 \neq 0 \tag{17}$$

it is convenient to define

$$R = \frac{A_1}{A_2} \tag{18}$$

and work with

$$\mathcal{F} = \frac{F(\phi)}{A_2}, \quad \mathcal{U} = \frac{U(\phi)}{A_2}. \tag{19}$$

We then have

$$\mathcal{F} = R\sin\phi - \sin 2\phi \tag{20}$$

$$\mathcal{U} = R\cos\phi - \frac{1}{2}\cos 2\phi, \quad \mathcal{U}' = -R\sin\phi + \sin 2\phi \tag{21}$$

$$\mathcal{U}'' = -R\cos\phi + 2\cos 2\phi, \quad \mathcal{U}''' = R\sin\phi - 4\sin 2\phi \tag{22}$$

$$\mathcal{U}'''' = R\cos\phi - 8\cos 2\phi, \quad \mathcal{U}''''' = -R\sin\phi + 16\sin 2\phi \tag{23}$$

$$\mathcal{U}'''''' = -R\cos\phi + 32\cos 2\phi, \quad \mathcal{U}''''''' = R\sin\phi - 64\sin 2\phi \tag{24}$$

and

$$\mathcal{F} = -\mathcal{U}' \tag{25}$$

where the primes denote differentiation with respect to ϕ.

Introducing the notation

$$C = \cos\phi, \quad S = \sin\phi \tag{26}$$

we have the identities

$$\sin 2\phi = 2CS \tag{27}$$

$$\cos 2\phi = 2C^2 - 1 \tag{28}$$

which give

$$\mathcal{U} = RC - \frac{1}{2}\left(2C^2 - 1\right) \tag{29}$$

$$\mathcal{U}' = -S(R - 2C) \tag{30}$$

$$\mathcal{U}'' = -RC + 2\left(2C^2 - 1\right). \tag{31}$$

For phases with subscript "a" we write

$$C_a = \cos\phi_a, \quad S_a = \sin\phi_a \tag{32}$$

$$\mathcal{U}_a = RC_a - \frac{1}{2}\left(2C_a^2 - 1\right) \tag{33}$$

$$\mathcal{U}_a' = -S_a(R - 2C_a) \tag{34}$$

$$\mathcal{U}_a'' = -RC_a + 2\left(2C_a^2 - 1\right). \tag{35}$$

6 Fixed point phases

Fixed point phases ϕ_f are those for which

$$\mathcal{U}'_f = 0. \tag{36}$$

The fixed point phase is then stable if

$$\mathcal{U}''_f < 0 \tag{37}$$

and unstable if

$$\mathcal{U}''_f > 0. \tag{38}$$

The function $-\mathcal{U}(\phi)$ then has a local minimum at a stable fixed point and a local maximum at an unstable fixed point. It is therefore useful to think in terms of the "potential well" $-\mathcal{U}(\phi)$ rather than $\mathcal{U}(\phi)$.

We use subscripts "s" and "u" to denote fixed points that are stable and unstable respectively.

7 Unstable fixed point phases $\phi_u = \pm\pi$

For these phases equations (21) through (24) give

$$\mathcal{U}' = 0, \quad \mathcal{U}''' = 0, \quad \mathcal{U}''''' = 0, \quad \mathcal{U}''''''' = 0 \tag{39}$$

$$\mathcal{U}'' = R + 2, \quad \mathcal{U}'''' = -R - 8, \quad \mathcal{U}'''''' = R + 32. \tag{40}$$

Here

$$R \geq 0 \tag{41}$$

and therefore

$$\mathcal{U}'' > 0. \tag{42}$$

We therefore have unstable fixed point phases

$$\phi_u = \pm\pi. \tag{43}$$

The function $-\mathcal{U}(\phi)$ reaches a local maximum at these phases.

8 Fixed point phase $\phi_f = 0$

For this phase, equations (21) through (24) give

$$\mathcal{U}' = 0, \quad \mathcal{U}''' = 0, \quad \mathcal{U}''''' = 0, \quad \mathcal{U}''''''' = 0 \tag{44}$$

$$\mathcal{U}'' = 2 - R, \quad \mathcal{U}'''' = R - 8, \quad \mathcal{U}'''''' = 32 - R. \tag{45}$$

We therefore have fixed point phase

$$\phi_f = 0 \tag{46}$$

which is stable if

$$R > 2 \tag{47}$$

and unstable if

$$R < 2. \tag{48}$$

If

$$R = 2 \tag{49}$$

then (45) becomes

$$\mathcal{U}'' = 0, \quad \mathcal{U}'''' = -6, \quad \mathcal{U}'''''' = 30 \tag{50}$$

which gives a stable fixed point.

9 Stable fixed point phase $0 < \phi_s < \pi/2$

If

$$0 < R < 2 \tag{51}$$

there is, according to (34), a fixed point phase that satisfies

$$R - 2C_f = 0. \tag{52}$$

This gives

$$C_f = \frac{R}{2} \tag{53}$$

$$\mathcal{U}''_f = -\frac{R^2}{2} + R^2 - 2 \tag{54}$$

$$\mathcal{U}''_f = \frac{1}{2}\left(R^2 - 4\right) \tag{55}$$

and therefore

$$\mathcal{U}_f'' < 0. \tag{56}$$

There is therefore a stable fixed point phase

$$\phi_s = \arccos\left(\frac{R}{2}\right) \tag{57}$$

with

$$0 < \phi_s < \frac{\pi}{2}. \tag{58}$$

The function $-\mathcal{U}(\phi)$ has a local minimum at ϕ_s. For

$$R = 0, \quad 1, \quad \sqrt{2}, \quad \sqrt{3}, \quad 2 \tag{59}$$

we have

$$\phi_s = \frac{\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{\pi}{4}, \quad \frac{\pi}{6}, \quad 0 \tag{60}$$

showing that ϕ_s goes from $\pi/2$ to 0 as R goes from 0 to 2. It follows from the identities

$$\mathcal{U}(-\phi) = \mathcal{U}(\phi), \quad \mathcal{U}'(-\phi) = -\mathcal{U}'(\phi), \quad \mathcal{U}''(-\phi) = \mathcal{U}''(\phi) \tag{61}$$

that we also have stable fixed point phase $-\phi_s$. The particles of the unmerged bunches occupy the two potential wells centered on phases $\pm\phi_s$. As R goes from 0 to 2, these phases move from $\pm\pi/2$ to zero. The width and depth of the wells also decrease to zero. If this is done sufficiently slowly, the particles follow the wells and the two bunches move toward one another. Particles also flow out of the top of the wells and into a single larger well as the width and depth of the smaller wells decrease. We look at this in detail in the next three Sections (10, 11, 12).

10 Width and depth of the potential wells

The maximum turning point phase ϕ_e associated with oscillations about the stable fixed point phase (57) is given by

$$\mathcal{U}_e = \mathcal{U}(\phi_e) = \mathcal{U}(\phi_u) = \mathcal{U}_u \tag{62}$$

where

$$\phi_u = 0. \tag{63}$$

The width of the potential wells is then

$$\Delta\phi = \phi_e - \phi_u = \phi_e. \tag{64}$$

Here

$$\mathcal{U}_u = R - \frac{1}{2} \tag{65}$$

and

$$\mathcal{U}_e = RC_e - \frac{1}{2}\left(2C_e^2 - 1\right). \tag{66}$$

Equating (65) and (66) gives

$$RC_e - C_e^2 + \frac{1}{2} = R - \frac{1}{2} \tag{67}$$

$$C_e^2 - RC_e + R - 1 = 0 \tag{68}$$

$$C_e = \frac{R}{2} \pm \frac{1}{2}\sqrt{R^2 - 4R + 4} \tag{69}$$

and

$$C_e = \frac{R}{2} \pm \frac{1}{2}\left(2 - R\right). \tag{70}$$

Taking the minus sign gives

$$C_e = R - 1 \tag{71}$$

and

$$\phi_e = \arccos\left(R - 1\right). \tag{72}$$

For

$$0 < \phi_s \tag{73}$$

and

$$0 < R < 2 \tag{74}$$

we have

$$\phi_s < \phi_e < \pi. \tag{75}$$

Specifically, for

$$R = 0, \quad \frac{1}{2}, \quad 1, \quad \frac{3}{2}, \quad 2 \tag{76}$$

we have

$$\phi_s = \frac{\pi}{2}, \quad 0.4196\,\pi, \quad \frac{\pi}{3}, \quad 0.2301\pi, \quad 0 \tag{77}$$

and

$$\phi_e = \pi, \quad \frac{2\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{\pi}{3}, \quad 0. \tag{78}$$

It follows from the identities

$$\mathcal{U}(-\phi) = \mathcal{U}(\phi), \quad \mathcal{U}'(-\phi) = -\mathcal{U}'(\phi), \quad \mathcal{U}''(-\phi) = \mathcal{U}''(\phi) \tag{79}$$

that we also have turning point phases $-\phi_e$. As R goes from 0 to 2, the phases $\pm\phi_e$ go from $\pm\pi$ to zero and the width of the wells goes to zero.

The depth of the wells is

$$\Delta\mathcal{U} = -\mathcal{U}_u - (-\mathcal{U}_s) \tag{80}$$

where

$$\mathcal{U}_u = R - \frac{1}{2} \tag{81}$$

$$\mathcal{U}_s = RC_s - \frac{1}{2}\left(2C_s^2 - 1\right) \tag{82}$$

and

$$C_s = \frac{R}{2}. \tag{83}$$

Thus

$$\mathcal{U}_s = \frac{R^2}{4} + \frac{1}{2} \tag{84}$$

and we have

$$\Delta\mathcal{U} = 1 - R + \frac{R^2}{4} \tag{85}$$

showing that the depth of the wells goes from 1 to 0 as R goes from 0 to 2.

11 The inner separatrix

The unstable fixed point phase associated with oscillations about the stable fixed point phase (57) is

$$\phi_u = 0. \tag{86}$$

The inner separatrix is the separatrix that passes through this phase. It is given by the curve $W_e(\phi)$, where

$$W_e^2(\phi) = \frac{2A_2}{a}\left(\mathcal{U}_u - \mathcal{U}\right) \tag{87}$$

$$a = \left(\frac{h^2\omega^2\eta}{\beta^2 E}\right) = \left(\frac{h^2c^2\eta}{\mathcal{R}^2 E}\right), \quad A_2 = \frac{eQV_2}{2\pi h} \tag{88}$$

$$\mathcal{U} = RC - \frac{1}{2}\left(2C^2 - 1\right), \quad \mathcal{U}_u = R - \frac{1}{2} \tag{89}$$

$$\mathcal{U} - \mathcal{U}_u = 1 - R(1-C) - C^2 \tag{90}$$

$$R = \frac{A_1}{A_2} = \frac{V_1}{V_2}, \quad C = \cos\phi. \tag{91}$$

Using (88) we have

$$\left(\frac{2A_2}{|a|}\right)^{1/2} = \frac{\mathcal{R}}{hc}\left\{\frac{eQV_2E}{\pi h|\eta|}\right\}^{1/2} \tag{92}$$

which gives

$$W_e(\phi) = \pm\frac{1}{4}\,\mathcal{A}_2\,(\mathcal{U} - \mathcal{U}_u)^{1/2} \tag{93}$$

where

$$\mathcal{A}_2 = 4\,\frac{\mathcal{R}}{hc}\left\{\frac{eQV_2E}{\pi h|\eta|}\right\}^{1/2} \tag{94}$$

is the harmonic $2h$ stationary bucket area. The curve $W_e(\phi)$ is a closed curve that consists of two lobes on either side of the phase $\phi_u = 0$. The left lobe extends from phase $-\phi_e$ to 0 and the right from 0 to ϕ_e. Because of the symmetry $\mathcal{U}(-\phi) = \mathcal{U}(\phi)$, the areas enclosed by the two lobes are equal. The area of either lobe is

$$\mathcal{A}_e = 2\left(\frac{2A_2}{|a|}\right)^{1/2}\int_0^{\phi_e}(\mathcal{U} - \mathcal{U}_u)^{1/2}\,d\phi \tag{95}$$

which we can write as

$$\mathcal{A}_e = \frac{1}{4}\,\mathcal{A}_2\,\mathcal{B}_e \tag{96}$$

where

$$\mathcal{B}_e = 2\int_0^{\phi_e}(\mathcal{U} - \mathcal{U}_u)^{1/2}\,d\phi \tag{97}$$

$$\mathcal{U} - \mathcal{U}_u = 1 - R(1-C) - C^2 \tag{98}$$

and

$$\phi_e = \arccos\left(R - 1\right). \tag{99}$$

These equations show that the single-lobe area $\mathcal{A}_e$ is proportional to the harmonic $2h$ stationary bucket area $\mathcal{A}_2$ and to $\mathcal{B}_e$ which depends only on the ratio R.

For

$$R = 0, \quad \frac{1}{2}, \quad 1, \quad \frac{3}{2}, \quad 2 \tag{100}$$

we have turning point phases

$$\phi_e = \pi, \quad \frac{2\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{\pi}{3}, \quad 0 \tag{101}$$

and

$$\mathcal{B}_e = 4, \ \ 2.147143, \ \ 1.065680, \ \ 0.352081, \ \ 0. \tag{102}$$

For

$$R = 0, \quad 1, \quad \sqrt{2}, \quad \sqrt{3}, \quad 2 \tag{103}$$

we have stable fixed point phases

$$\phi_s = \frac{\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{\pi}{4}, \quad \frac{\pi}{6}, \quad 0 \tag{104}$$

and

$$\mathcal{B}_e = 4, \ \ 1.065680, \ \ 0.451194, \ \ 0.134490, \ \ 0. \tag{105}$$

Thus, as R goes from 0 to 2, the stable fixed point phase ϕ_s goes from $\pi/2$ to 0, the turning point phase ϕ_e goes from π to 0, and the lobe area $\mathcal{A}_e$ goes from $\mathcal{A}_2$ to 0.

12 The merge

When

$$R = 0 \tag{106}$$

the single-lobe area (96) is

$$\mathcal{A}_e = \mathcal{A}_2 \tag{107}$$

and the unmerged bunches are sitting in stationary harmonic $2h$ buckets with area $\mathcal{A}_2$. We call this the **start of merge**. As R increases from 0, the bunches follow the phases $\pm\phi_s$ and move toward phase zero as shown in Section 9. The lobe area (96) decreases, and when it equals the area occupied by the bunch, particles begin to be squeezed out of the inner separatrix and into the area enclosed by an outer separatrix. The outer separatrix is associated with unstable fixed point phases $\pm\pi$. It is discussed in Section 15. As R approaches and reaches 2, the lobe area shrinks to zero and all the particles of both bunches end up in the RF bucket formed by the outer separatrix. At this point the bunches have been brought together and occupy a single RF bucket. We call this point the **end of merge**. After this there is an additional period of time during which the harmonic $2h$ voltage is brought down to zero while the harmonic h voltage is held constant. This leaves the merged bunch sitting in a single harmonic h bucket and the merge is said to be **complete**.

13 RF voltages at the start and end of merge

We use subscripts I and F to denote parameters at the **start** and **end of merge**. At the start of merge we have harmonic h and $2h$ voltages

$$V_1 = V_{1I}, \quad V_2 = V_{2I} \tag{108}$$

and at the end

$$V_1 = V_{1F}, \quad V_2 = V_{2F}. \tag{109}$$

Since the merge starts and ends with $R = 0$ and 2, respectively, we have

$$R_I = \frac{V_{1I}}{V_{2I}} = 0, \quad R_F = \frac{V_{1F}}{V_{2F}} = 2 \tag{110}$$

which gives

$$V_{1I} = 0, \quad V_{1F} = 2V_{2F}. \tag{111}$$

The harmonic h and $2h$ stationary bucket areas [3]

$$\mathcal{A}_1 = 8\,\frac{\mathcal{R}}{hc}\left\{\frac{2eQV_1E}{\pi h|\eta|}\right\}^{1/2}, \quad \mathcal{A}_2 = 4\,\frac{\mathcal{R}}{hc}\left\{\frac{eQV_2E}{\pi h|\eta|}\right\}^{1/2} \tag{112}$$

give

$$\frac{\mathcal{A}_1}{\mathcal{A}_2} = 2\sqrt{2}\left(\frac{V_1}{V_2}\right)^{1/2} \tag{113}$$

and

$$\frac{\mathcal{A}_{1F}}{\mathcal{A}_{2I}} = 2\sqrt{2}\left(\frac{V_{1F}}{V_{2I}}\right)^{1/2}. \tag{114}$$

Now, to ensure that the merged bunch fits in the harmonic h bucket upon **completion** of the merge, it is sufficient to require that

$$\mathcal{A}_{1F} \geq 2\mathcal{A}_{2I}. \tag{115}$$

If the unmerged bunches happen to completely fill the initial harmonic $2h$ buckets, then (115) becomes a necessary condition. If we make the further requirement that ratio of the merged bunch area to bucket area $\mathcal{A}_{1F}$ be the same as the ratio of the unmerged bunch area to bucket area $\mathcal{A}_{2I}$, then we must have

$$\mathcal{A}_{1F} = 2\mathcal{A}_{2I} \tag{116}$$

and it follows from (114) that

$$V_{1F} = \frac{1}{2}\,V_{2I}. \tag{117}$$

Using the second of equations (111),

$$V_{1F} = 2V_{2F}, \tag{118}$$

we then have

$$V_{2F} = \frac{1}{4}\,V_{2I}. \tag{119}$$

The end of merge harmonic h and $2h$ voltages, V_{1F} and V_{2F}, are then completely determined by the initial harmonic $2h$ voltage V_{2I}.

14 Normalized Parameters

It is convenient to normalize voltages and bucket areas with respect to initial harmonic $2h$ voltage V_{2I} and corresponding bucket area $\mathcal{A}_{2I}$. Thus we define normalized voltages and lobe area

$$V_1^N = \frac{V_1}{V_{2I}}, \quad V_2^N = \frac{V_2}{V_{2I}}, \quad \mathcal{A}_e^N = \frac{\mathcal{A}_e}{\mathcal{A}_{2I}}. \tag{120}$$

We also define normalized bunch area

$$b_e^N = \frac{b_e}{\mathcal{A}_{2I}} \tag{121}$$

where b_e is the area of the bunch matched to either lobe of the inner separatrix. Using

$$\mathcal{A}_e = \frac{1}{4}\,\mathcal{A}_2\,\mathcal{B}_e \tag{122}$$

and

$$\mathcal{A}_2 = 4\,\frac{\mathcal{R}}{hc}\left\{\frac{eQV_2E}{\pi h|\eta|}\right\}^{1/2} \tag{123}$$

we have normalized lobe area

$$\mathcal{A}_e^N = \frac{1}{4} \left(V_2^N \right)^{1/2} \mathcal{B}_e \tag{124}$$

where $\mathcal{B}_e$ depends only on

$$R = \frac{V_1}{V_2} = \frac{V_1^N}{V_2^N}. \tag{125}$$

Using

$$V_{1I} = 0, \quad V_{1F} = \frac{1}{2} V_{2I}, \quad V_{2F} = \frac{1}{4} V_{2I} \tag{126}$$

and

$$\mathcal{B}_{eI} = 4, \quad \mathcal{B}_{eF} = 0 \tag{127}$$

we have normalized voltages

$$V_{1I}^N = 0, \quad V_{1F}^N = \frac{1}{2}, \quad V_{2I}^N = 1, \quad V_{2F}^N = \frac{1}{4} \tag{128}$$

and normalized areas

$$\mathcal{A}_{eI}^N = 1, \quad \mathcal{A}_{eF}^N = 0. \tag{129}$$

The normalized inner separatrix is given by

$$W_e^N = \frac{W_e}{\mathcal{A}_{2I}} \tag{130}$$

where

$$W_e = \pm \frac{1}{4} \mathcal{A}_2 \left(\mathcal{U} - \mathcal{U}_u \right)^{1/2} \tag{131}$$

as shown in Section 11. This gives

$$W_e^N = \pm \frac{1}{4} \left(\frac{V_2}{V_{2I}} \right)^{1/2} \left(\mathcal{U} - \mathcal{U}_u \right)^{1/2} \tag{132}$$

and therefore

$$W_e^N = \pm \frac{1}{4} \left(V_2^N \right)^{1/2} \left(\mathcal{U} - \mathcal{U}_u \right)^{1/2}. \tag{133}$$

15 The outer separatrix and area enclosed

The outer separatrix is the separatrix that passes through the unstable fixed point phases $\phi_u = \pm\pi$. These give

$$\mathcal{U}_u = -R - \frac{1}{2} \tag{134}$$

and therefore

$$\mathcal{U} - \mathcal{U}_u = 1 + R(1 + C) - C^2. \tag{135}$$

The separatrix is then given by

$$W_O = \pm\frac{1}{4}\,\mathcal{A}_2\,(\mathcal{U} - \mathcal{U}_u)^{1/2} \tag{136}$$

and the area enclosed by the right or left half is

$$\mathcal{A}_O = \frac{1}{4}\,\mathcal{A}_2\,\mathcal{B}_O \tag{137}$$

where

$$\mathcal{B}_O = 2\int_0^\pi (\mathcal{U} - \mathcal{U}_u)^{1/2}\,d\phi. \tag{138}$$

The normalized area is

$$\mathcal{A}_O^N = \frac{\mathcal{A}_O}{\mathcal{A}_{2I}} \tag{139}$$

and therefore

$$\mathcal{A}_O^N = \frac{1}{4}\left(V_2^N\right)^{1/2}\mathcal{B}_O \tag{140}$$

where $\mathcal{B}_O$ depends only on

$$R = \frac{V_1}{V_2} = \frac{V_1^N}{V_2^N}. \tag{141}$$

The normalized separatrix is given by

$$W_O^N = \frac{W_O}{\mathcal{A}_{2I}} \tag{142}$$

and therefore

$$W_O^N = \pm\frac{1}{4}\left(V_2^N\right)^{1/2}(\mathcal{U} - \mathcal{U}_u)^{1/2}. \tag{143}$$

16 Synchrotron frequency during the merge

Because the RF voltages V_1 and V_2 are changing in time during the merge, the instantaneous synchrotron frequency, or its inverse the synchrotron period, are useful for estimating the amount of time required to make the merge as adiabatic as practical. The synchrotron frequency of small oscillations about the stable fixed point phase

$$\phi_s = \arccos\left(\frac{R}{2}\right) \tag{144}$$

is given by [4]

$$F_s = \frac{1}{2\pi}\sqrt{ak} \tag{145}$$

where

$$a = \left(\frac{h^2\omega^2\eta}{\beta^2 E}\right) = \left(\frac{h^2c^2\eta}{\mathcal{R}^2 E}\right) \tag{146}$$

and

$$k = -F'(\phi_s). \tag{147}$$

Here

$$F(\phi) = A_1 \sin\phi - A_2 \sin 2\phi \tag{148}$$

and

$$F'(\phi) = A_1 \cos\phi - 2A_2 \cos 2\phi. \tag{149}$$

Using

$$\cos 2\phi_s = 2\cos^2\phi_s - 1 \tag{150}$$

we have

$$F'(\phi_s) = A_1 \cos\phi_s - 2A_2\left(2\cos^2\phi_s - 1\right) \tag{151}$$

which gives

$$k = -2A_2 - A_1\cos\phi_s + 4A_2\cos^2\phi_s. \tag{152}$$

Using

$$R = \frac{A_1}{A_2}, \quad \cos\phi_s = \frac{R}{2} \tag{153}$$

we then have

$$k = A_2\left\{-2 - R\left(\frac{R}{2}\right) + 4\left(\frac{R}{2}\right)^2\right\} \tag{154}$$

$$k = A_2\left(\frac{R^2}{2} - 2\right) \tag{155}$$

where

$$A_2 = \frac{eQV_2}{2\pi h}. \tag{156}$$

Thus

$$ak = \left(\frac{h^2c^2\eta}{\mathcal{R}^2E}\right)\left(\frac{eQV_2}{2\pi h}\right)\left(\frac{R^2}{2} - 2\right) \tag{157}$$

which gives

$$F_s = \frac{c}{2\pi\mathcal{R}}\left\{\frac{2h|\eta|\, eQV_2}{2\pi E}\right\}^{1/2}\left(1 - \frac{R^2}{4}\right)^{1/2} \tag{158}$$

where

$$0 \le R \le 2. \tag{159}$$

The corresponding synchrotron period is

$$P_s = \frac{1}{F_s}. \tag{160}$$

At the beginning of the merge we have

$$V_1 = V_{1I} = 0, \quad V_2 = V_{2I}, \quad R_I = \frac{V_{1I}}{V_{2I}} = 0 \tag{161}$$

and at the end

$$V_1 = V_{1F}, \quad V_2 = V_{2F}, \quad R_F = \frac{V_{1F}}{V_{2F}} = 2. \tag{162}$$

The initial synchrotron frequency is

$$F_{sI} = \frac{c}{2\pi\mathcal{R}}\left\{\frac{2h|\eta|\, eQV_{2I}}{2\pi E}\right\}^{1/2} \tag{163}$$

which is just the frequency of small oscillations in a stationary harmonic $2h$ bucket. The initial synchrotron period is

$$P_{sI} = \frac{1}{F_{sI}}. \tag{164}$$

We define normalized synchrotron frequency and period

$$F_s^N = \frac{F_s}{F_{sI}}, \quad P_s^N = \frac{P_s}{P_{sI}} = \frac{F_{sI}}{F_s} \tag{165}$$

which gives

$$F_s^N = \left(\frac{V_2}{V_{2I}}\right)^{1/2} \left(1 - \frac{R^2}{4}\right)^{1/2} , \quad P_s^N = \frac{1}{F_s^N} \tag{166}$$

and

$$F_s = F_{sI} F_s^N, \quad P_s = P_{sI} P_s^N. \tag{167}$$

The initial and final normalized synchrotron frequencies are

$$F_{sI}^N = \left(\frac{V_{2I}}{V_{2I}}\right)^{1/2} \left(1 - \frac{R_I^2}{4}\right)^{1/2} = 1 \tag{168}$$

and

$$F_{sF}^N = \left(\frac{V_{2F}}{V_{2I}}\right)^{1/2} \left(1 - \frac{R_F^2}{4}\right)^{1/2} = 0. \tag{169}$$

This means that the merge becomes less adiabatic as it proceeds. The conditions for the adiabaticity of the merge are discussed in Section 19.

17 Synchrotron frequency after end of merge

At the end of the merge we have

$$R = 2. \tag{170}$$

At this point the area enclosed by the inner separatrix reaches zero, and stable fixed point phase ϕ_s given in Section 9 becomes zero. After the end of merge we have

$$R > 2 \tag{171}$$

and the fixed point phase $\phi_f = 0$ is stable as shown in Section 8. The harmonic h voltage V_1 is held fixed at V_{1F} while the harmonic $2h$ voltage is brought from V_{2F} down to zero. During this time we take $\phi_s = 0$ in (152) which gives

$$k = -2A_2 - A_1 + 4A_2 \tag{172}$$

$$k = A_1(2P - 1) \tag{173}$$

where

$$P = \frac{1}{R} = \frac{A_2}{A_1} = \frac{V_2}{V_1}, \quad A_1 = \frac{eQV_1}{2\pi h} \tag{174}$$

and

$$0 \leq P \leq \frac{1}{2}. \tag{175}$$

Following the work in the previous section we then have

$$ak = \left(\frac{h^2 c^2 \eta}{\mathcal{R}^2 E}\right)\left(\frac{eQV_1}{2\pi h}\right)(2P - 1) \tag{176}$$

which gives synchrotron frequency

$$G_s = \frac{c}{2\pi\mathcal{R}}\left\{\frac{h|\eta|\, eQV_1}{2\pi E}\right\}^{1/2}(1 - 2P)^{1/2}. \tag{177}$$

When the harmonic $2h$ voltage has been brought down to zero we have

$$P = 0 \tag{178}$$

and the synchrotron frequency is

$$G_{sF} = \frac{c}{2\pi\mathcal{R}}\left\{\frac{h|\eta|\, eQV_{1F}}{2\pi E}\right\}^{1/2}. \tag{179}$$

As explained in Section 13, we have

$$V_{1F} = \frac{1}{2}\, V_{2I} \tag{180}$$

which gives

$$G_{sF} = \frac{c}{2\pi\mathcal{R}}\left\{\frac{2h|\eta|\, eQV_{2I}}{4(2\pi E)}\right\}^{1/2} \tag{181}$$

and therefore

$$G_{sF} = \frac{1}{2}\, F_{sI} \tag{182}$$

where F_{sI} is given by (163). This would be the synchrotron frequency at the start of the next 2 to 1 merge if there is one. The second merge therefore would need to proceed at half the rate of the first merge in order to be as adiabatic as the first.

18 The RF voltage program

As shown in Sections 9 through 12, the merge starts when $R = 0$ and ends when $R = 2$. Let T be the time taken for R to go from 0 to 2. We assume that the harmonic h and harmonic $2h$ voltages vary linearly with time t. Then for

$$0 \leq t \leq T \tag{183}$$

we have

$$V_1 = V_{1F} \frac{t}{T} \tag{184}$$

and

$$V_2 = V_{2I} - (V_{2I} - V_{2F}) \frac{t}{T}. \tag{185}$$

In these equations we must have

$$V_{1F} = 2V_{2F} \tag{186}$$

in order to have

$$R_F = \frac{V_{1F}}{V_{2F}} = 2. \tag{187}$$

After the end of merge we want to bring the harmonic $2h$ voltage down to zero while keeping the harmonic h voltage fixed. This will complete the merge. Thus for

$$T < t \leq \mathcal{T} \tag{188}$$

we take

$$V_1 = V_{1F} \tag{189}$$

and

$$V_2 = V_{2I} - (V_{2I} - V_{2F}) \frac{t}{T}. \tag{190}$$

We require that $V_2 = 0$ when $t = \mathcal{T}$, which gives

$$\mathcal{T} = \left(\frac{V_{2I}}{V_{2I} - V_{2F}} \right) T. \tag{191}$$

This is the total time required for the merge. We also have

$$\mathcal{T} - T = \left(\frac{V_{2F}}{V_{2I} - V_{2F}} \right) T \tag{192}$$

which is the additional time required after the end of merge. If the total time $\mathcal{T}$ is given then the time from start to end of merge is

$$T = \left(\frac{V_{2I} - V_{2F}}{V_{2I}} \right) \mathcal{T}. \tag{193}$$

The additional time needed to bring the harmonic $2h$ voltage down to zero is

$$\mathcal{T} - T = \frac{V_{2F}}{V_{2I}} \mathcal{T}. \tag{194}$$

Now, as already discussed in Section 13, we want to have

$$V_{1F} = \frac{1}{2} V_{2I} \tag{195}$$

and this together with (186) gives

$$V_{2F} = \frac{1}{4} V_{2I}. \tag{196}$$

Equations (193) and (194) then become

$$T = \frac{3}{4} \mathcal{T}, \quad \mathcal{T} - T = \frac{1}{4} \mathcal{T} \tag{197}$$

and we see that three fourths of the total time $\mathcal{T}$ is used to go from start to end of merge. One fourth of the total time is used to complete the merge by bringing V_2 from V_{2F} down to zero. The harmonic h and $2h$ voltages in (184) and (185) become

$$V_1 = \frac{V_{2I}}{2} \left(\frac{t}{T} \right), \quad V_2 = V_{2I} \left(1 - \frac{3}{4} \frac{t}{T} \right). \tag{198}$$

These give

$$R = \frac{V_1}{V_2} = \frac{1}{2} \frac{t}{T} \left(1 - \frac{3}{4} \frac{t}{T} \right)^{-1} \tag{199}$$

and the inverse

$$\frac{t}{T} = \frac{4R}{2 + 3R}. \tag{200}$$

In terms of the total time $\mathcal{T}$ we have

$$V_2 = \left(1 - \frac{t}{\mathcal{T}} \right) V, \text{ for } 0 \le \frac{t}{\mathcal{T}} \le 1 \tag{201}$$

$$V_1 = \frac{4}{3} \left(\frac{t}{\mathcal{T}} \right) \frac{V}{2}, \text{ for } 0 \le \frac{t}{\mathcal{T}} \le \frac{3}{4} \tag{202}$$

$$V_1 = \frac{V}{2}, \text{ for } \frac{3}{4} \le \frac{t}{\mathcal{T}} \le 1 \tag{203}$$

where

$$V = V_{2I}. \tag{204}$$

The corresponding time derivatives are

$$\frac{dV_2}{dt} = -\frac{V}{\mathcal{T}}, \text{ for } 0 \le \frac{t}{\mathcal{T}} \le 1 \tag{205}$$

$$\frac{dV_1}{dt} = \frac{2}{3}\frac{V}{\mathcal{T}}, \text{ for } 0 \le \frac{t}{\mathcal{T}} \le \frac{3}{4} \tag{206}$$

and

$$\frac{dV_1}{dt} = 0, \text{ for } \frac{3}{4} \le \frac{t}{\mathcal{T}} \le 1. \tag{207}$$

19 Adiabaticity of the merge

According to the treatment of adiabatic invariants in [5], we have the condition

$$P_s \left| \frac{dV}{dt} \right| \ll V \tag{208}$$

for the adiabaticity of the 2 to 1 merge. For small oscillations about the stable fixed point phase

$$\phi_s = \arccos\left(\frac{R}{2}\right) \tag{209}$$

we have, as shown in Section 16,

$$P_s = \frac{1}{F_s}, \quad F_s = F_{sI} F_s^N \tag{210}$$

$$F_s^N = \left(\frac{V_2}{V_{2I}}\right)^{1/2} \left(1 - \frac{R^2}{4}\right)^{1/2}, \quad R = \frac{V_1}{V_2} \tag{211}$$

and

$$F_{sI} = \frac{c}{2\pi\mathcal{R}} \left\{ \frac{2h|\eta|\, eQV_{2I}}{2\pi E} \right\}^{1/2}. \tag{212}$$

Taking

$$P_s = P_{sI}, \quad \left| \frac{dV}{dt} \right| = \frac{V_{2I}}{\mathcal{T}}, \quad V = V_{2I} \tag{213}$$

in (208) we have

$$P_{sI}\left(\frac{V_{2I}}{\mathcal{T}}\right) \ll V_{2I} \tag{214}$$

which gives the requirement

$$P_{sI} \ll \mathcal{T}. \tag{215}$$

Thus the total merge time $\mathcal{T}$ must be much greater than the initial synchrotron period P_{sI}.

For a 2 to 1 merge of polarized proton bunches sitting in harmonic 12 buckets in AGS at

$$G\gamma = 45.5, \tag{216}$$

the initial synchrotron period is

$$P_{sI} = 384 \text{ ms} \tag{217}$$

when the harmonic 12 voltage is adjusted to give bucket area

$$\mathcal{A}_{2I} = 0.648 \text{ eV s}. \tag{218}$$

The adiabaticity condition (215) then becomes

$$384 \text{ ms} \ll \mathcal{T}. \tag{219}$$

The simulations discussed in Section 25 show that taking

$$\mathcal{T} = 4000 \text{ ms} \tag{220}$$

gives a reasonably good (though not perfect) merge.

20 Illustrations of the RF voltage program and resulting parameters

Figures 1 through **6** in the following pages illustrate the RF voltage program and resulting bucket parameters, $\mathcal{A}_e$, $\mathcal{A}_O$, F_s, P_s, ϕ_s and ϕ_e. Here $\mathcal{A}_e$ and $\mathcal{A}_O$ are the areas enclosed by the inner and outer separatrices respectively. F_s and P_s are the synchrotron frequency and period of oscillations about the stable fixed points associated with the inner separatrix. ϕ_s is the stable fixed point phase and ϕ_e the associated maximum turning point phase. These parameters have been introduced in Sections 9–12, 15, and 16.

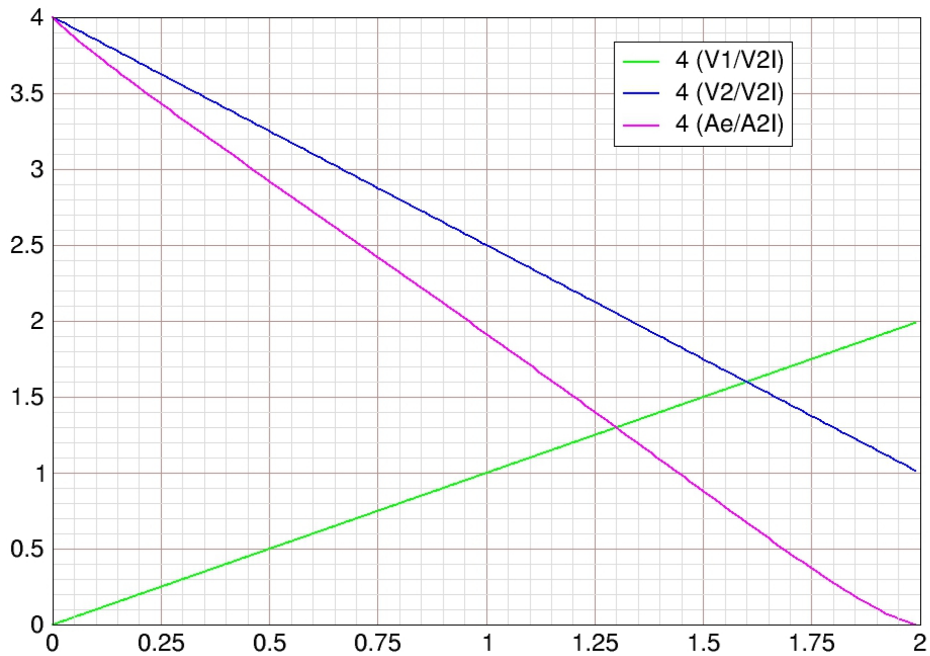


Figure 1: Here the ratios $4V_1/V_{2I}$, $4V_2/V_{2I}$, and $4\mathcal{A}_e/\mathcal{A}_{2I}$ are plotted as functions of time. The harmonic h and $2h$ voltages, V_1 and V_2, are given by equations (198). V_{2I} is the initial harmonic $2h$ voltage and $\mathcal{A}_{2I}$ is the corresponding bucket area given by (94) with $V_2 = V_{2I}$. The lobe area $\mathcal{A}_e$ is the area of either lobe of the inner separatrix as explained in Section 11. The horizontal axis gives the time in units for which the time T from start to end of merge is 2. For any given initial harmonic $2h$ voltage V_{2I} and corresponding bucket area $\mathcal{A}_{2I}$, one can obtain voltages V_1 and V_2, and lobe area $\mathcal{A}_e$ from the green, blue, and pink curves respectively. The plot shows that $\mathcal{A}_e$ decreases approximately linearly with respect to time, which means that particles will be squeezed out of the inner separatrix at a nearly constant rate if the merge is done adiabatically. The approximate linear behavior of $\mathcal{A}_e$ is due to the fact that the RF voltages vary linearly with time and have the particular proportions given by (198).

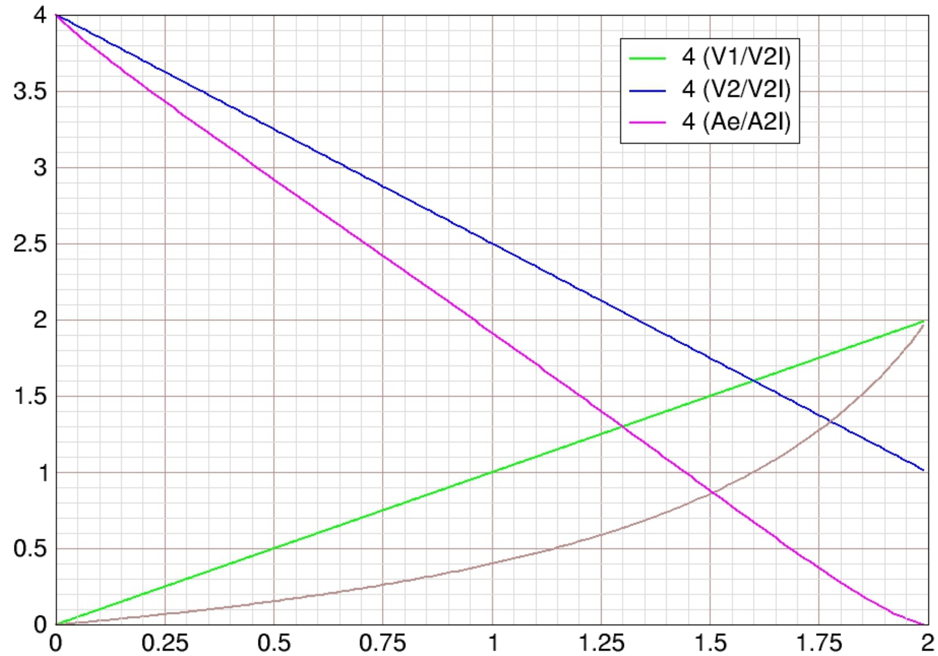


Figure 2: Here the ratio $R = V_1/V_2$ (shown as the brown curve) is plotted along with the curves from Figure 1. As explained in Section 12, the merge starts and ends with $R = 0$ and 2 respectively. When $R = 2$ is reached, the harmonic h voltage V_1 stops increasing and is held constant. The harmonic $2h$ voltage V_2 continues to ramp down to zero. The merge is complete when V_2 has reached zero.

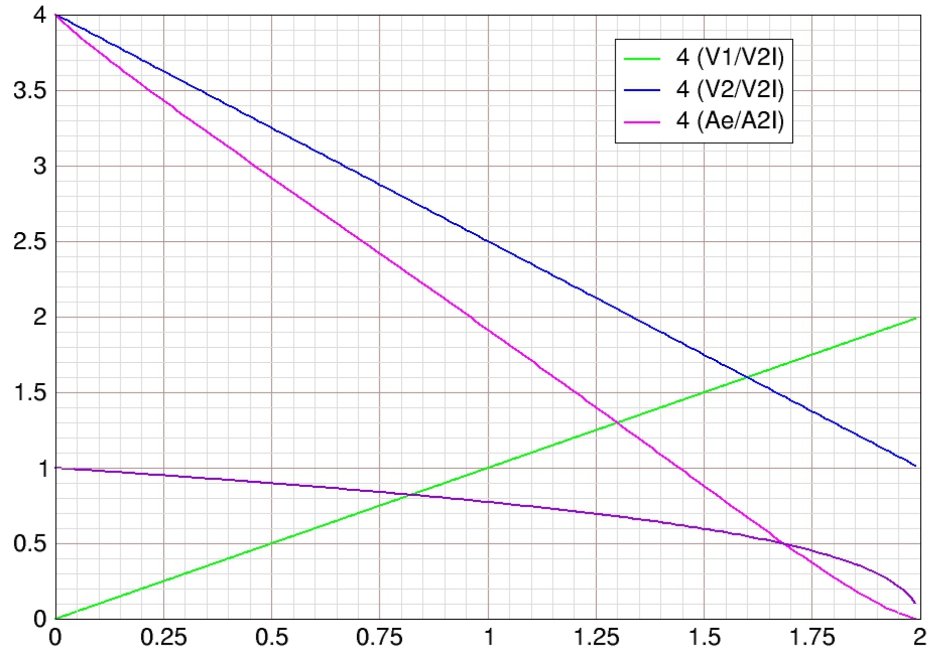


Figure 3: Here the ratio F_s/F_{sI} (shown as the violet curve) is plotted along with the curves from Figure 1. The synchrotron frequency F_s is the frequency of small amplitude oscillations about the stable fixed point phase of either lobe of the inner separatrix as explained in Section 16. At time $t = 0$ this is equal to F_{sI}, the frequency of oscillation in the harmonic $2h$ bucket at the start of the merge. If F_{sI} is given, the frequency F_s can be obtained from the plotted values of F_s/F_{sI}. The merge becomes less and less adiabatic as F_s decreases from F_{sI} toward zero. However, the plot shows that for most of the merge F_s is greater than $F_{sI}/2$.

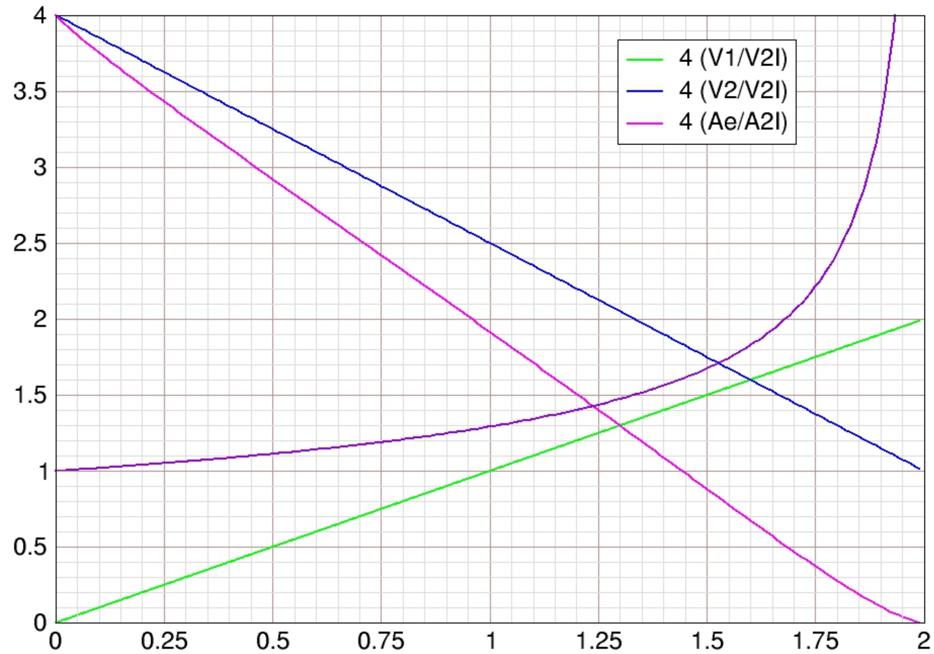


Figure 4: Here the ratio P_s/P_{sI} (shown as the violet curve) is plotted along with the curves from Figure 1. The synchrotron period P_s is the period of small amplitude oscillations about the stable fixed point phase of either lobe of the inner separatrix as explained in Section 16. At time $t = 0$ this is equal to P_{sI}, the period of oscillation in the harmonic $2h$ bucket at the start of the merge. If P_{sI} is given, the period P_s can be obtained from the plotted values of P_s/P_{sI}. The merge becomes less and less adiabatic as P_s increases from P_{sI} toward infinity. However, the plot shows that for most of the merge P_s is less than $2P_{sI}$.

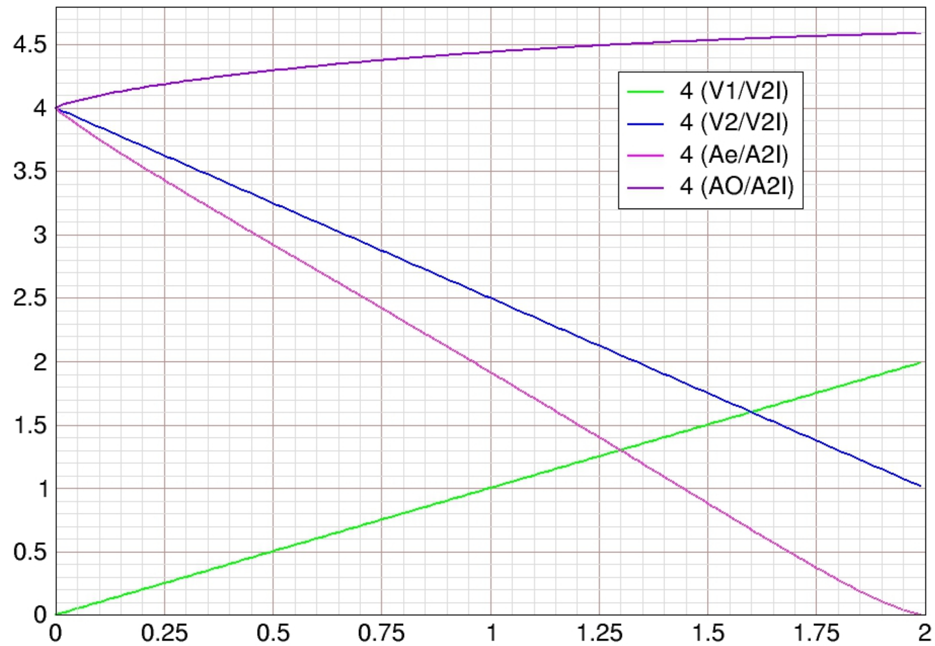


Figure 5: Here the ratio $4\mathcal{A}_O/\mathcal{A}_{2I}$ (shown as the violet curve) is plotted along with the curves from Figure 1. The area $\mathcal{A}_O$ is the area enclosed by the left or right half of the outer separatrix as explained in Section 15. The plot shows that this area increases monotonically from the initial harmonic $2h$ bucket area $\mathcal{A}_{2I}$ and therefore contains the bunches as they are merged.

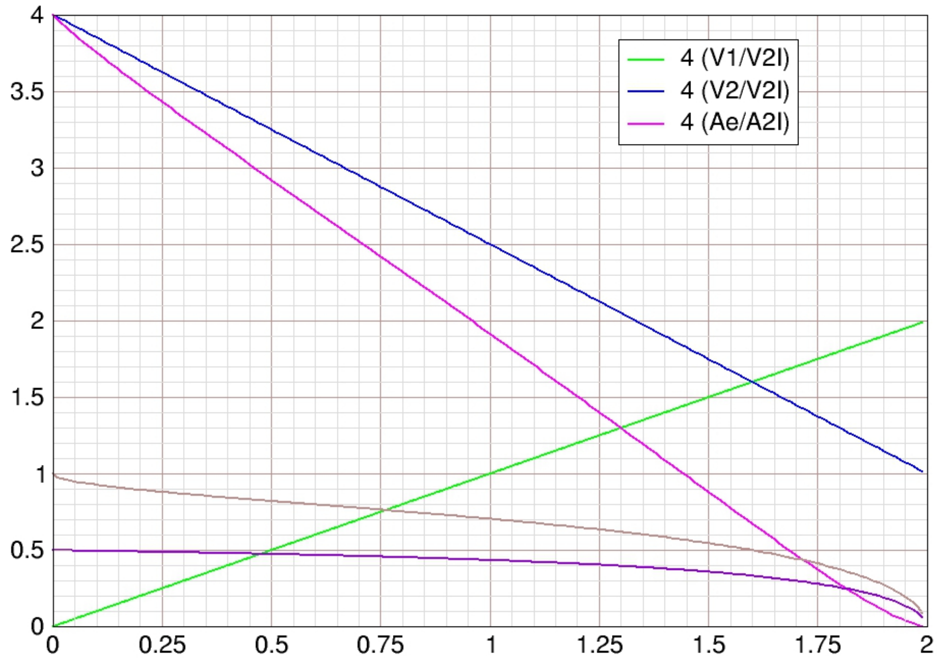


Figure 6: Here the ratios ϕ_s/π and ϕ_e/π (shown as the violet and brown curves respectively) are plotted along with the curves from Figure 1. The phase ϕ_s is the stable fixed point phase associated with the right lobe of the inner separatrix, and ϕ_e is the corresponding maximum turning point phase as discussed in Sections 9 and 10. The evolution of ϕ_s and ϕ_e during the merge also can be seen in **Figures 7** through **18** of the following section.

21 Illustration of the inner and outer separatrix evolution during the merge

The inner and outer separatrices are defined in Sections 11 and 15. **Figures 7** through **18** in the following pages show the evolution of the separatrices during the 2 to 1 merge. The upper figure on each page shows the separatrices. The lower figure shows the RF voltage program and inner separatrix lobe area $\mathcal{A}_e$. At the start of the merge the two separatrices coincide with the harmonic $2h$ bucket.

Figure 19 shows the corresponding evolution of the associated potential wells.

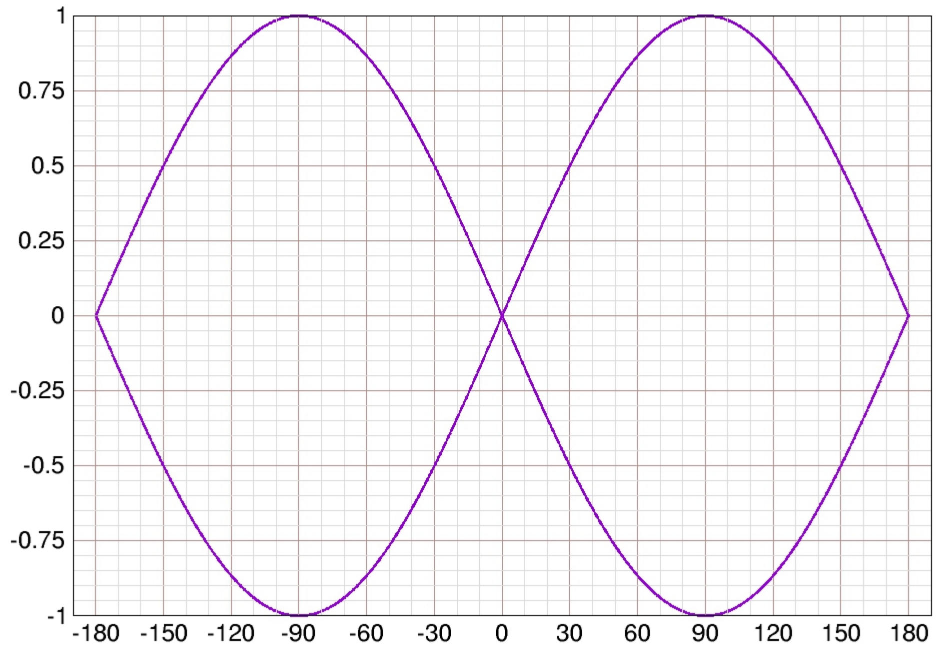
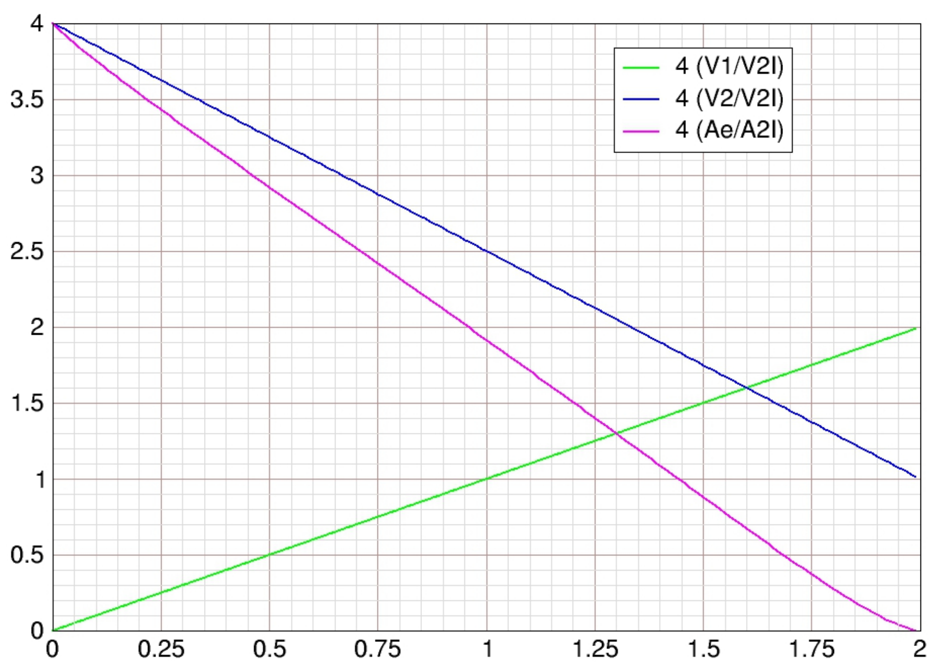


Figure 7: Inner separatrix at time 0.01



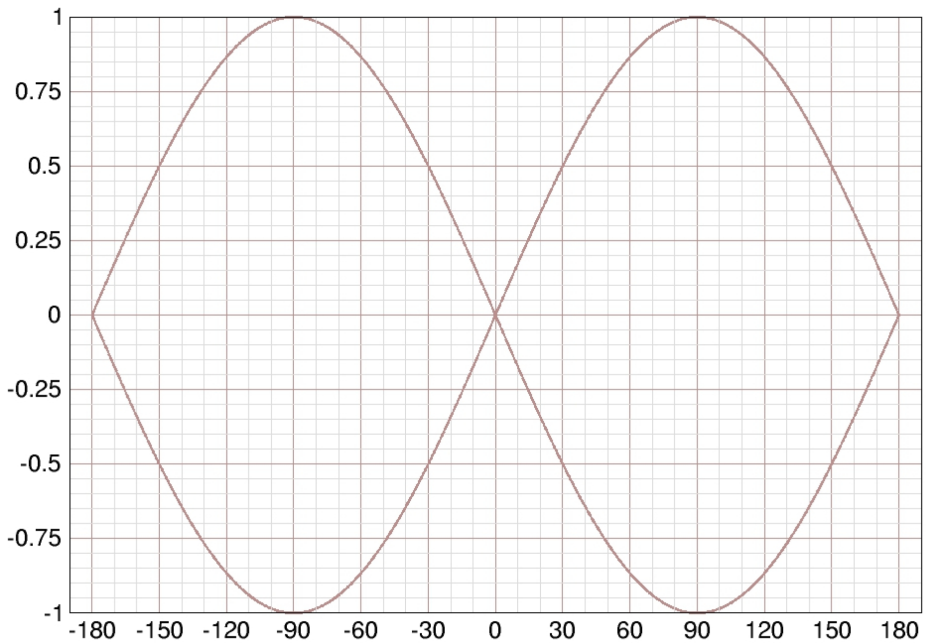
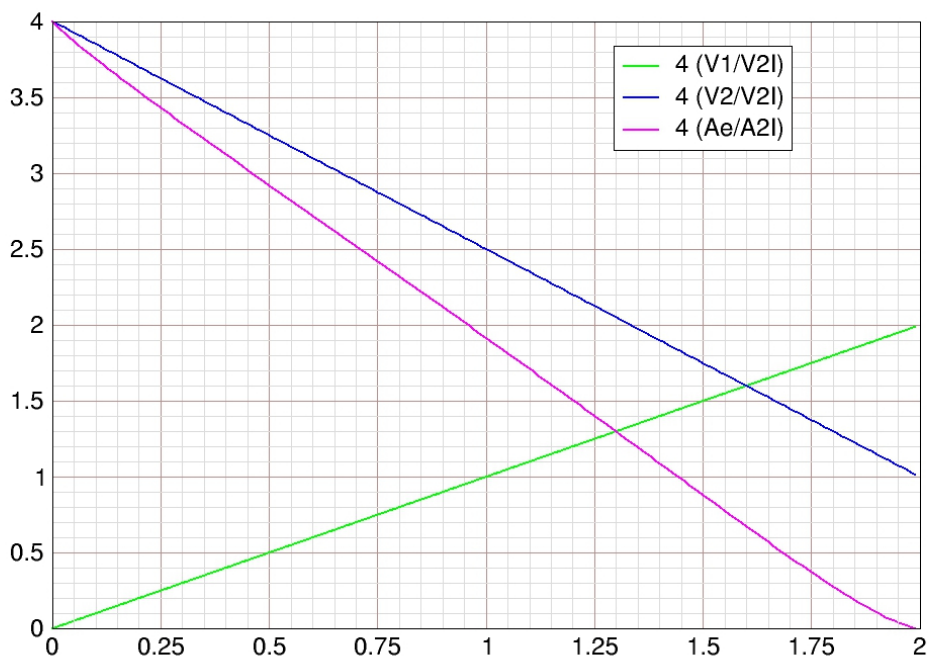


Figure 8: Outer separatrix at time 0.01



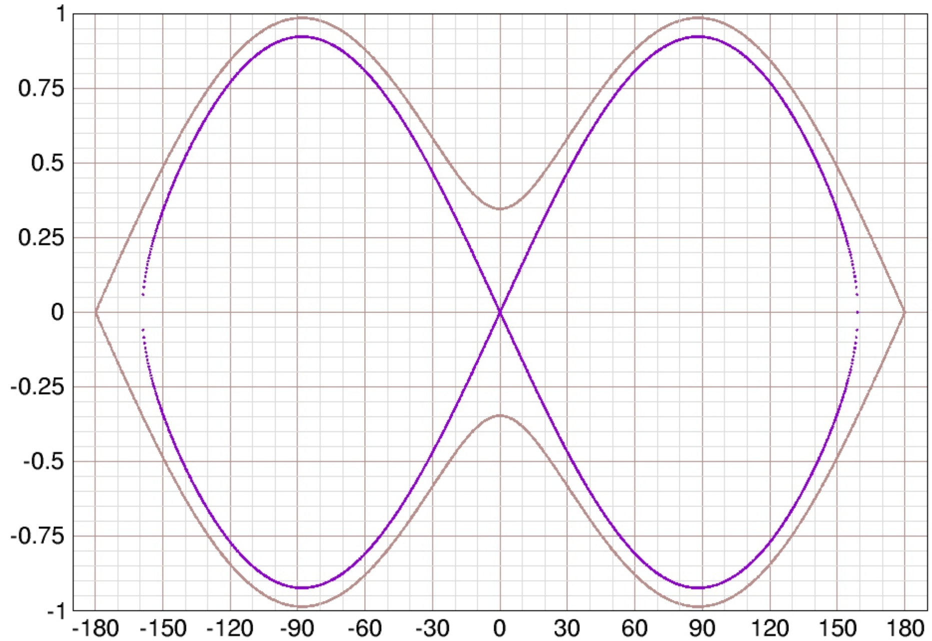
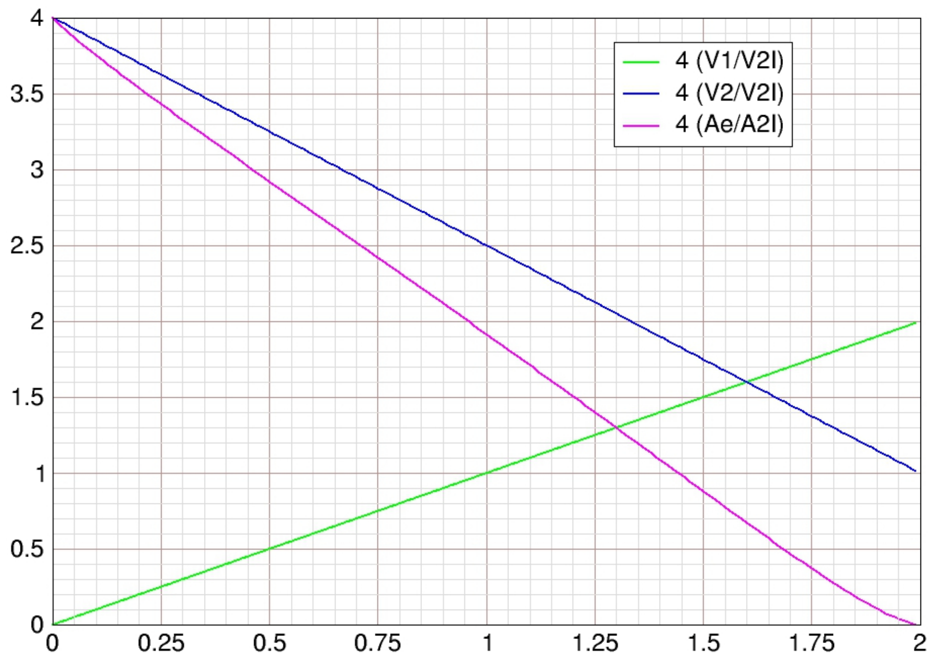


Figure 9: Inner (violet) and outer (brown) separatrices at time 0.25



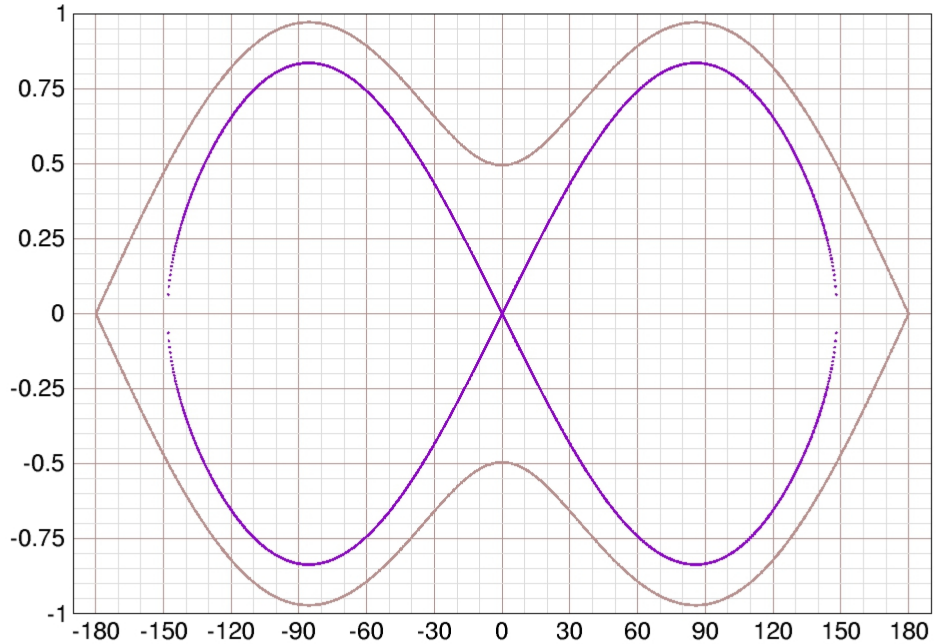
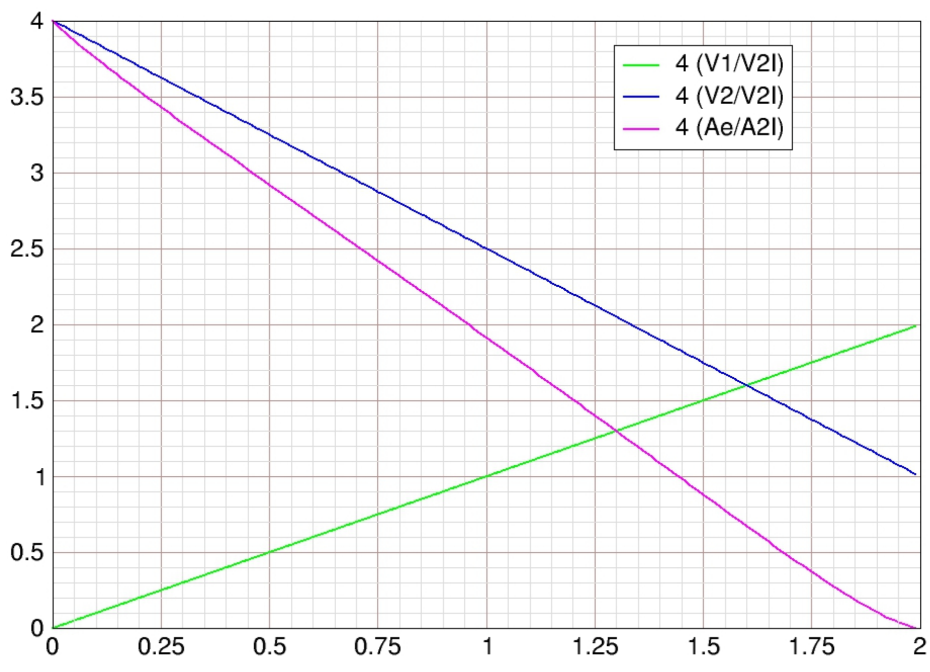


Figure 10: Inner (violet) and outer (brown) separatrices at time 0.50



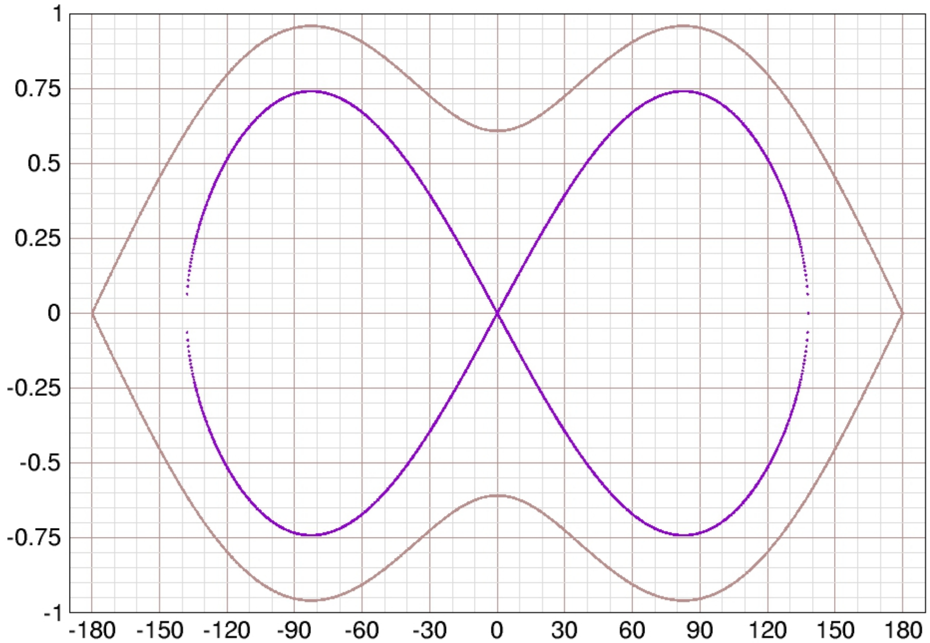
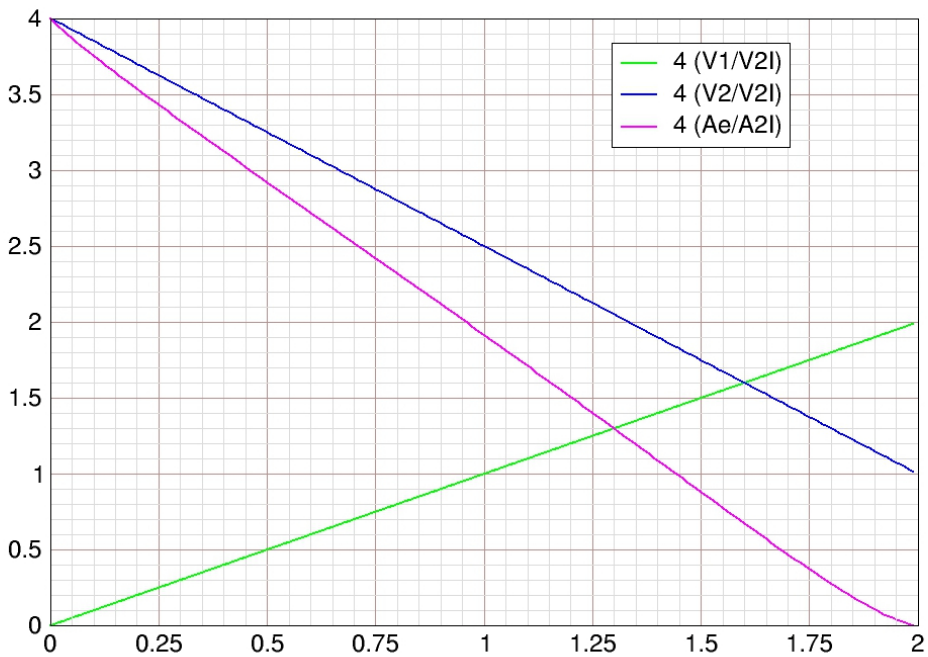


Figure 11: Inner (violet) and outer (brown) separatrices at time 0.75



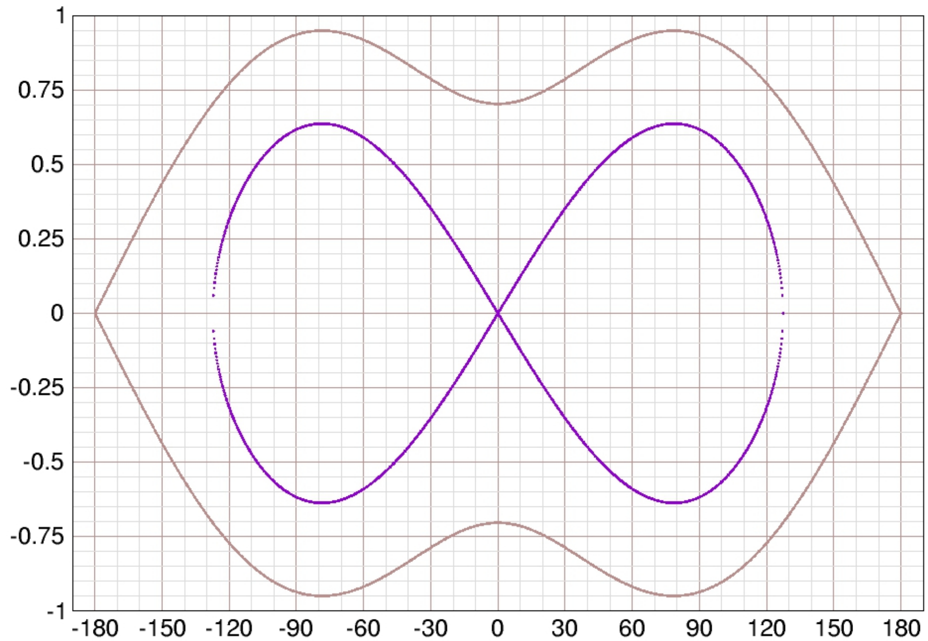
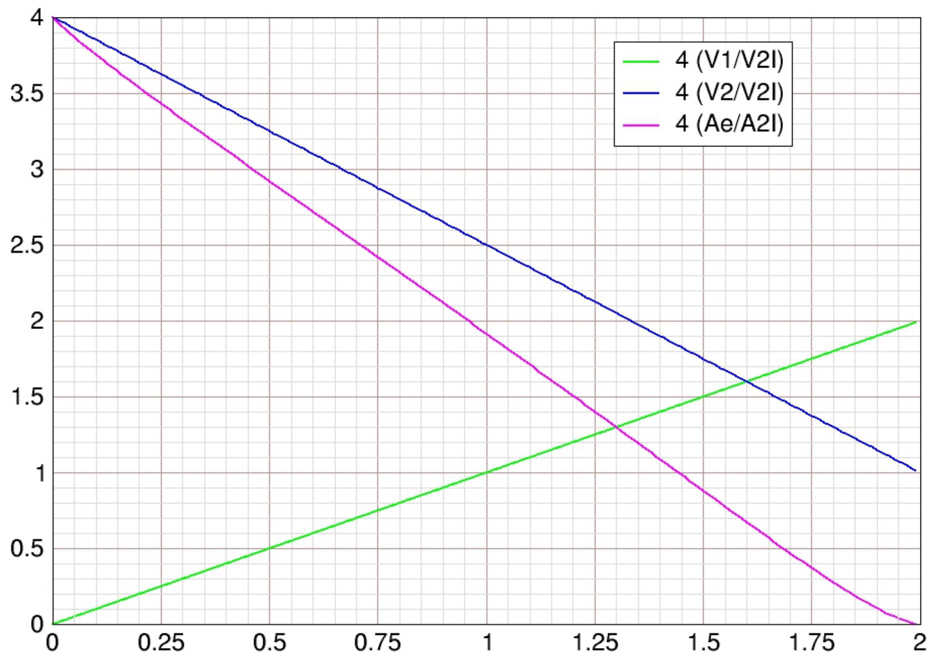


Figure 12: Inner (violet) and outer (brown) separatrices at time 1.00



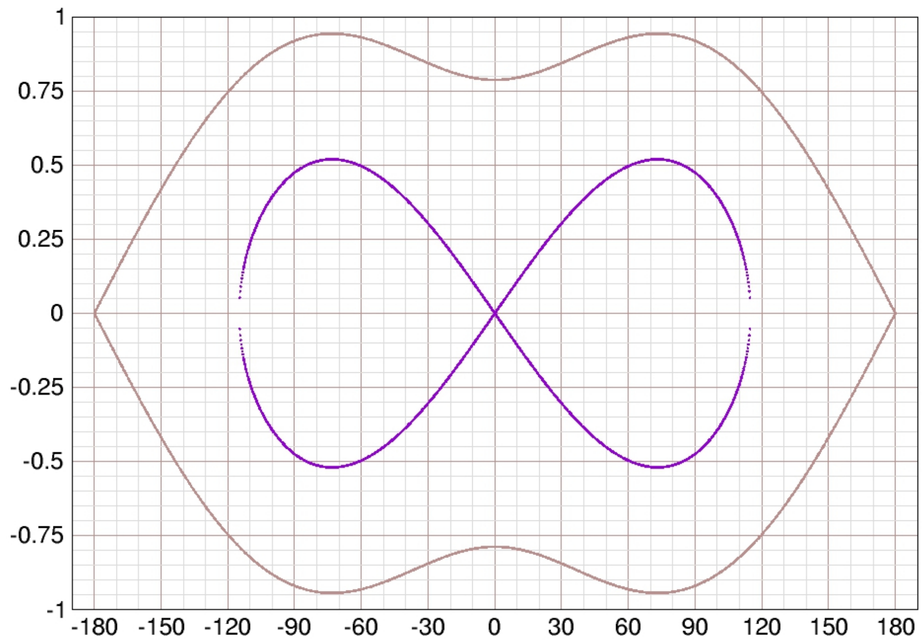
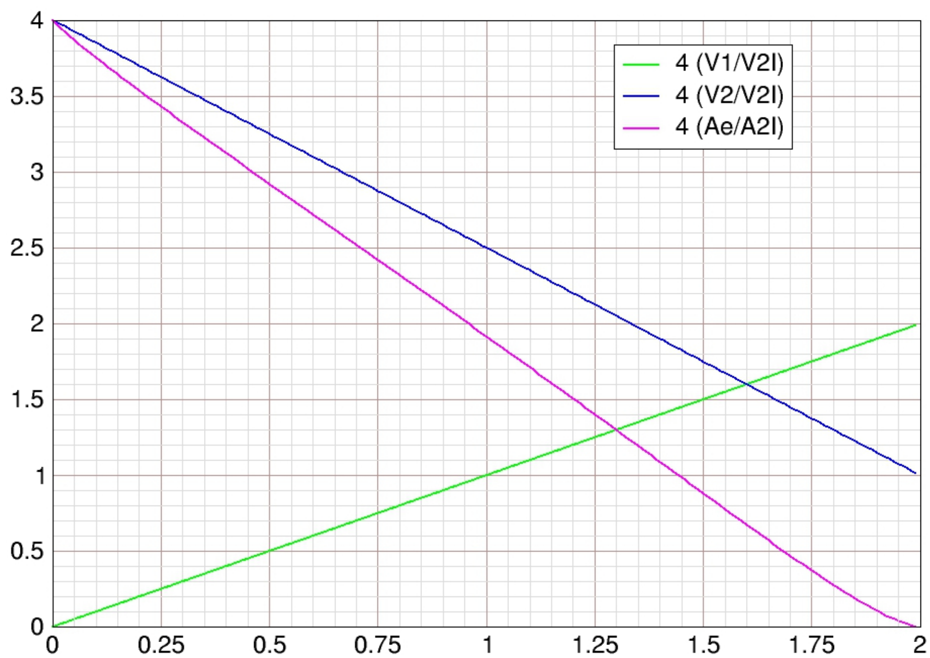


Figure 13: Inner (violet) and outer (brown) separatrices at time 1.25



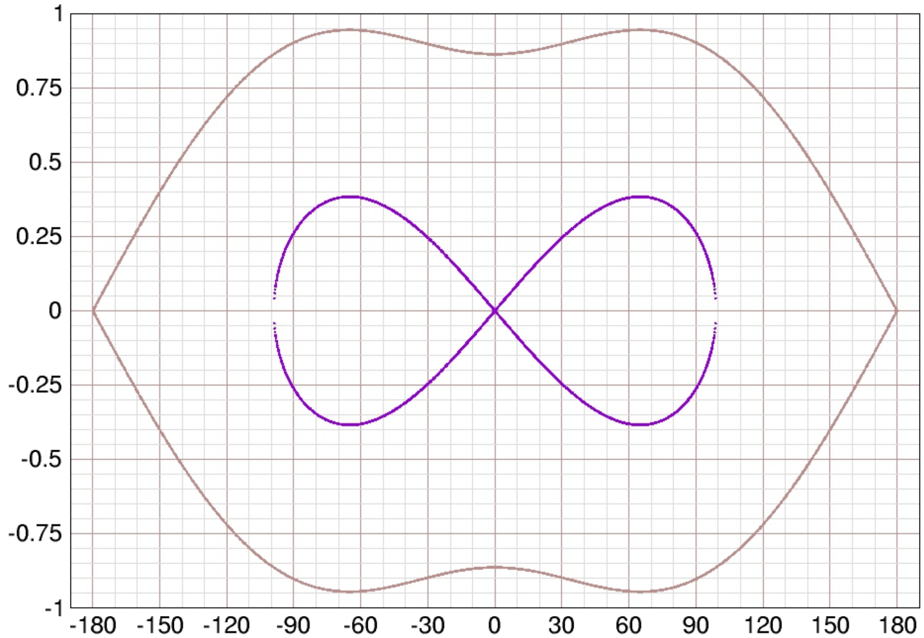
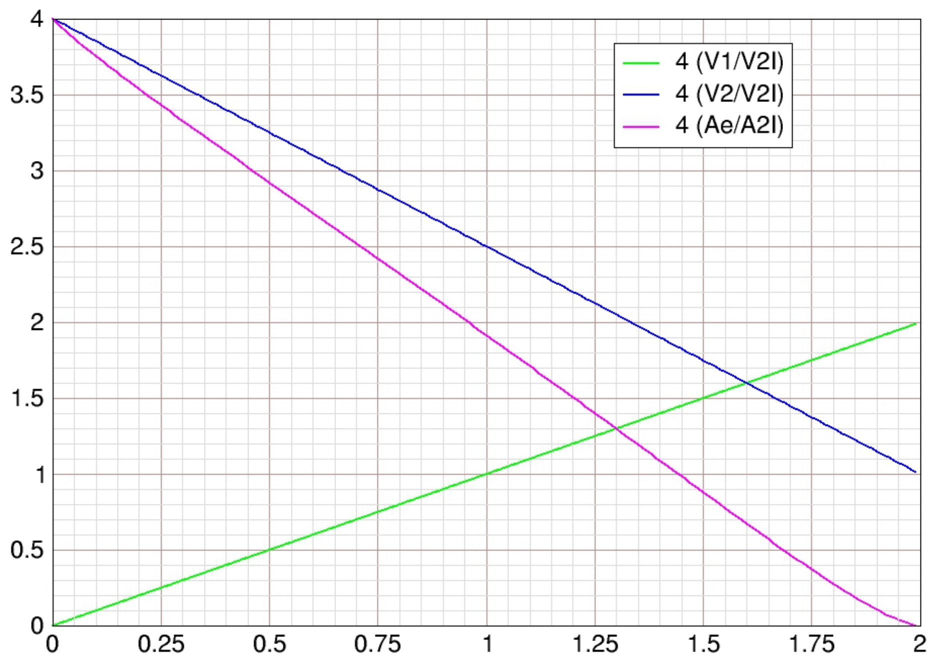


Figure 14: Inner (violet) and outer (brown) separatrices at time 1.50



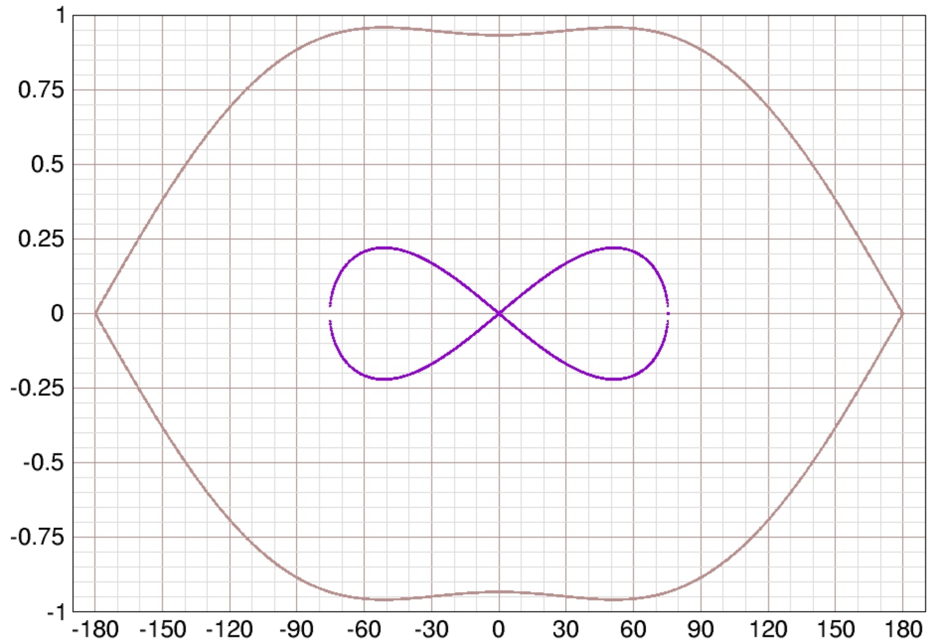
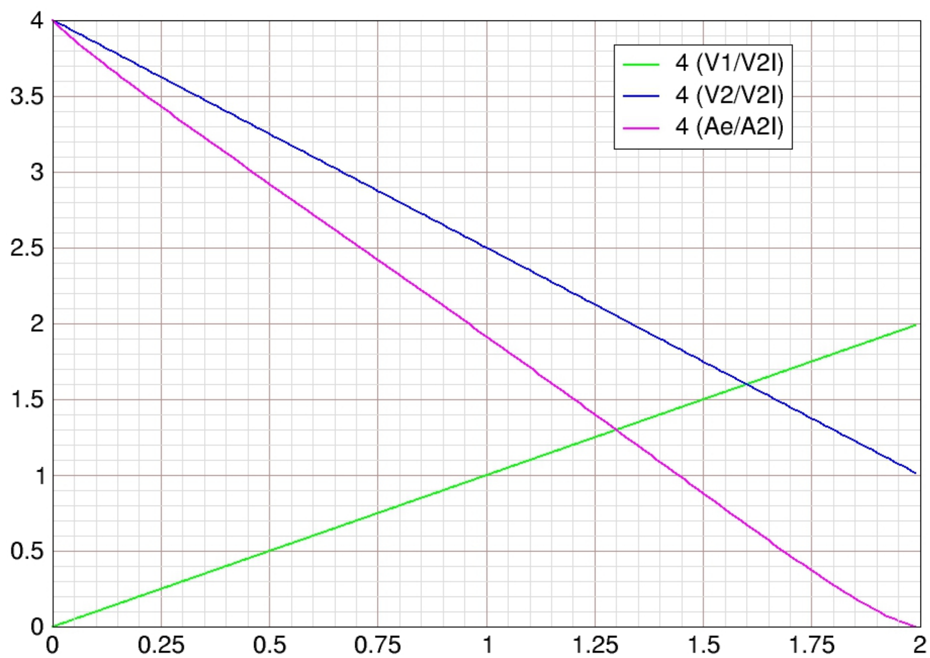


Figure 15: Inner (violet) and outer (brown) separatrices at time 1.75



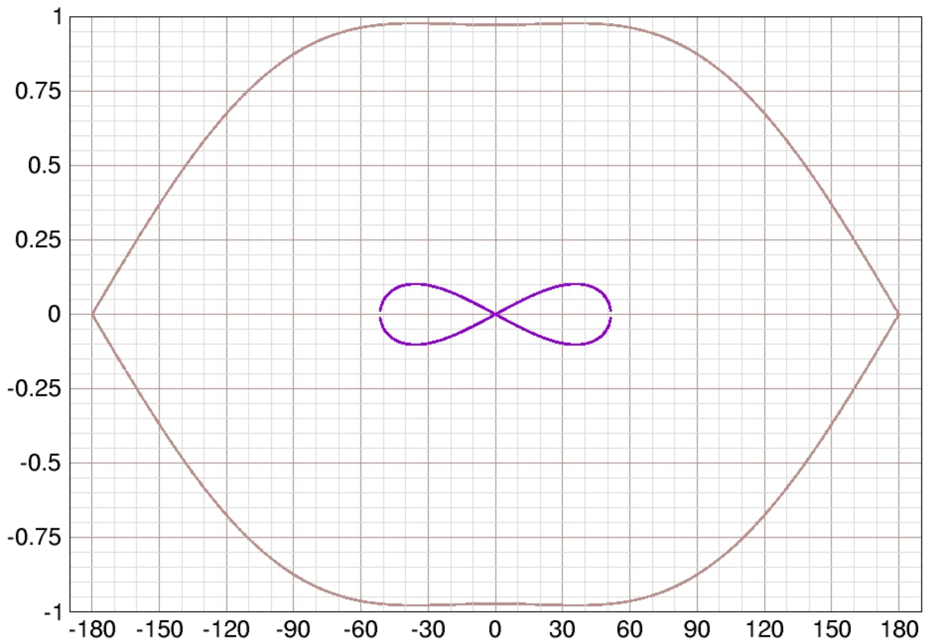
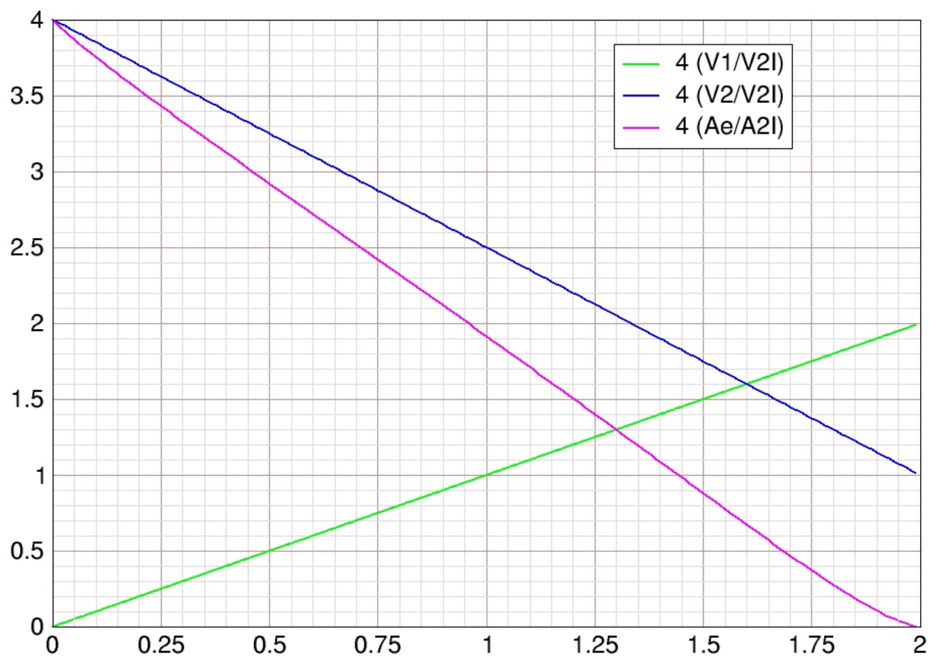


Figure 16: Inner (violet) and outer (brown) separatrices at time 1.90



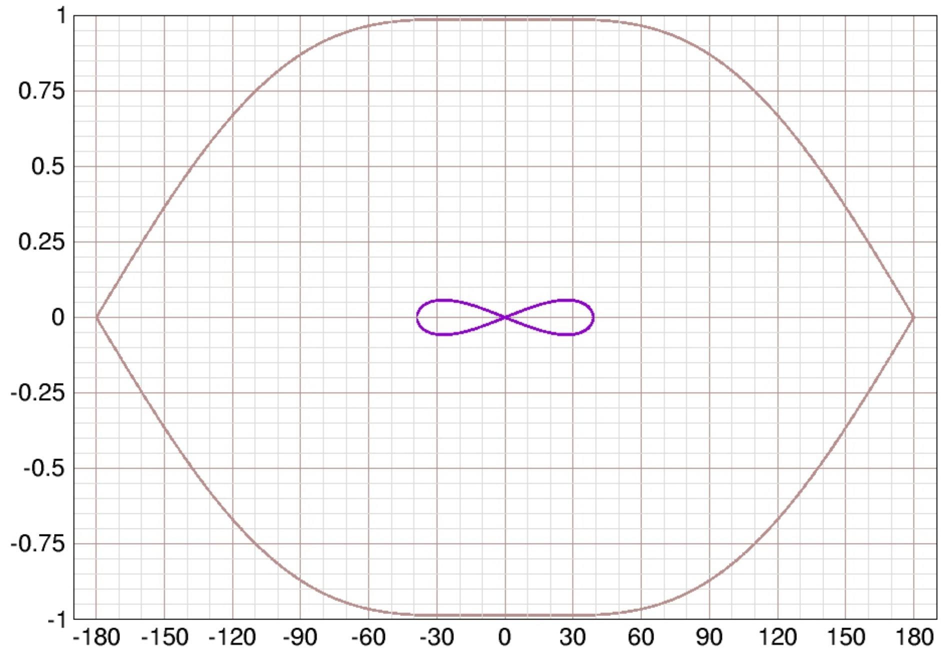
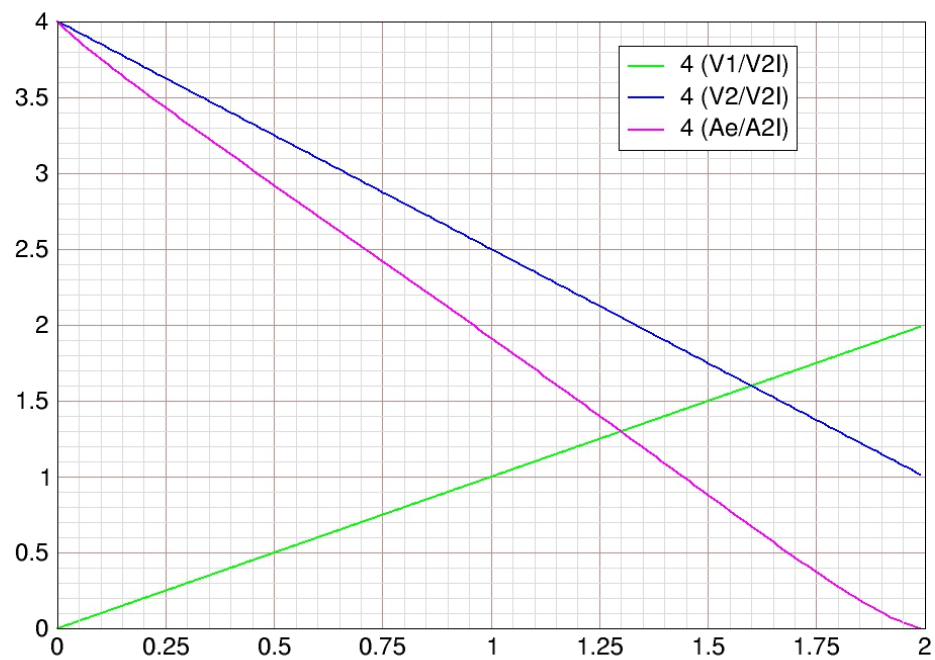


Figure 17: Inner (violet) and outer (brown) separatrices at time 1.95



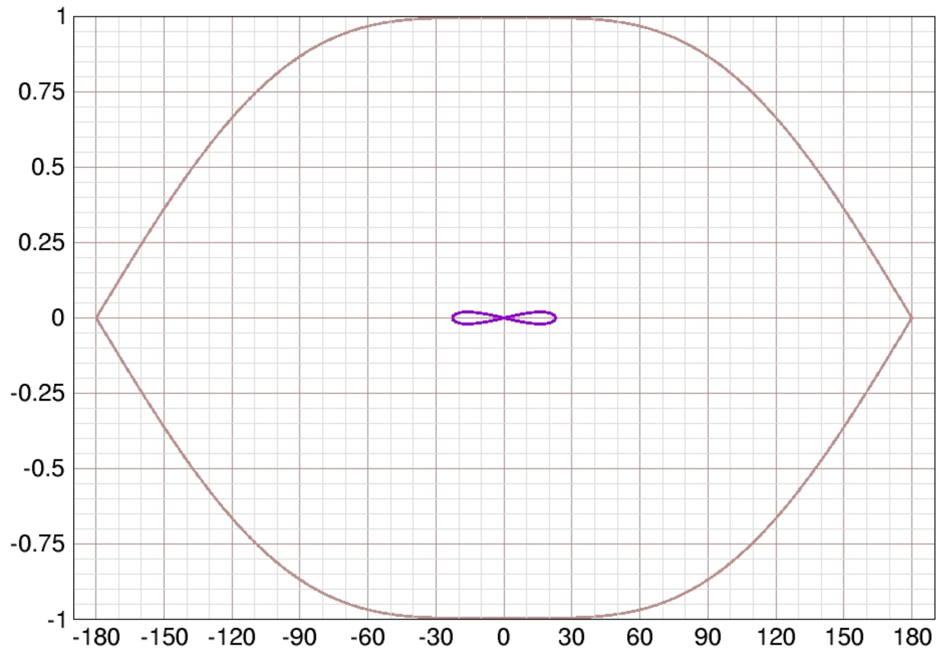
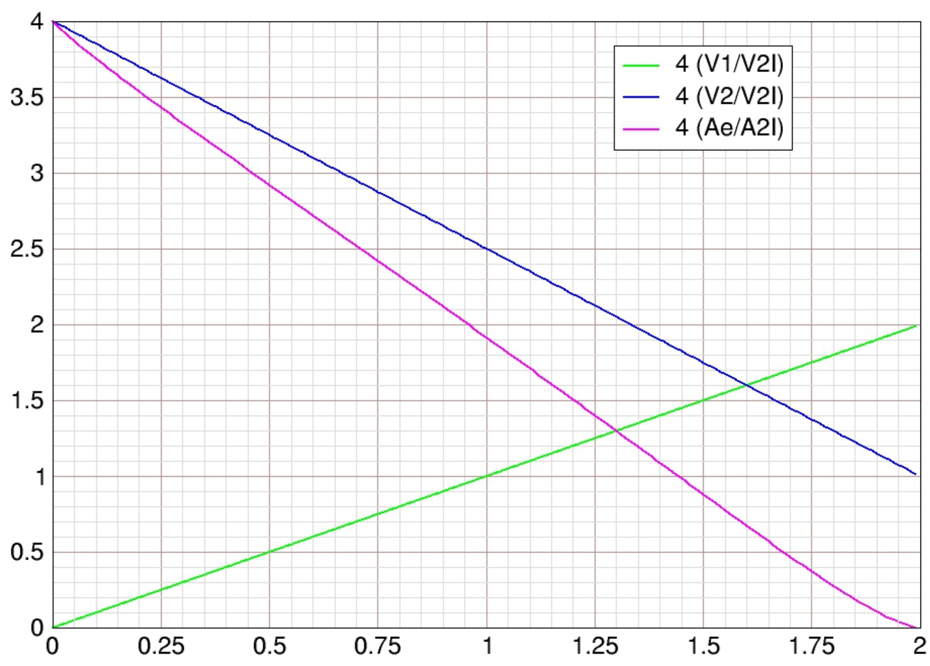


Figure 18: Inner (violet) and outer (brown) separatrices at time 1.99



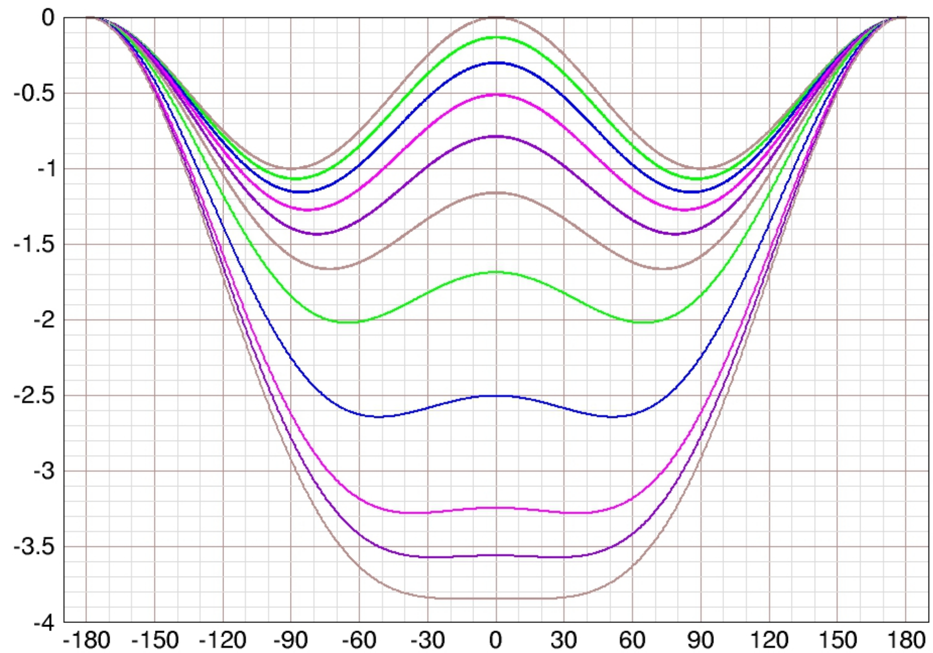


Figure 19: Evolution of potential well $-(\mathcal{U} - \mathcal{U}_u)$ during the 2 to 1 merge. The curves correspond to times 0.01, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 1.90, 1.95, and 1.99 in the previous figures.

22 Bunch movement and mixing

Let b_e be the area of a bunch matched to either lobe of the inner separatrix. When lobe area is equal to b_e, the bunch in one lobe will just touch the bunch in the other lobe, and as the lobe area is reduced the particles of the two bunches will be squeezed out of the lobes and into the area enclosed by the outer separatrix. We use subscript M to denote parameters at the start of this mixing. We then have lobe area

$$\mathcal{A}_{eM} = b_e \tag{221}$$

at the start of mixing, and corresponding normalized lobe area

$$\mathcal{A}_{eM}^N = \frac{b_e}{\mathcal{A}_{2I}}. \tag{222}$$

As a function of time the normalized lobe area is approximately linear with

$$\mathcal{A}_e^N = 1 - \frac{t}{T} \tag{223}$$

for

$$0 \leq t \leq T. \tag{224}$$

This is shown in **Figure 1**. The approximate linear behavior of $\mathcal{A}_e$ is due to the fact that the RF voltages vary linearly with time and have the particular proportions given by (198).

Let T_M be the time at the start of mixing. Then according to (223) we have

$$\mathcal{A}_{eM}^N = 1 - \frac{T_M}{T}. \tag{225}$$

Using (222) we then have

$$\frac{T_M}{T} = 1 - \frac{b_e}{\mathcal{A}_{2I}} \tag{226}$$

which gives the fraction

$$\mathcal{F}_M = \frac{T - T_M}{T} = \frac{b_e}{\mathcal{A}_{2I}} \tag{227}$$

of merge time T available for mixing. Here we see that if b_e is much smaller than the initial harmonic $2h$ bucket area $\mathcal{A}_{2I}$, then mixing does not begin until the merge is nearly finished. In that case the two bunches simply move toward one another with no mixing for most of the merge. On the other hand, if b_e is equal to $\mathcal{A}_{2I}$ then mixing begins as soon as the merge starts.

Suppose we adjust the initial harmonic $2h$ voltage V_{2I} so that the initial lobe area

$$\mathcal{A}_{eI} = \mathcal{A}_{2I} = b_e. \tag{228}$$

We then adjust the merge time T to obtain a good merge. Will we get as good a merge (or perhaps an even better one) if we increase V_{2I} and leave T unchanged? On the one hand, increasing V_{2I} gives an initial bucket area larger than b_e and therefore, according to (227), a shorter amount of time for mixing. On the other hand, increasing V_{2I} gives a higher synchrotron frequency which counteracts the effect of the shorter mixing time. Does one effect win over the other, or do the two essentially cancel one another? Consider the case in which V_{2I} is increased by a factor of 100. The bucket area $\mathcal{A}_{2I}$ then increases by a factor of 10 and, according to (227), the time available for mixing decreases by the same factor. The synchrotron frequency, on the other hand, increases by a factor of 10 throughout the merge as shown by the equations of Section 16. This means (presumably) that the adiabaticity of the mixing should stay about the same. The quality of the merge should then stay about the same. In Sections 23, 24, 25, and 27 examples are given for which this reasoning works.

One can also look at the adiabaticity condition

$$P_{sI} \ll \mathcal{T} \tag{229}$$

from Section 19. Here

$$P_{sI} = \frac{1}{F_{sI}} \tag{230}$$

where

$$F_{sI} = \frac{c}{2\pi\mathcal{R}} \left\{ \frac{2h|\eta|\, eQV_{2I}}{2\pi E} \right\}^{1/2} \tag{231}$$

is the initial synchrotron frequency in the stationary harmonic $2h$ bucket. If the condition (229) is satisfied for a given V_{2I} and $\mathcal{T}$, then it will also be satisfied if $\mathcal{T}$ is kept the same and V_{2I} is increased. If V_{2I} is increased by a factor of 100 then P_{2I} will be reduced by a factor of 10 and one might expect the quality of the merge to be better. However, the argument given above shows that one must also take into account the details of the particular process under consideration.

23 Illustration of bunch mixing

Here we illustrate the case in which bunch mixing begins as soon as the merge starts. The upper half of **Figure 20** shows the start of a simulation in which polarized proton bunches undergo two 2 to 1 merges on the AGS injection porch at $G\gamma = 4.5$. The bunches initially are sitting in four adjacent harmonic 12 buckets as shown in the figure. The longitudinal emittance of each bunch is 0.5 eV-s. The bunches are formed by adiabatic capture of a uniform distribution of unbunched particles as explained in [6]. The particles in the bunches are color coded according to their initial longitudinal oscillation amplitudes. In order of increasing amplitude, the colors are black, pink, blue, orange, green, and brown.

The first 2 to 1 merge is done over a period of 120 ms. The harmonic 12 and 6 voltages that bring about the merge are programmed in accordance with equations (198) through (204) and are shown by the blue and green lines (labeled V4 and V2 respectively) in the lower half of **Figure 20**. The harmonic 12 voltage falls from 4.4 to 0 kV in 120 ms. The harmonic 6 voltage rises from 0 to 2.2 kV in the first 90 ms and is then held constant from 90 to 120 ms. The initial area of each harmonic 12 bucket is 0.5564 eV-s. The initial synchrotron period is 3.744 ms. **Figures 20** through **30** show the evolution of the bunches from 0 to 90 ms. Here one sees that the merge proceeds slowly enough to keep the oscillation amplitude layers intact and the gross longitudinal emittance unchanged. The highest amplitude particles (brown) in the bunches are the first to mix together and the lowest amplitude ones (black) are the last to mix. At 90 ms the harmonic 12 voltage is half the harmonic 6 voltage and the bunches are contained in a single RF bucket. This is the end-of-merge point as explained in Section 12. The merge is complete when the harmonic 12 voltage has been brought down to zero and the merged bunch is sitting in a harmonic 6 bucket as shown in **Figure 31**. The oscillation amplitude layers in the bucket are shown individually in **Figures 32** through **37**. Here one sees slight filamentation in the blue layer and more pronounced filamentation in the pink layer. These could be reduced or eliminated by increasing the merge time.

The second 2 to 1 merge is done over a period of 240 ms starting at 120 ms as shown in the lower half of **Figure 37**. Here the harmonic 6 voltage (labeled V2) falls from 2.2 to 0 kV in 240 ms. The harmonic 3 voltage (labeled V1) rises from 0 to 1.1 kV in 180 ms and then is held constant for 60 ms, all in accordance with equations (198) through (204).

Figures 38 and **39** show the merged bunch at times 300 and 360 ms respectively. Here one sees again that the merge proceeds slowly enough to keep the oscillation amplitude layers intact and the gross longitudinal emittance unchanged. This requires twice as much time as the first merge because the initial synchrotron period is twice that of the first merge as explained in Section 17. The oscillation amplitude layers in the final bucket are shown individually in **Figures 40** through **45**. Here one sees, as before, slight filamentation in the blue layer and more pronounced filamentation in the pink layer. These can be reduced or elminated by increasing the merge time.

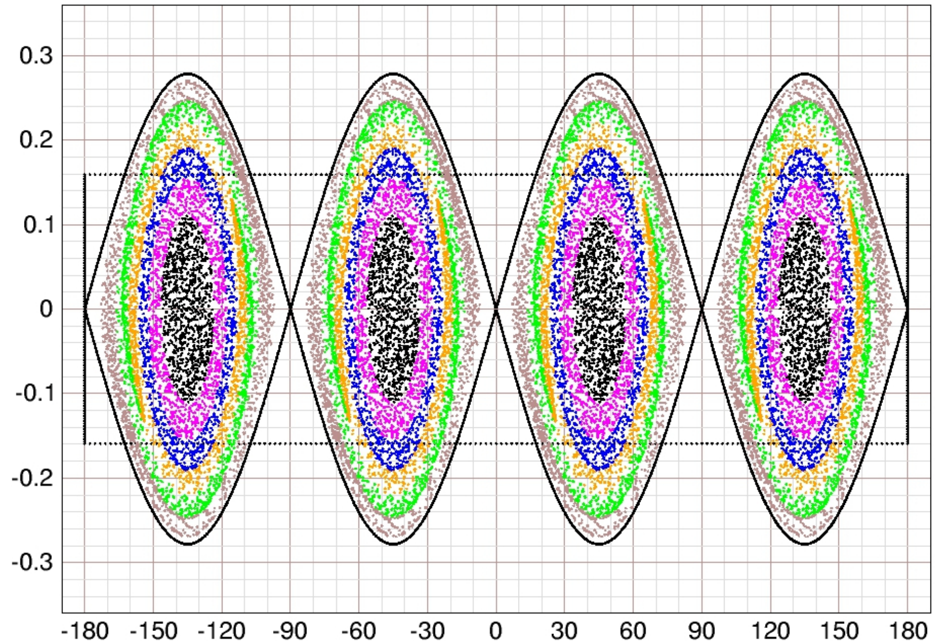
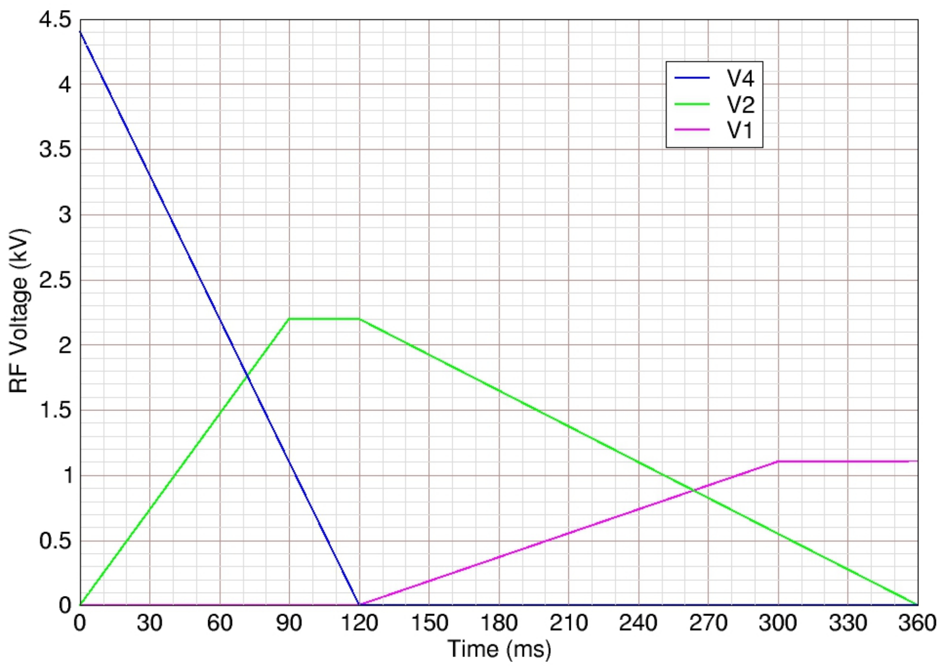


Figure 20: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



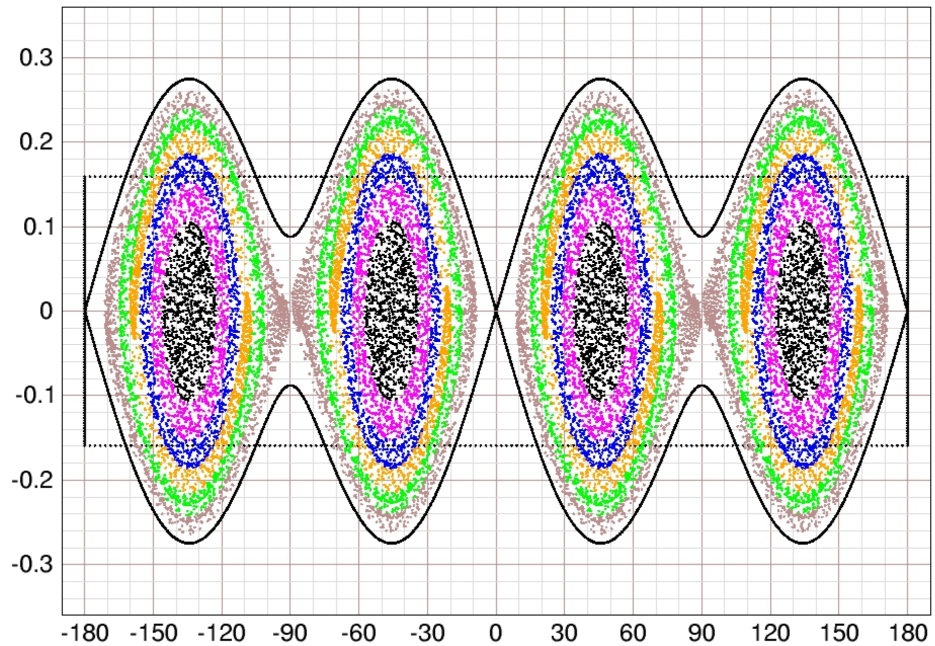
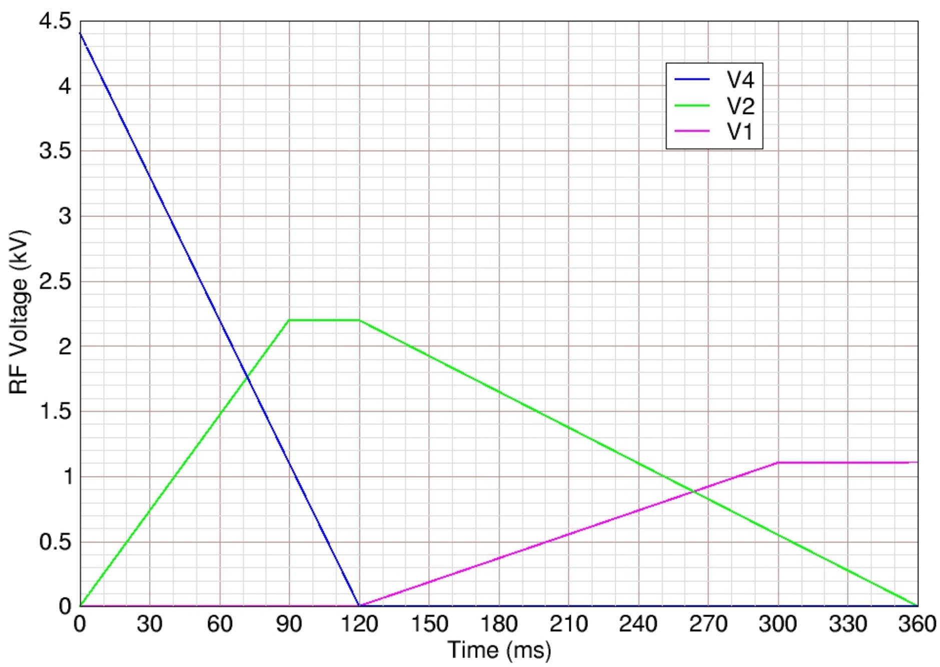


Figure 21: 4 to 2 to 1 merge at time $\underline{t = 9 \text{ ms}}$.



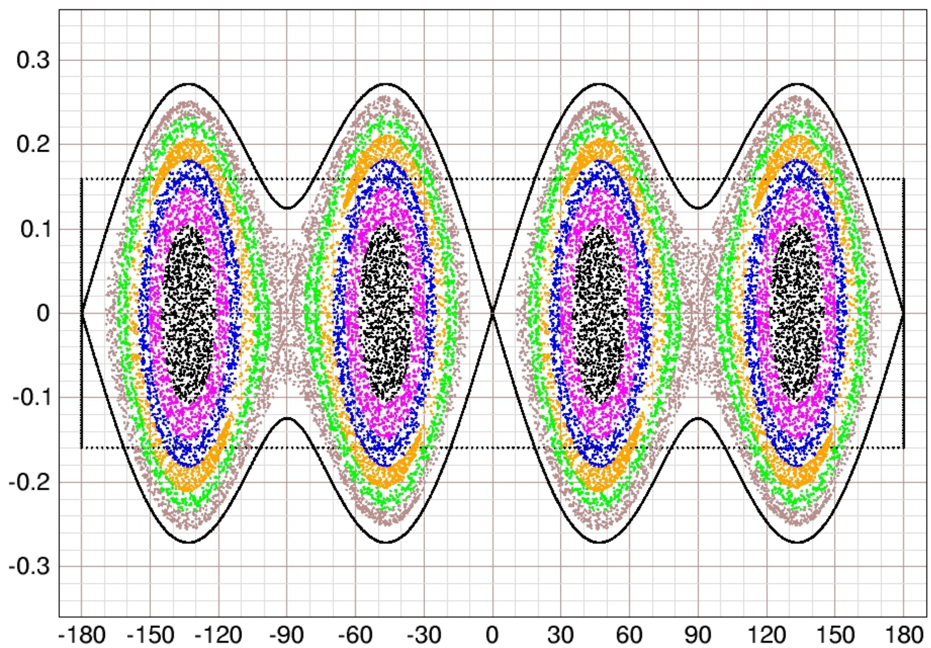
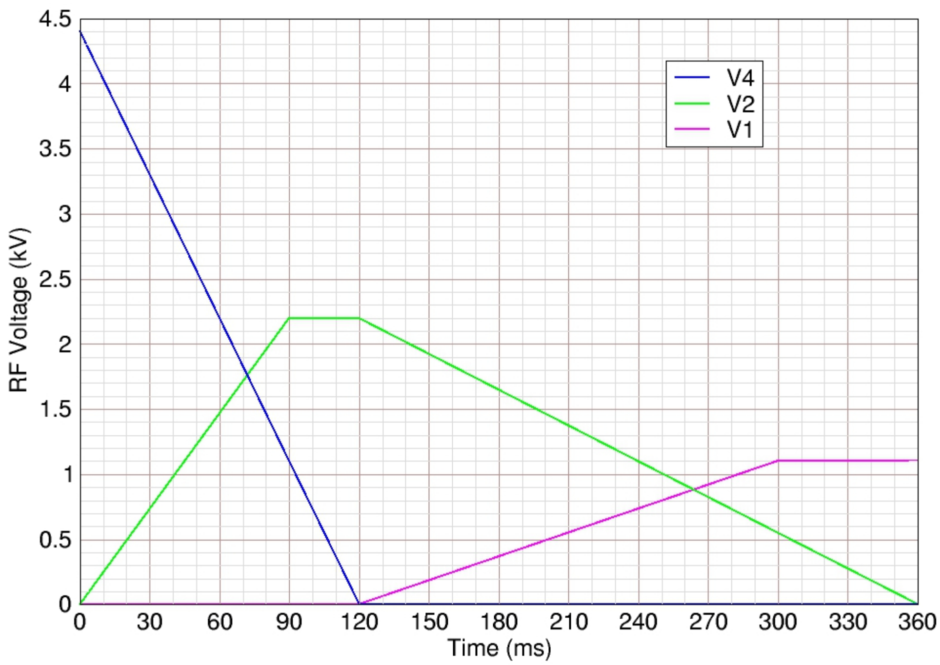


Figure 22: 4 to 2 to 1 merge at time $\underline{t = 18 \text{ ms}}$.



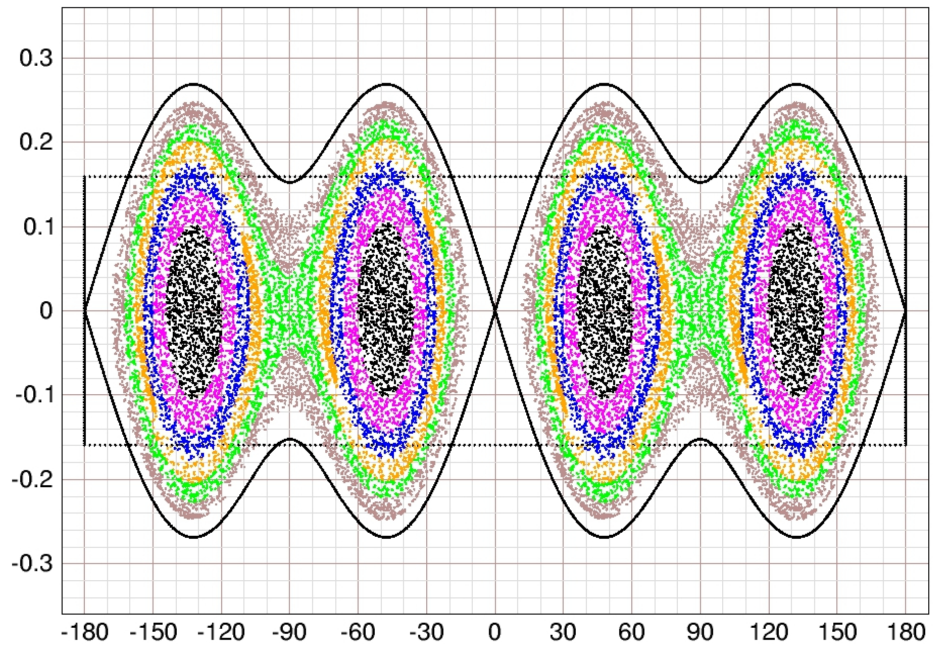
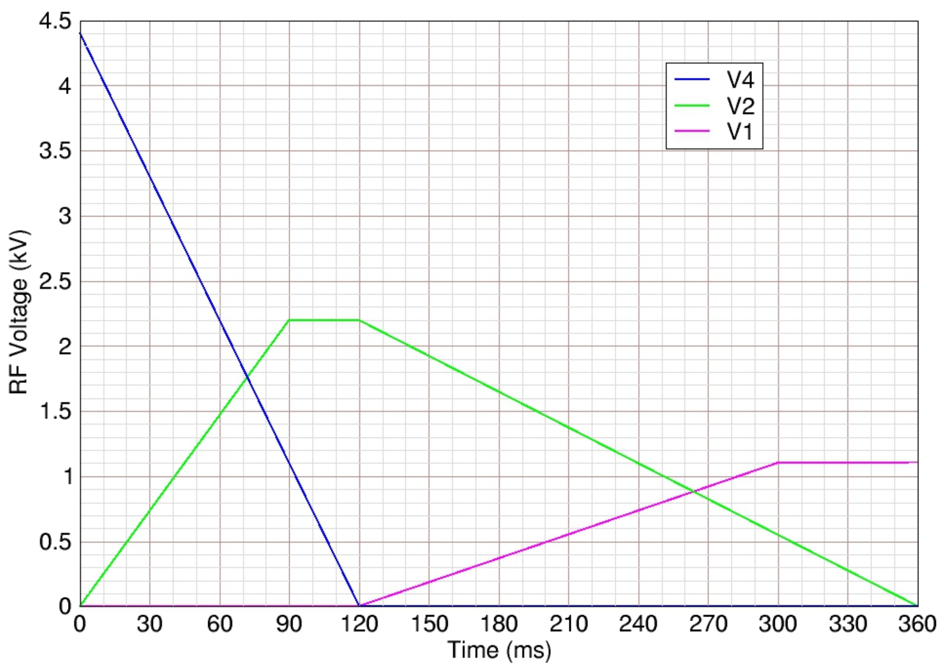


Figure 23: 4 to 2 to 1 merge at time $\underline{t = 27 \text{ ms}}$.



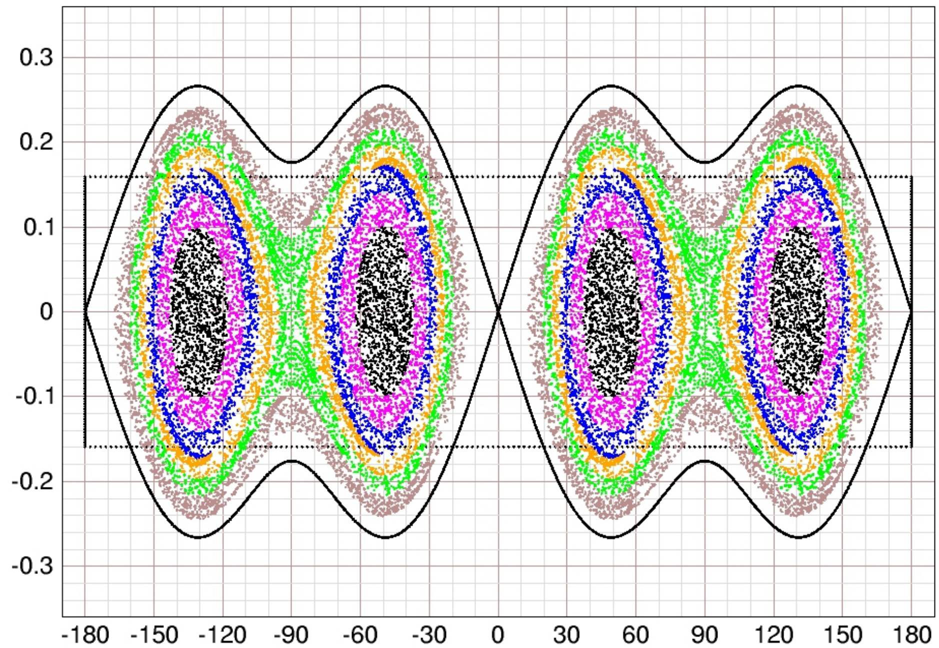
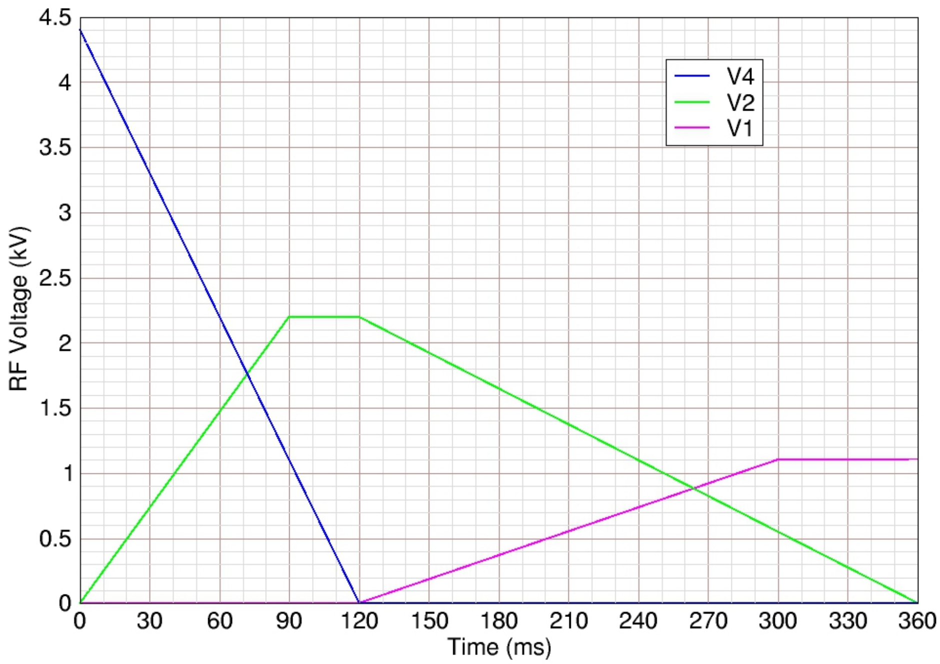


Figure 24: 4 to 2 to 1 merge at time <u>$t = 36$ ms</u>.



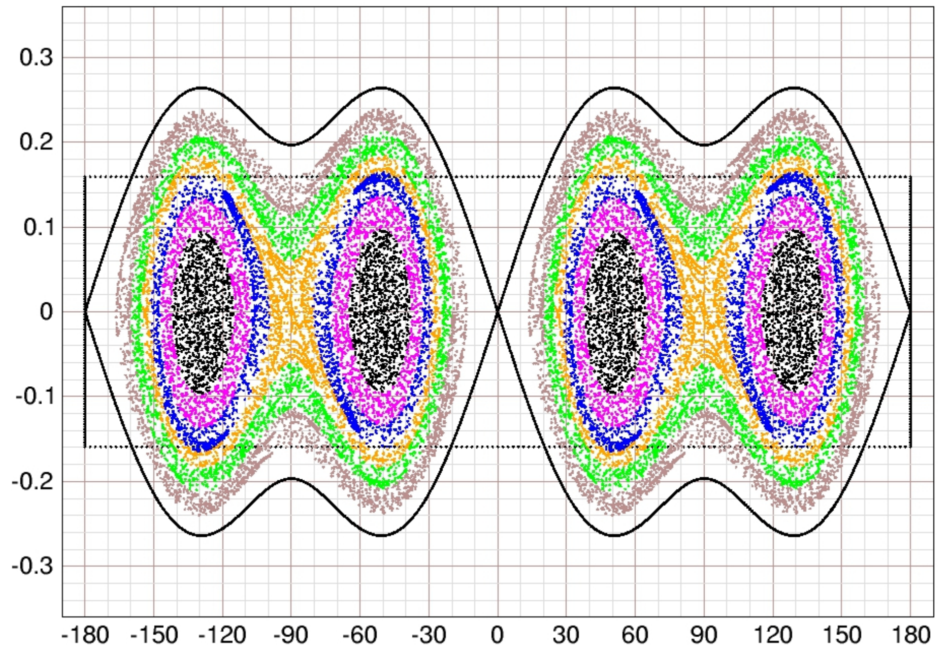
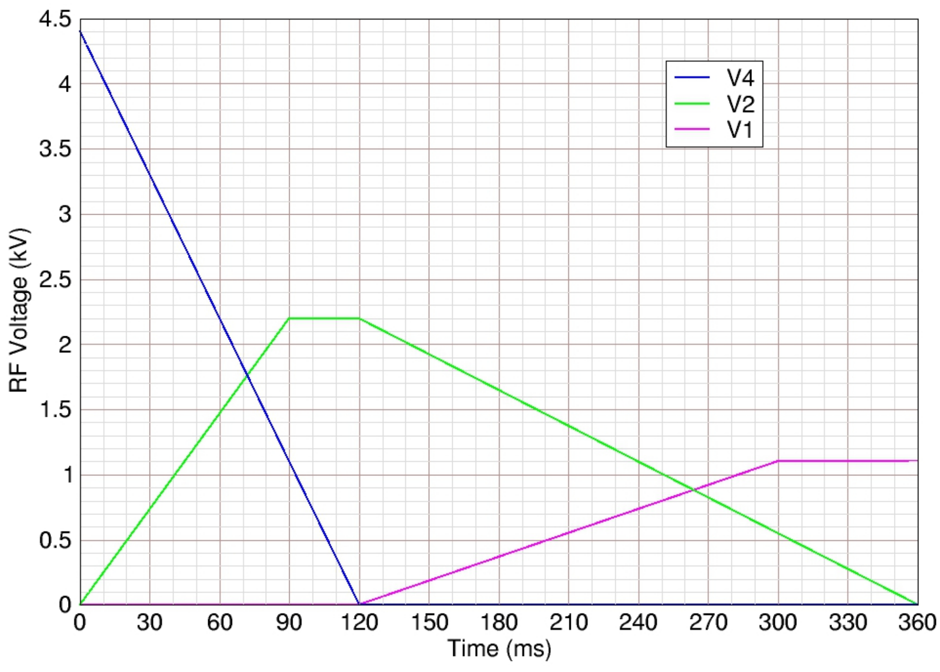


Figure 25: 4 to 2 to 1 merge at time $\underline{t = 45 \text{ ms}}$.



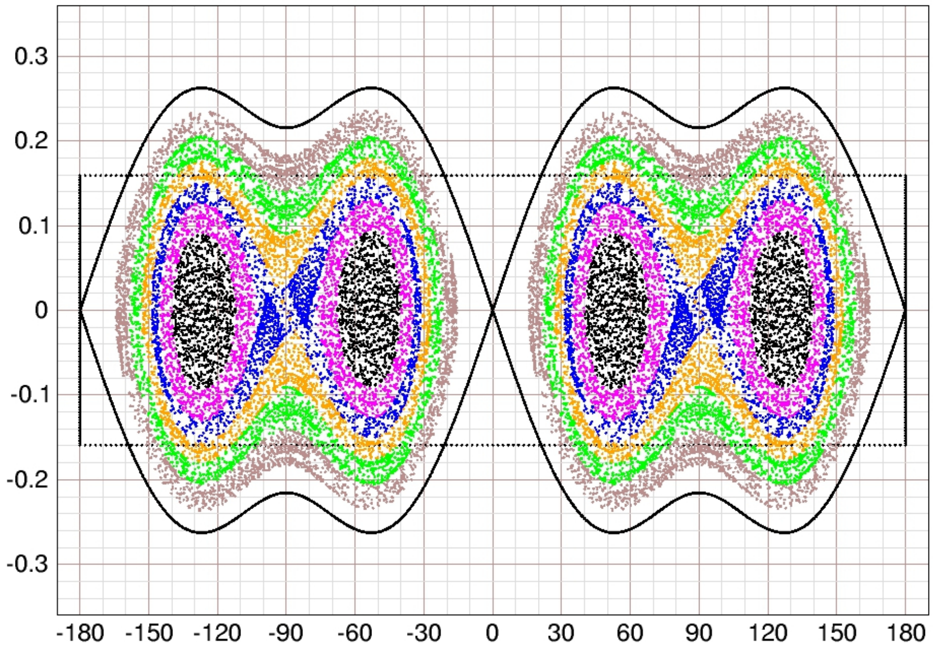
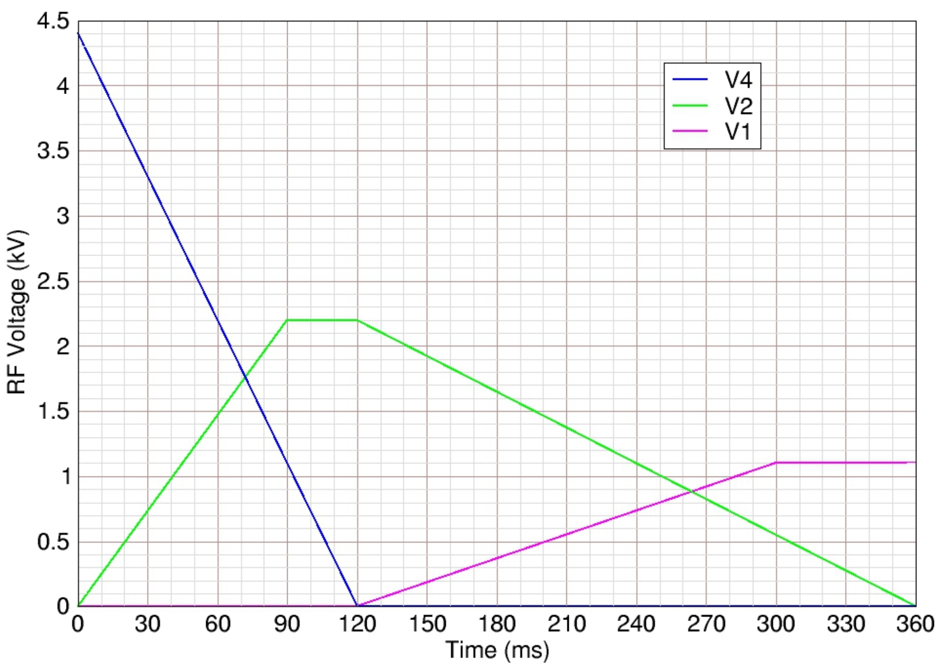


Figure 26: 4 to 2 to 1 merge at time $\underline{t = 54\ \text{ms}}$.



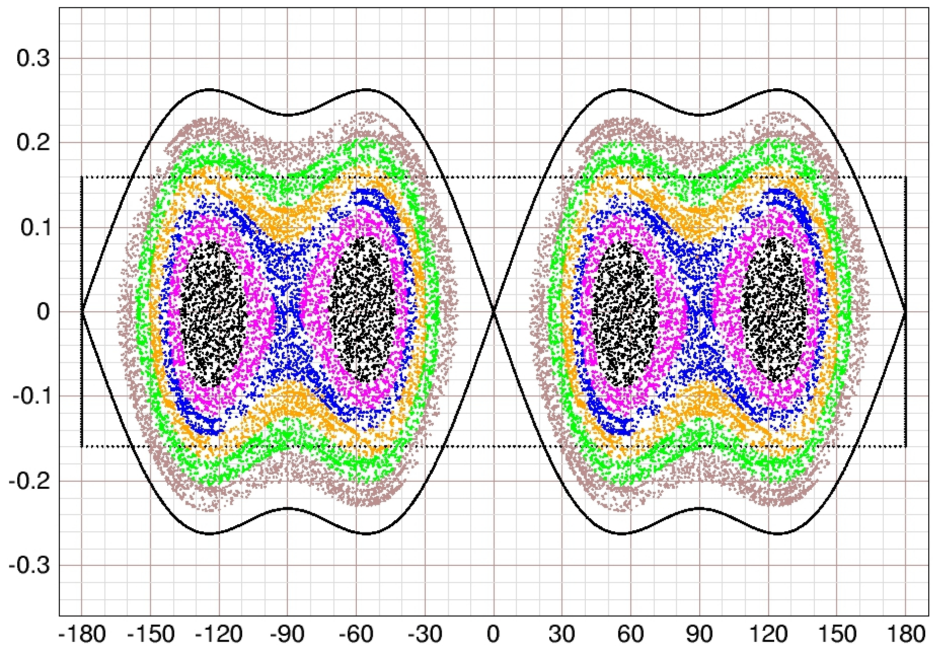
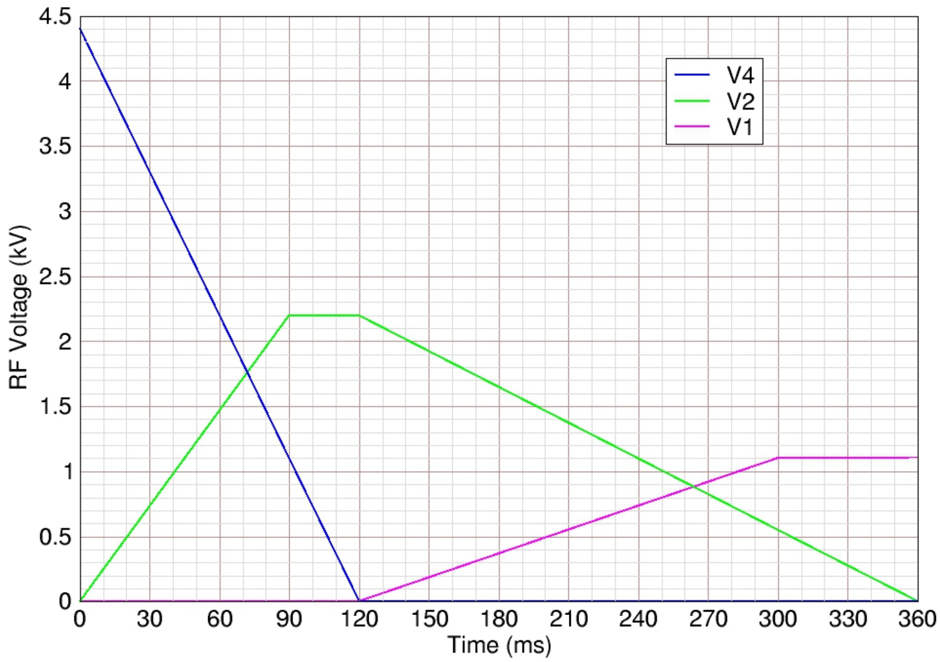


Figure 27: 4 to 2 to 1 merge at time $\underline{t = 63 \text{ ms}}$.



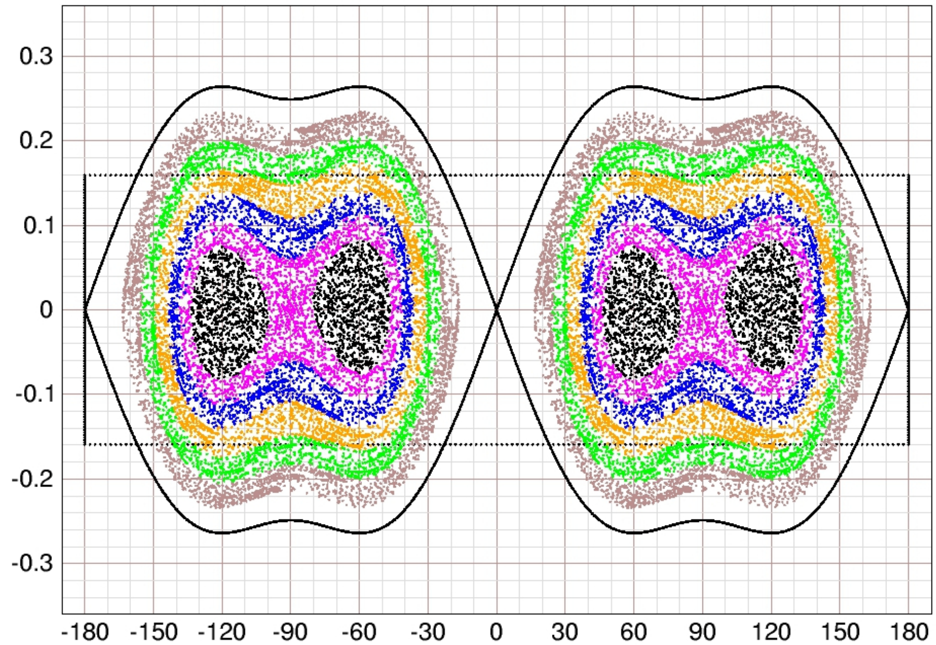
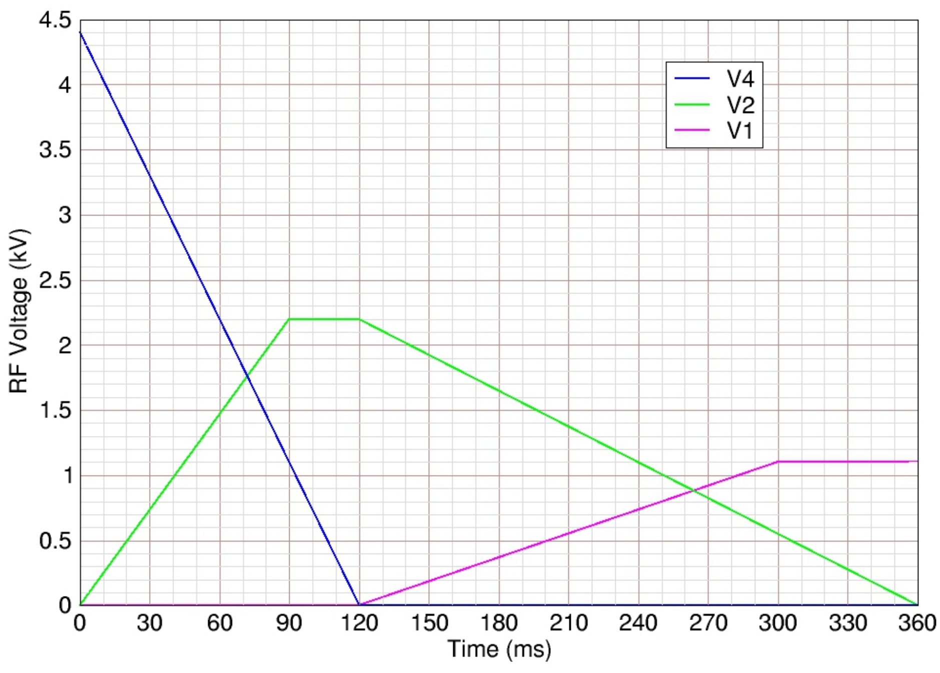


Figure 28: 4 to 2 to 1 merge at time $\underline{t = 72 \text{ ms}}$.



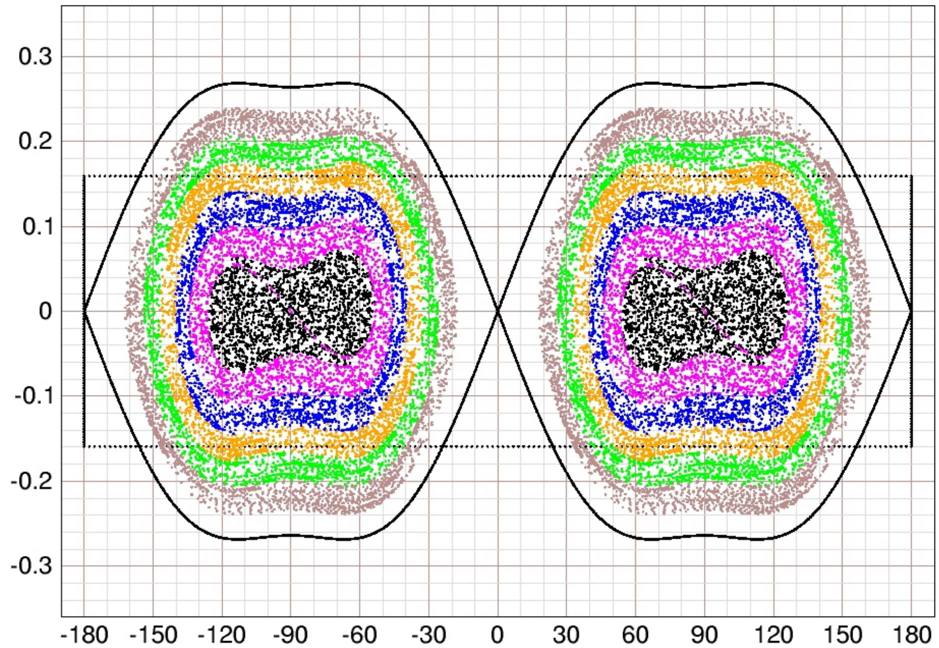
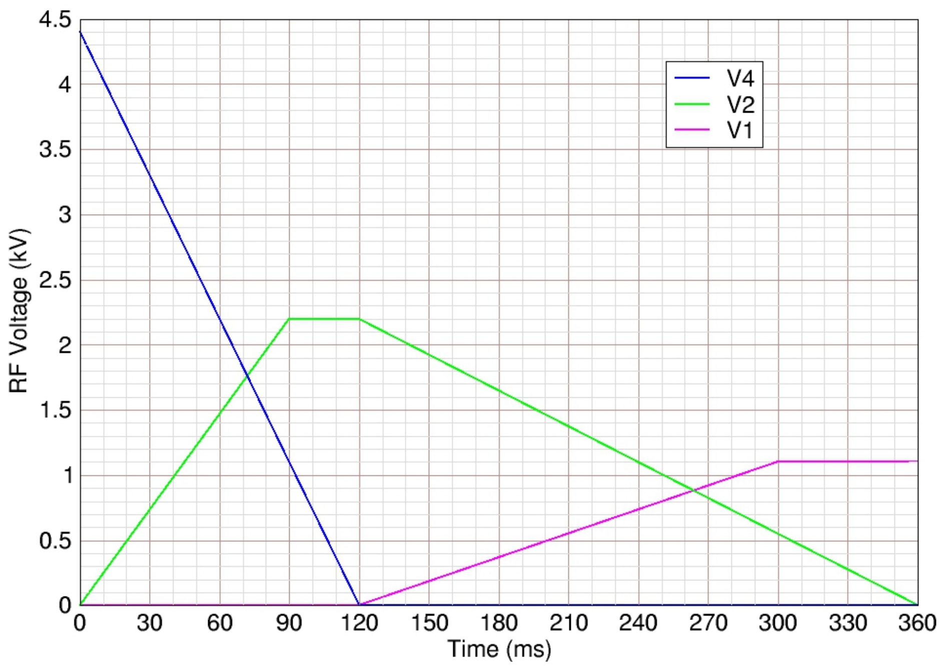


Figure 29: 4 to 2 to 1 merge at time $\underline{t = 81 \text{ ms}}$.



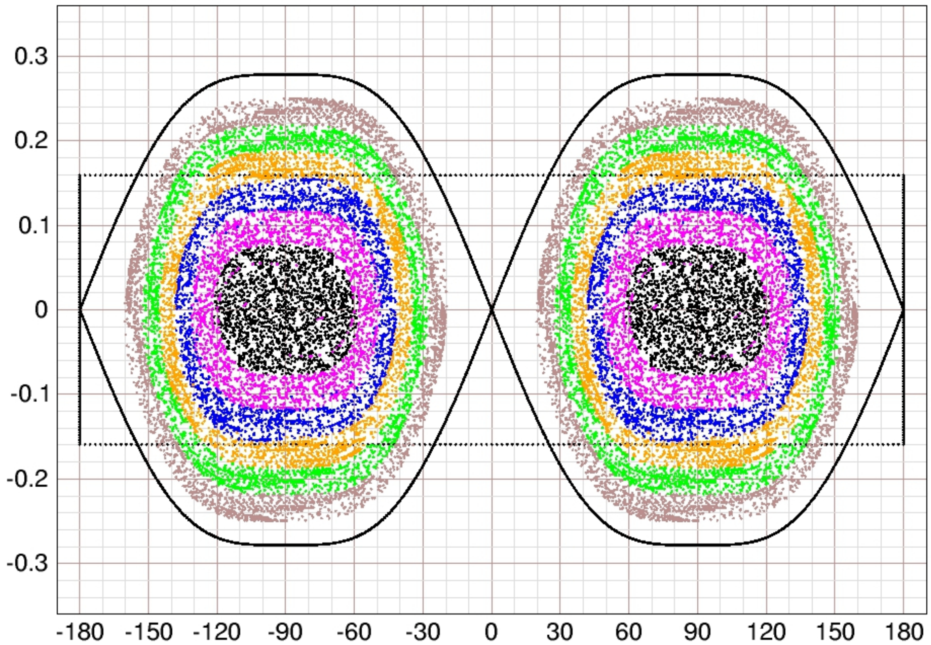
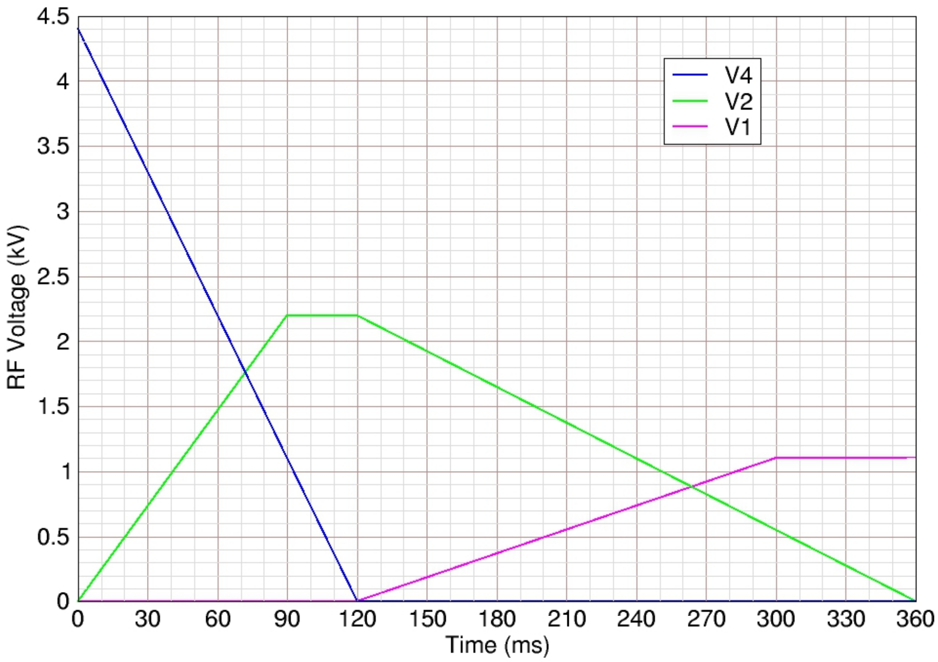


Figure 30: 4 to 2 to 1 merge at time $\underline{t = 90 \text{ ms}}$.



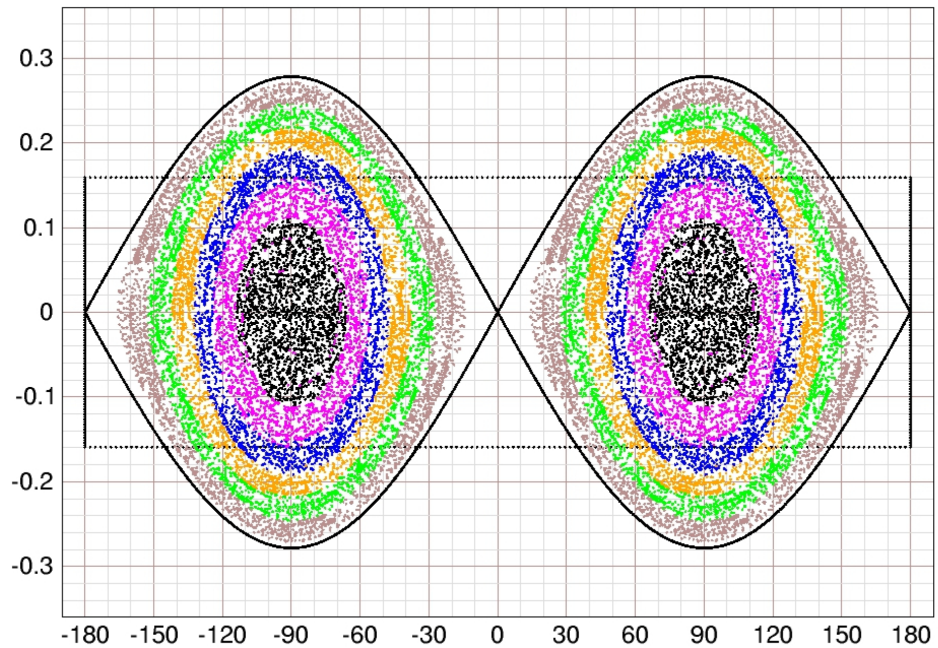
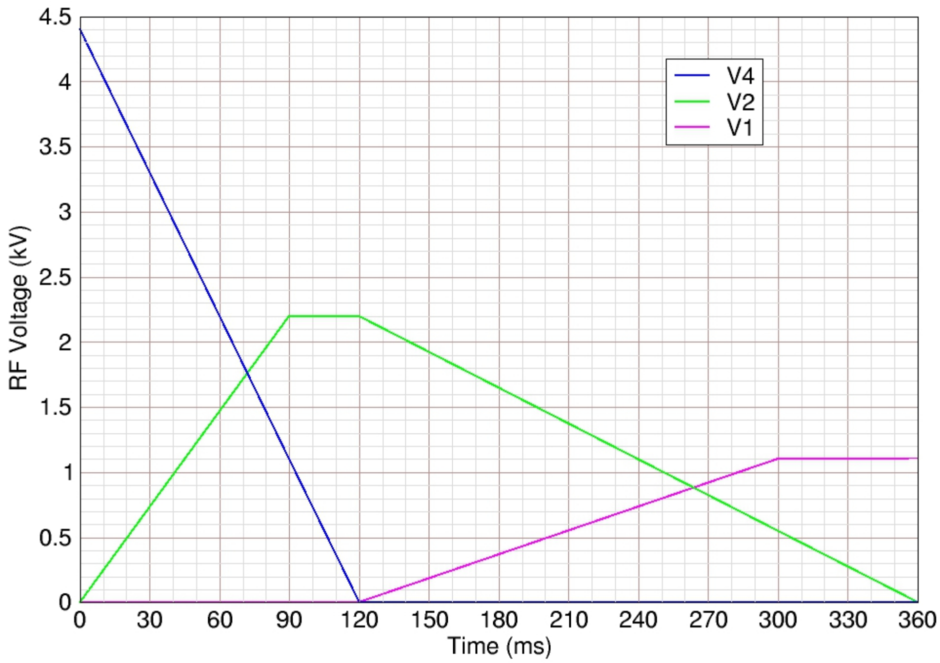


Figure 31: 4 to 2 to 1 merge at time $\underline{t = 120 \text{ ms}}$.



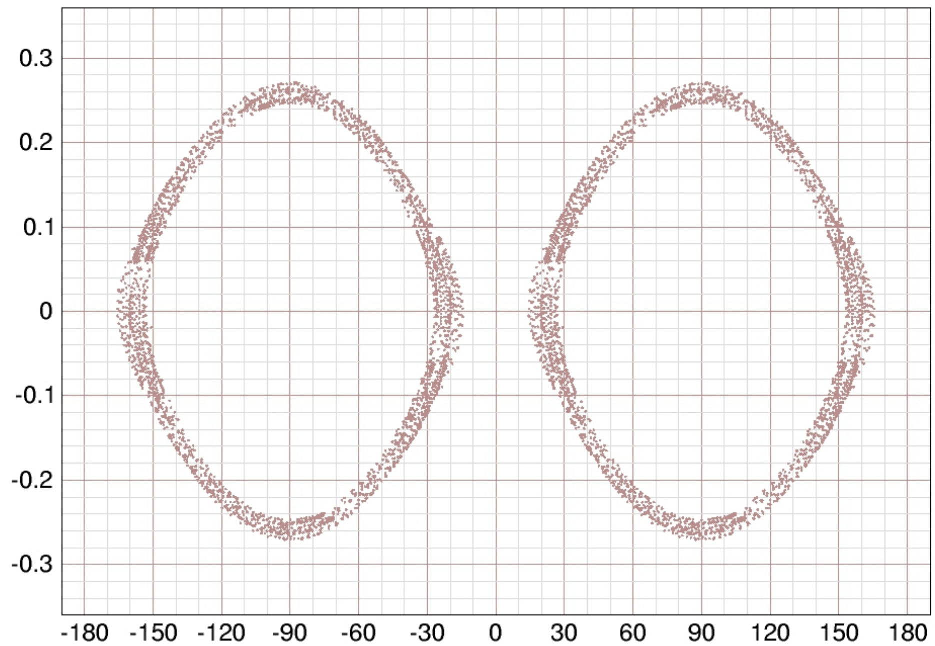
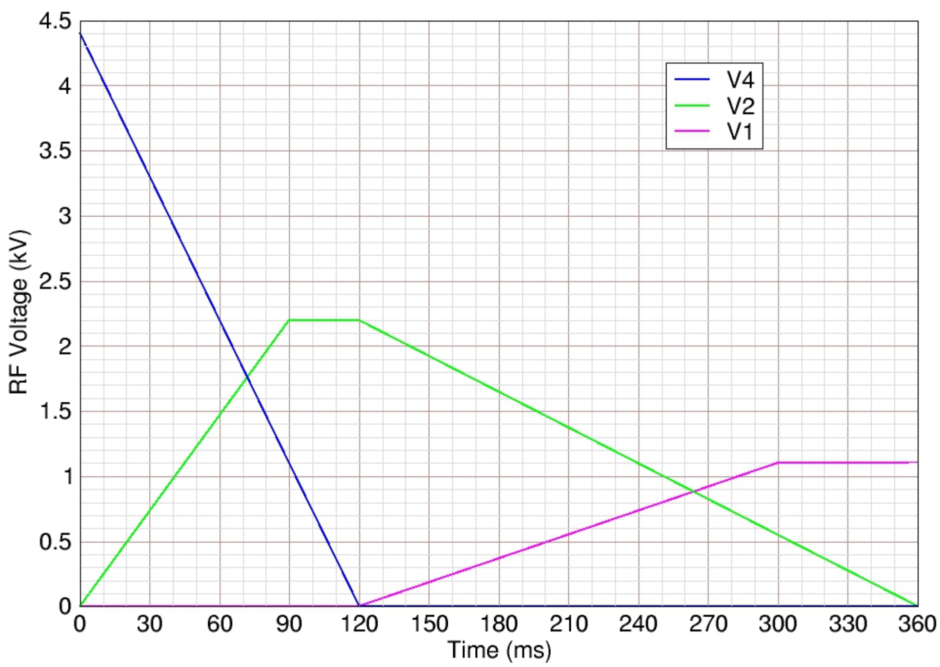


Figure 32: First layer (brown) deposited in harmonic 6 bucket.



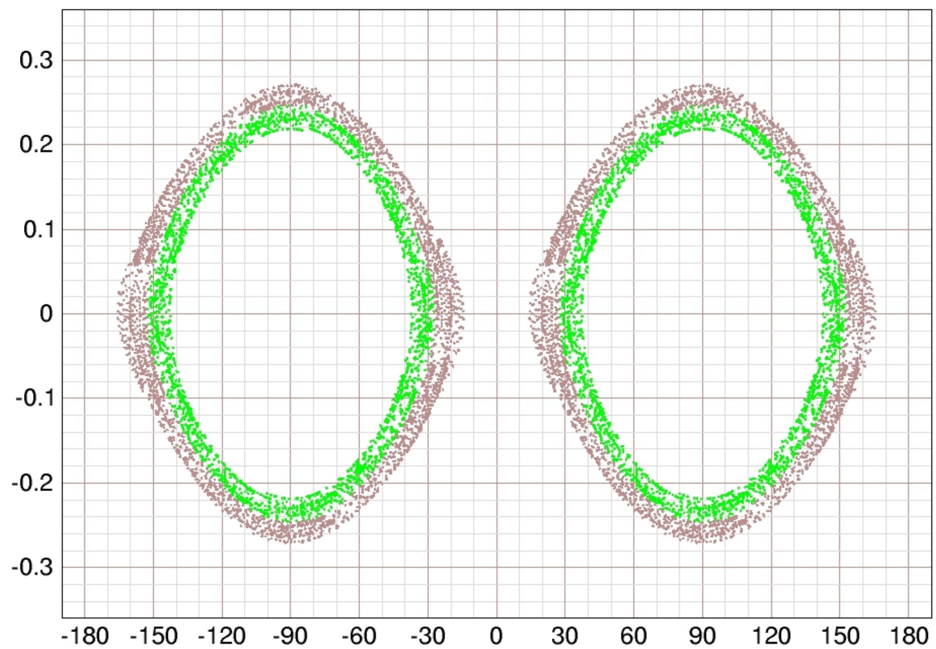
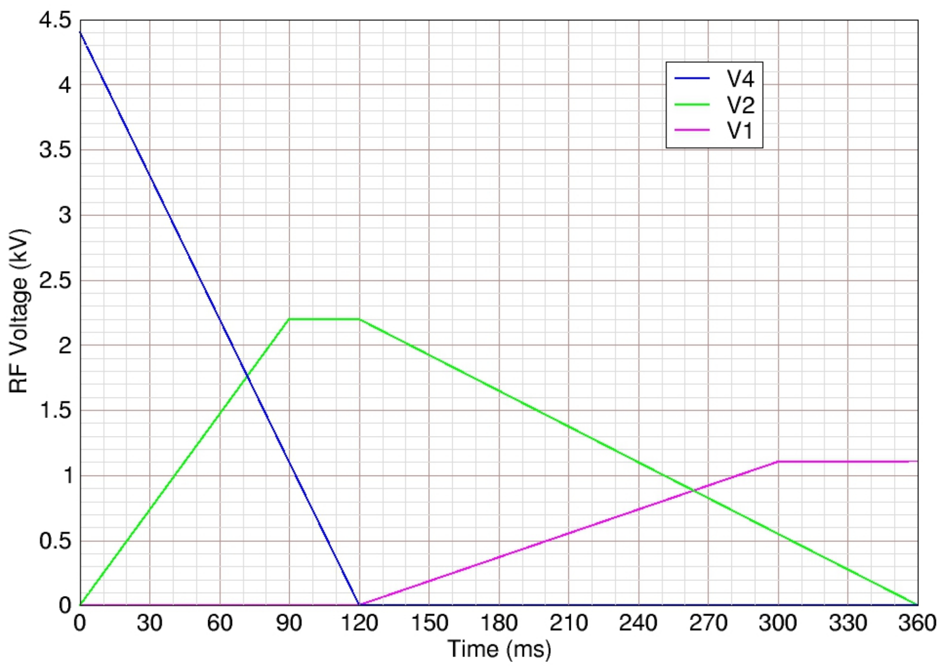


Figure 33: Second layer (green) deposited in harmonic 6 bucket.



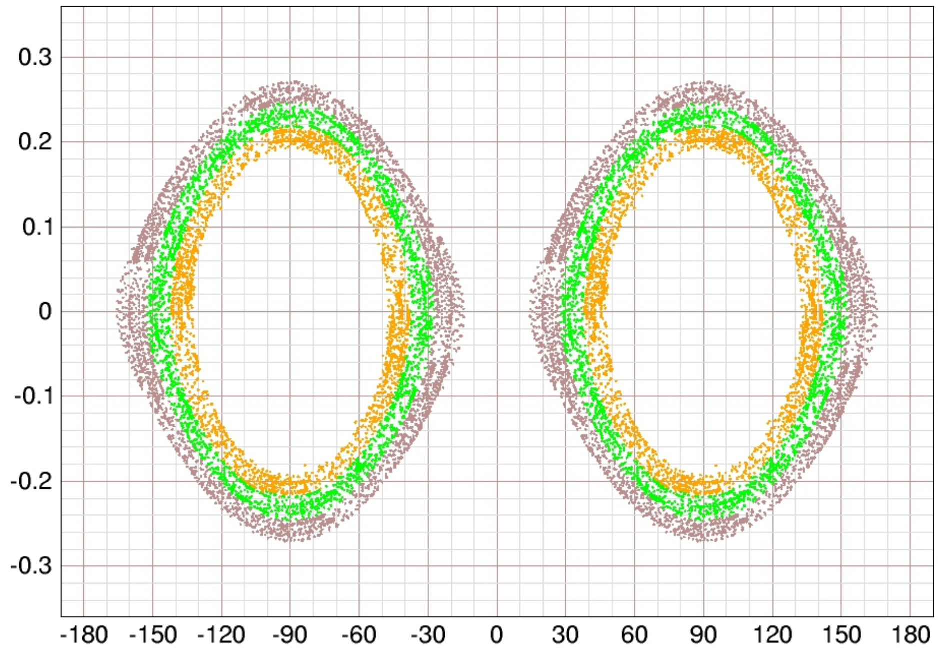
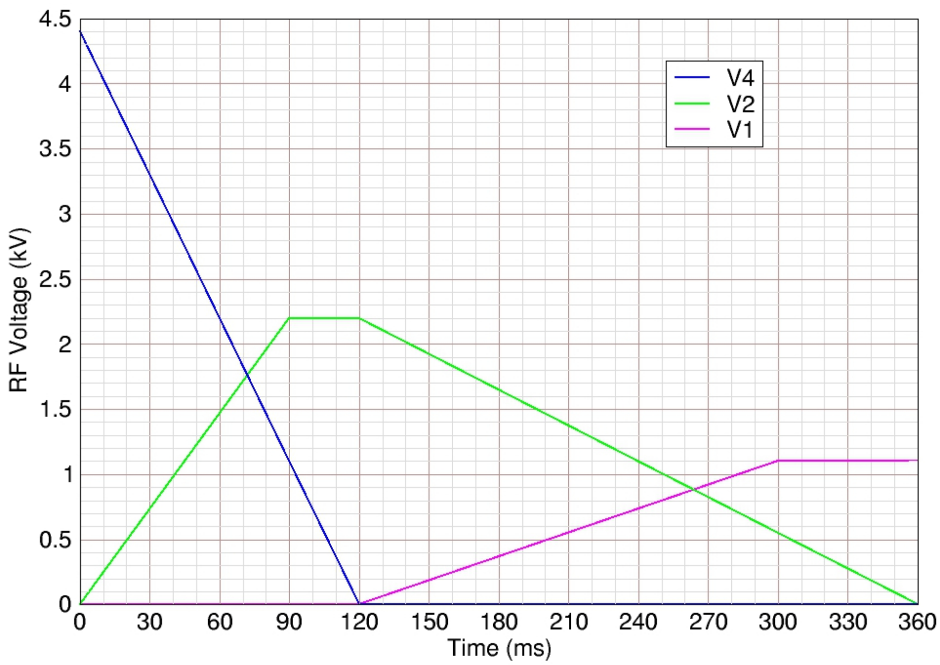


Figure 34: Third layer (orange) deposited in harmonic 6 bucket.



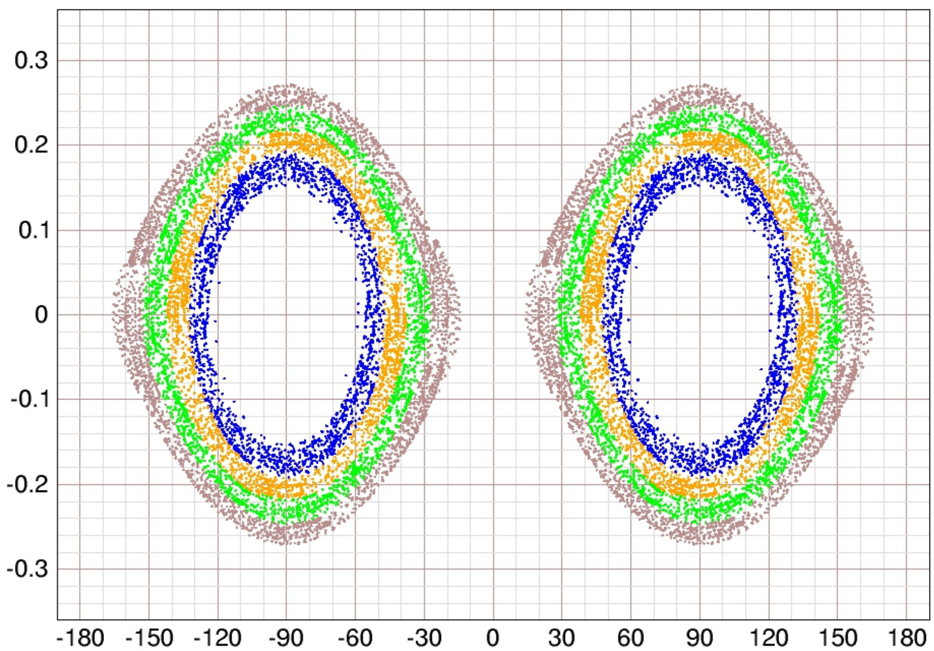
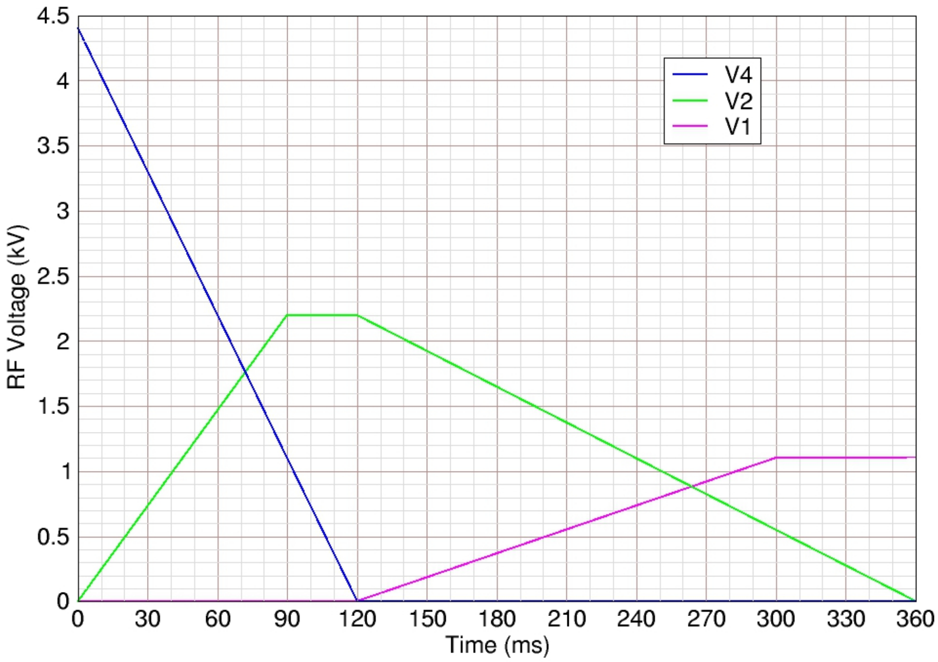


Figure 35: Fourth layer (blue) deposited in harmonic 6 bucket.



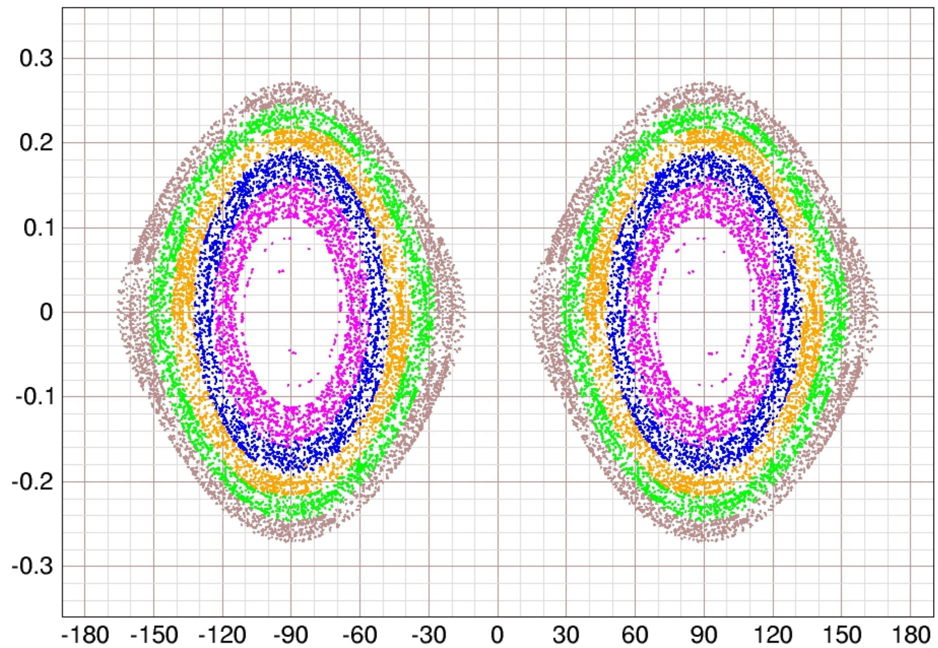
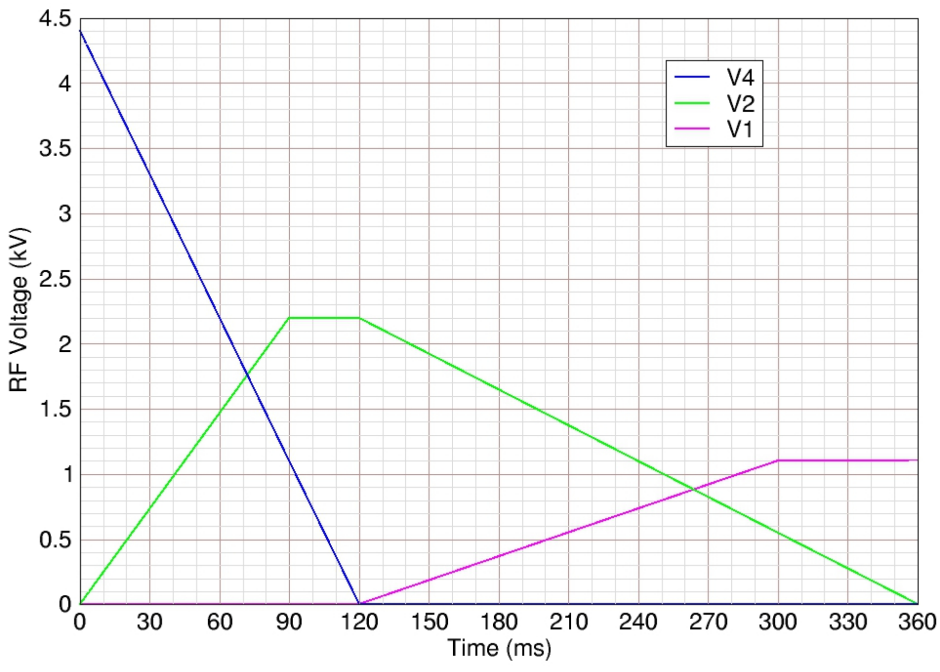


Figure 36: Fifth layer (pink) deposited in harmonic 6 bucket.



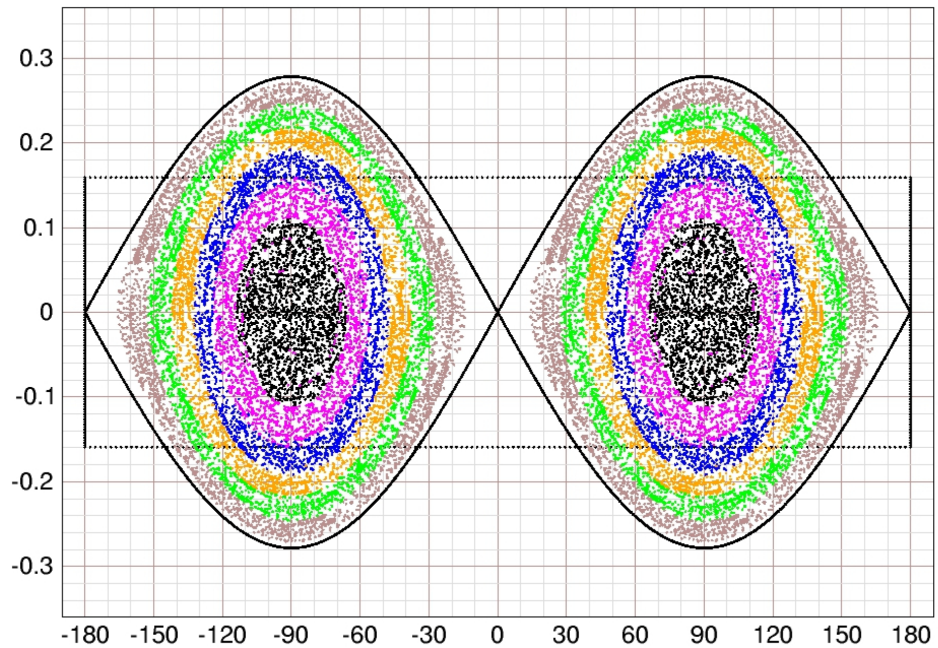
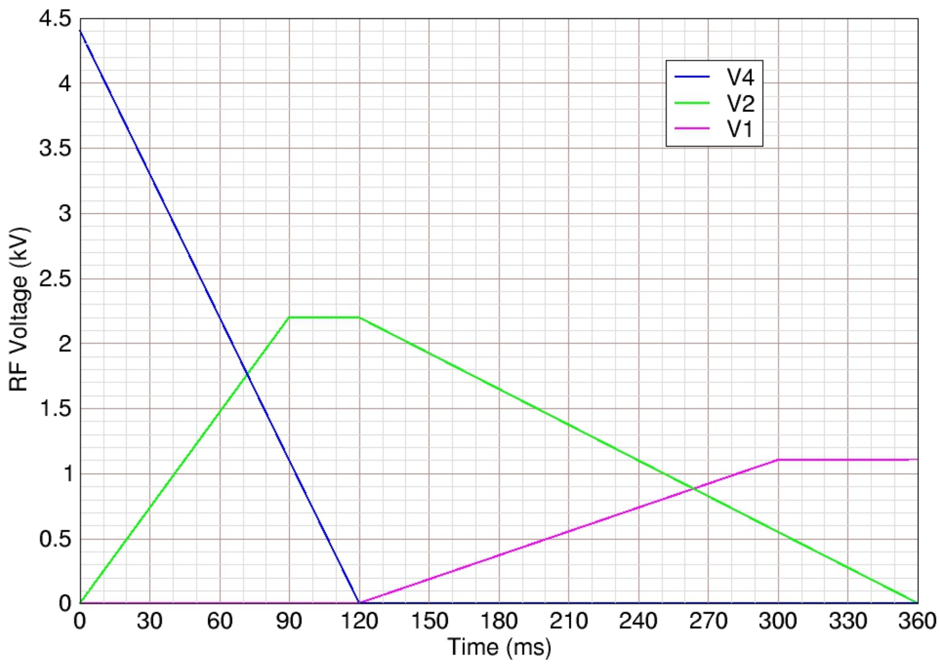


Figure 37: Final layer (black) deposited in harmonic 6 bucket.



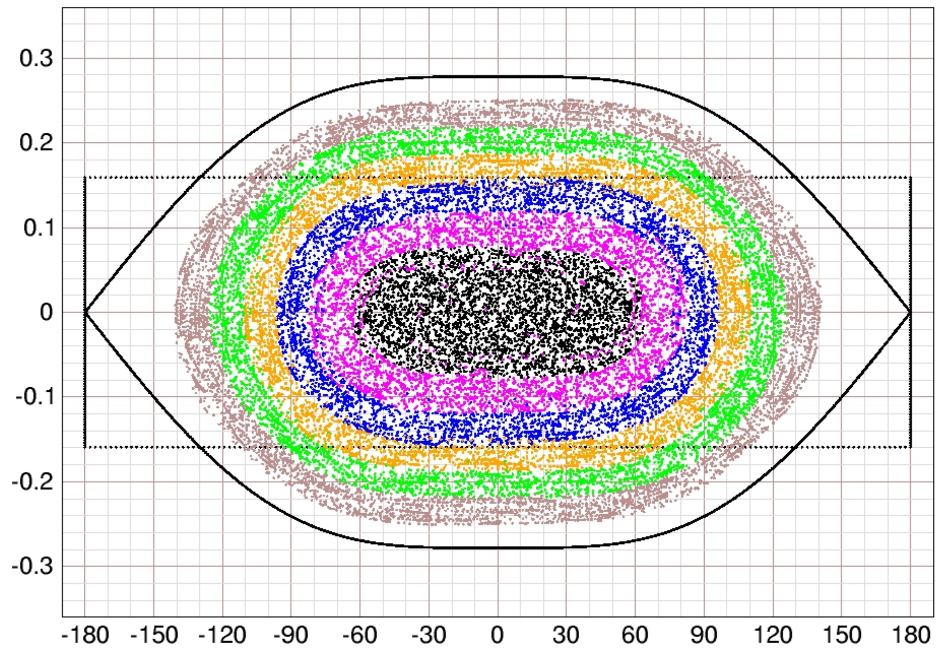
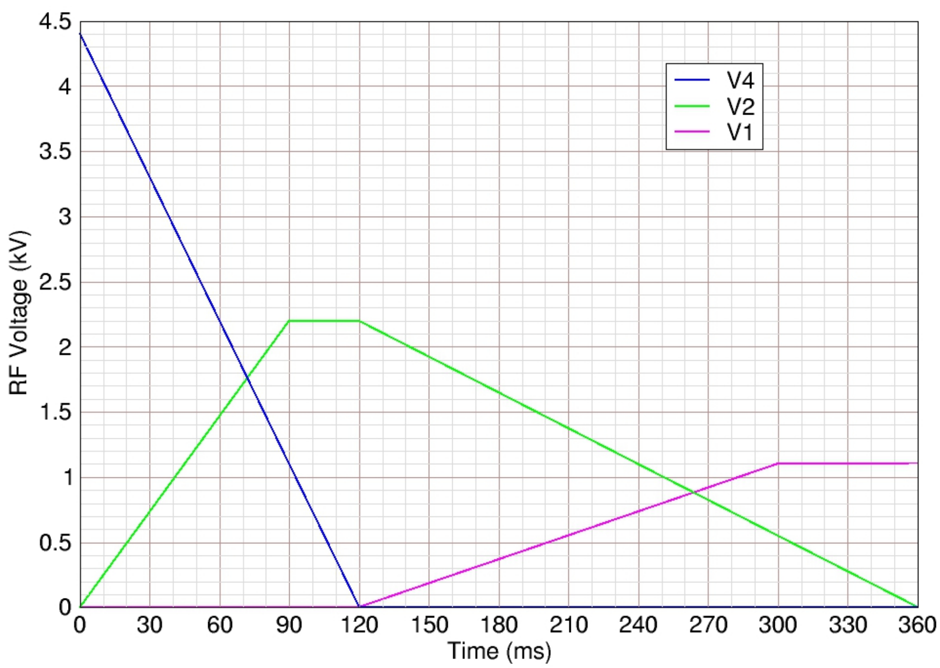


Figure 38: 4 to 2 to 1 merge at time $\underline{t = 300\text{ ms}}$.



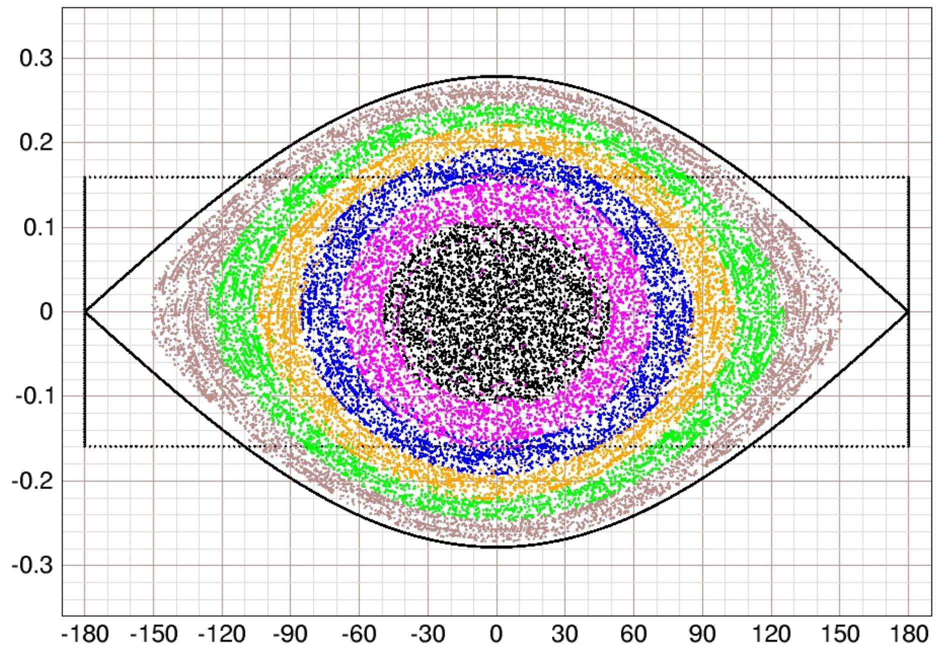
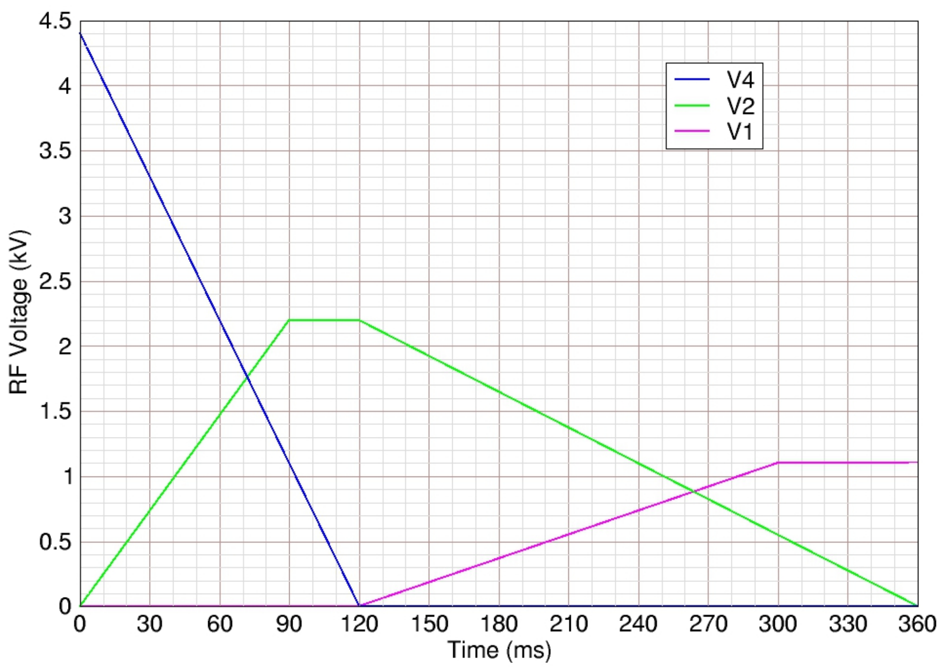


Figure 39: 4 to 2 to 1 merge at time $\underline{t = 360 \text{ ms}}$.



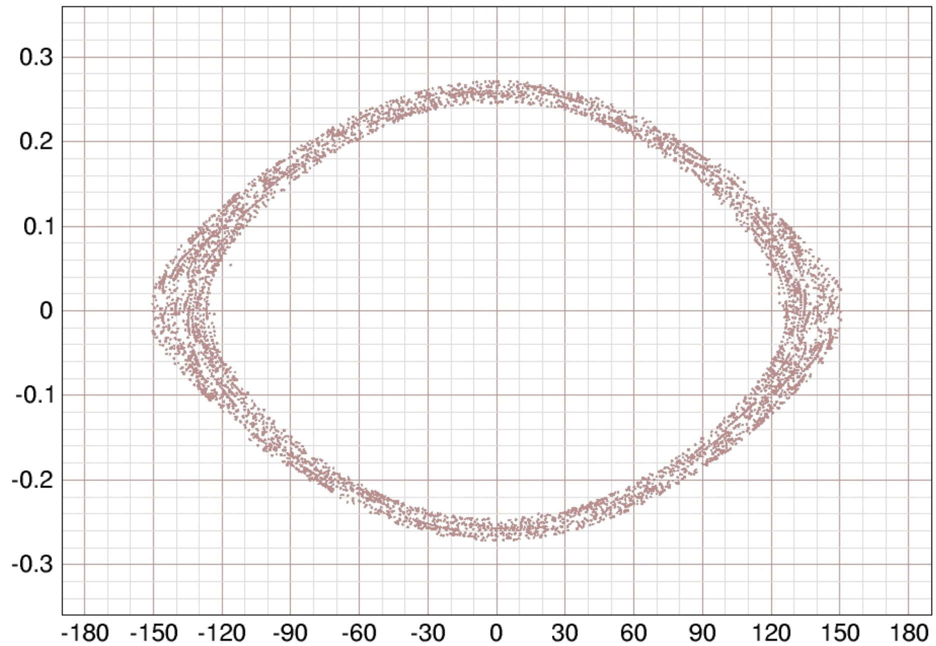
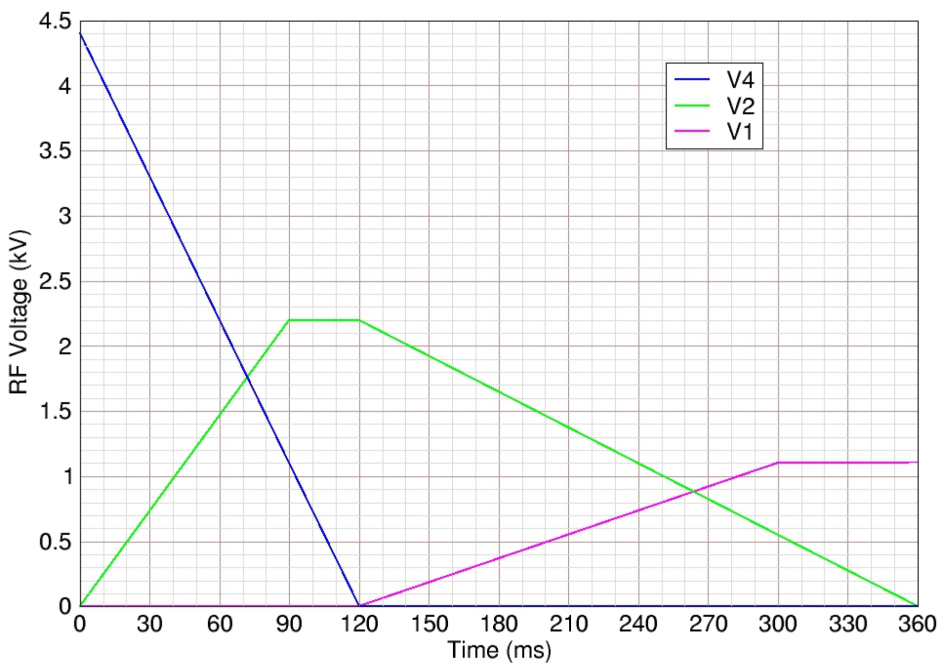


Figure 40: First layer deposited in harmonic 3 bucket.



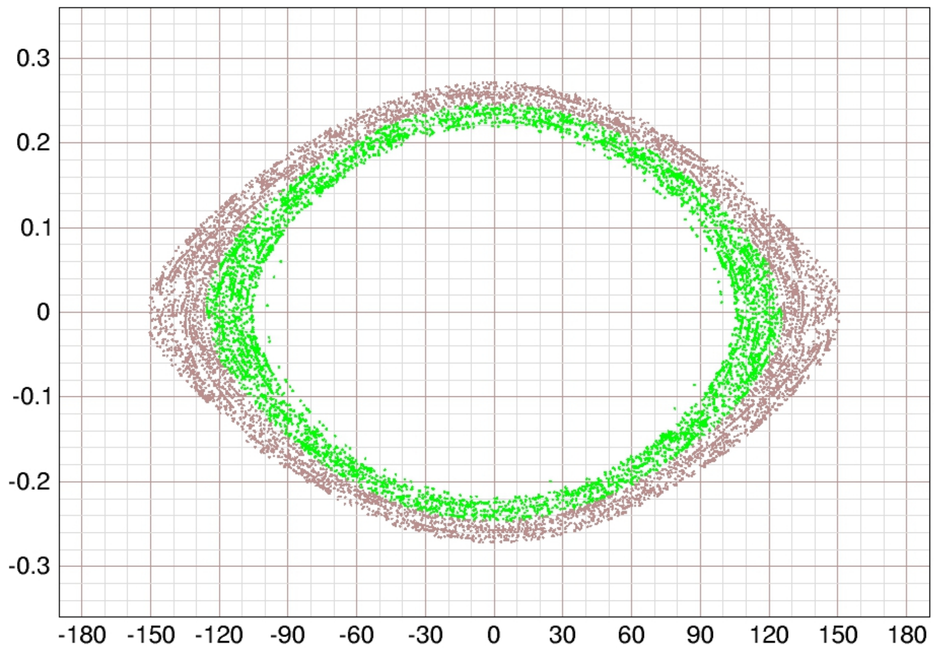
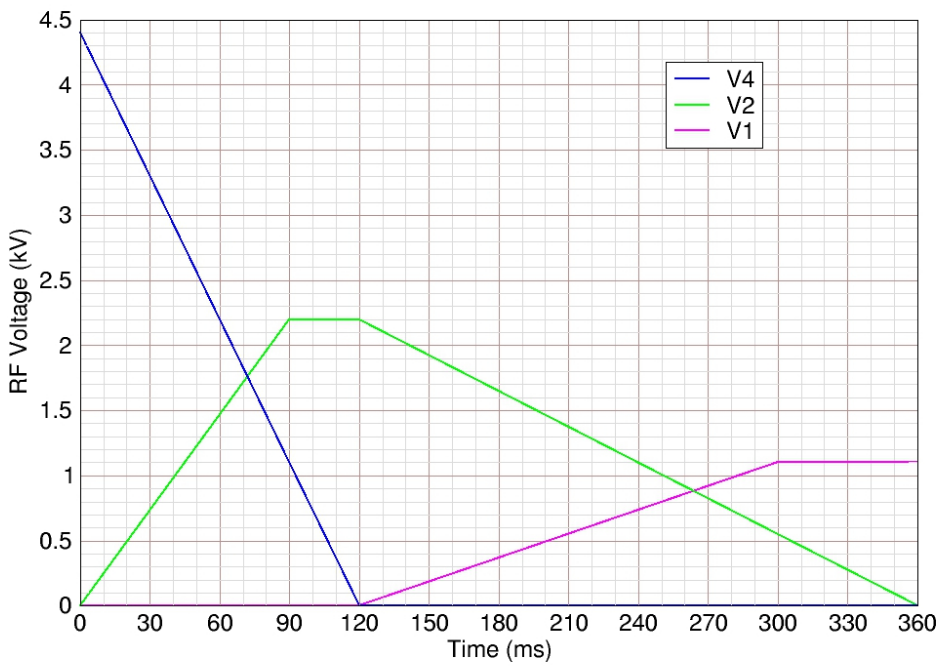


Figure 41: Second layer deposited in harmonic 3 bucket.



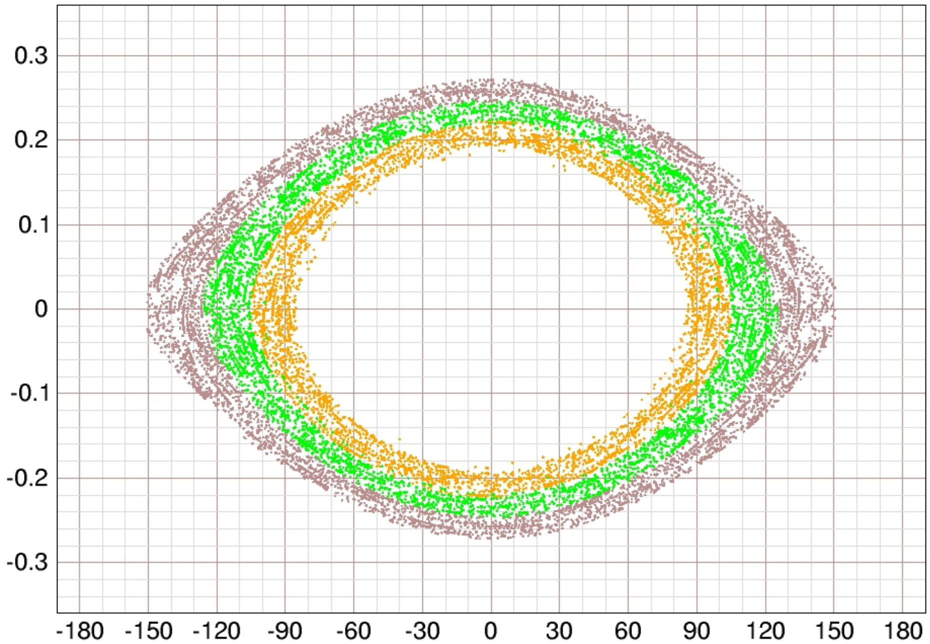
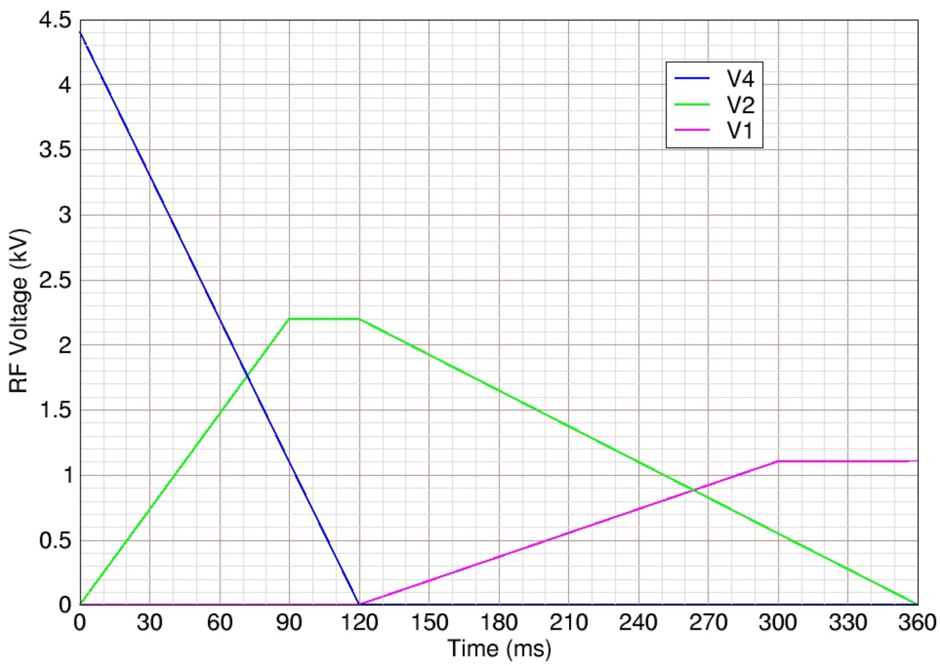


Figure 42: Third layer deposited in harmonic 3 bucket.



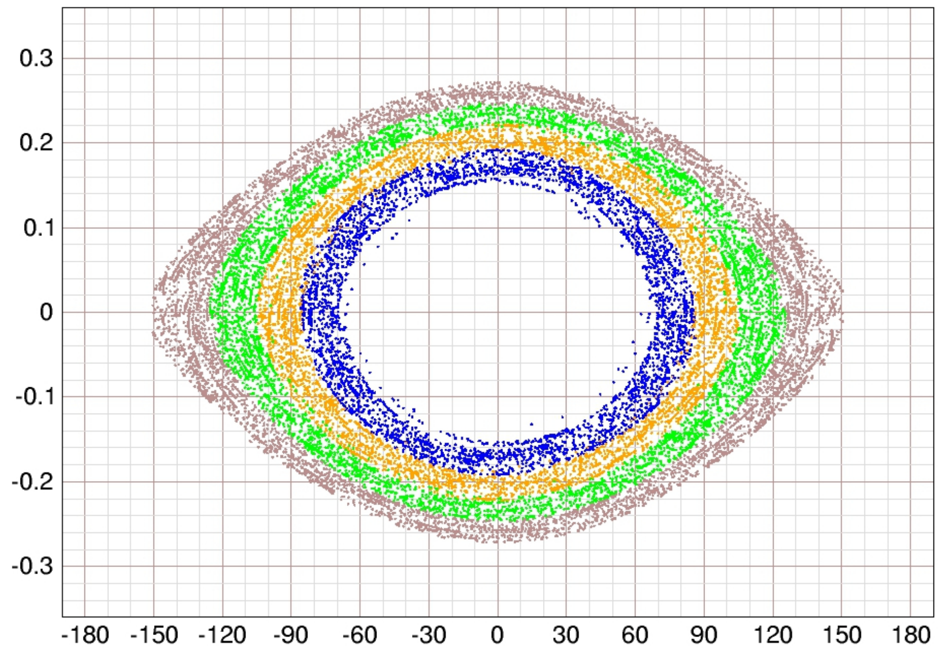
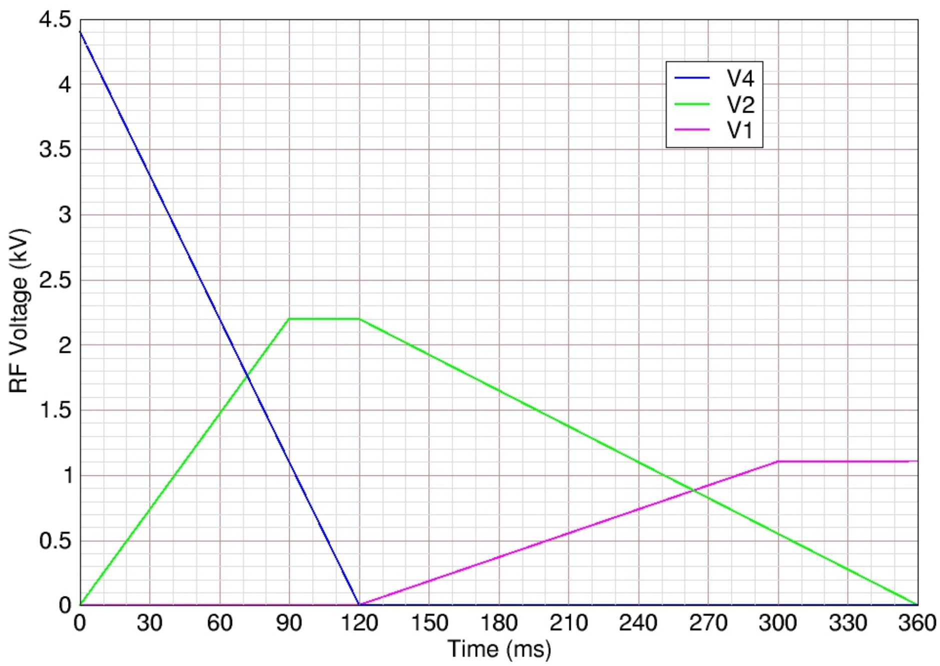


Figure 43: Fourth layer deposited in harmonic 3 bucket.



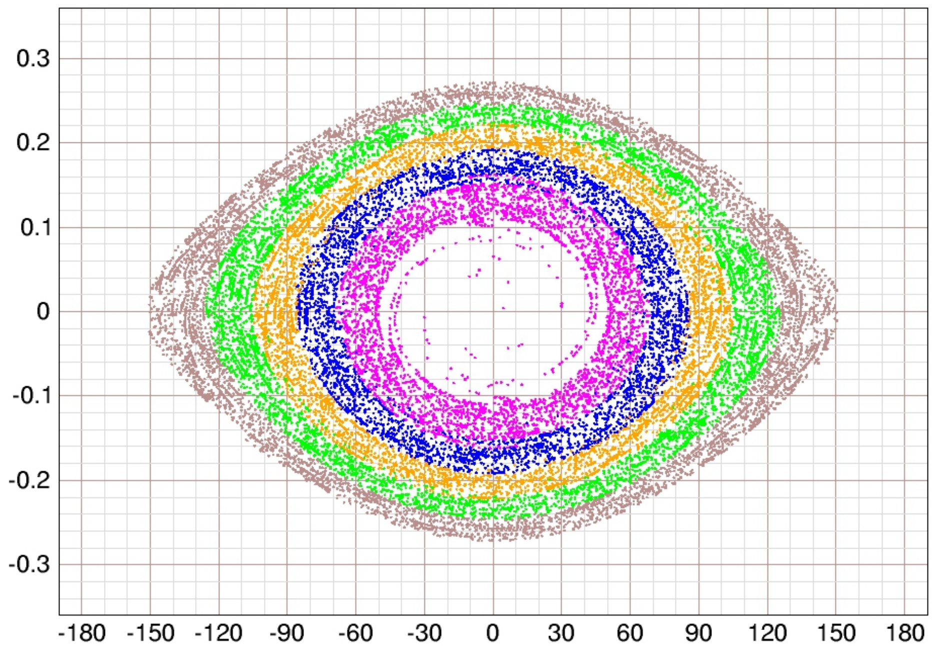
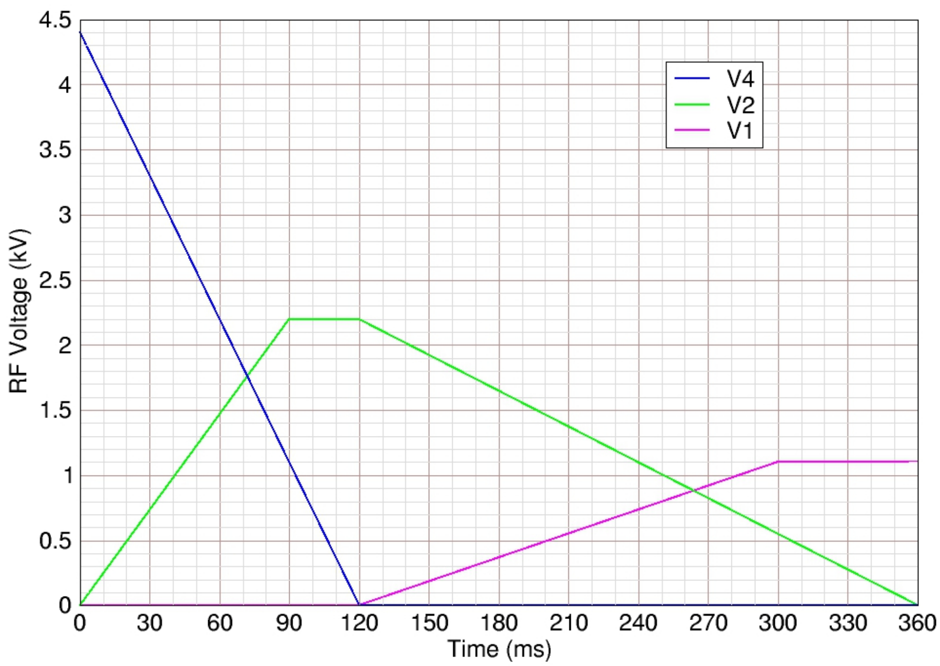


Figure 44: Fifth layer deposited in harmonic 3 bucket.



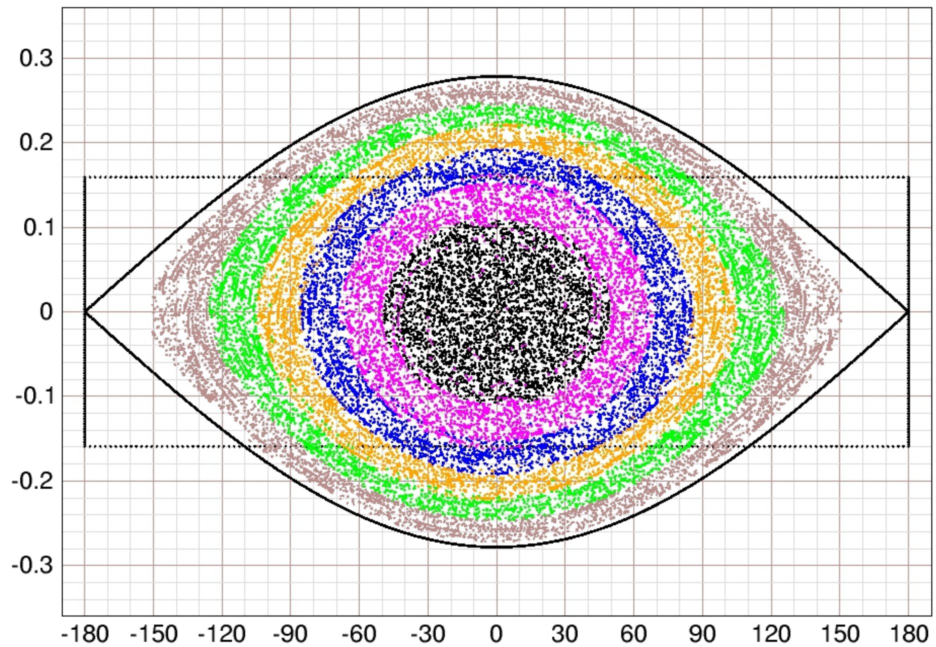
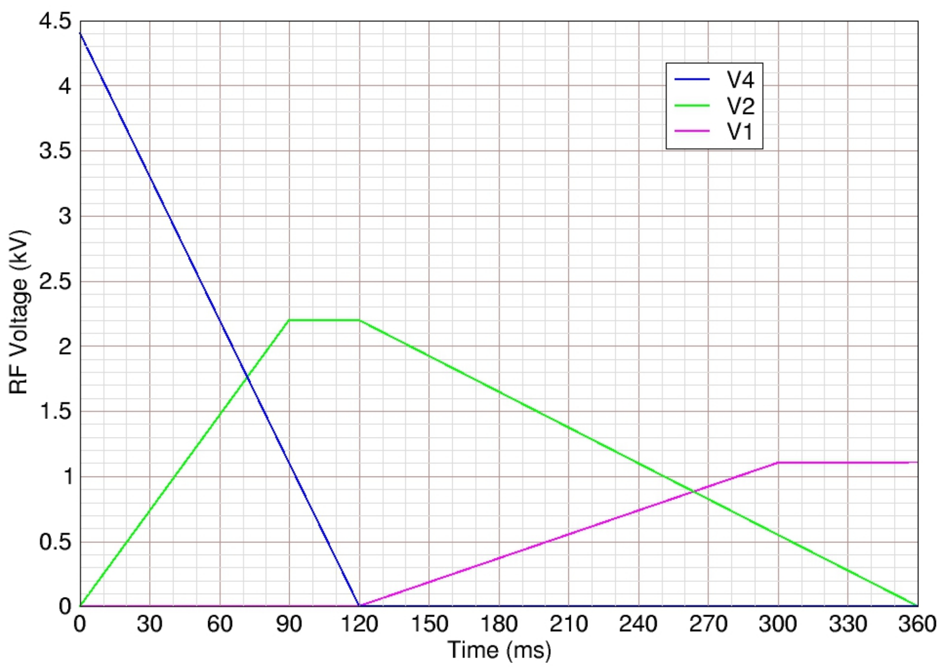


Figure 45: Final layer deposited in harmonic 3 bucket.



24 Illustration of delayed bunch mixing

We now carry out the exercise of the previous section with a much higher harmonic 12 voltage. The merge times are the same as before and the voltages are again programmed in accordance with equations (198) through (204). However, the initial harmonic 12 voltage is now 110 kV, which is 25 times larger than the corresponding 4.4 kV in the previous section. That means that the synchrotron period in the 110 kV bucket will be 5 times shorter. The lower half of **Figure 46** shows the programmed voltages. The upper half shows the initial bunches sitting in four harmonic 12 buckets. As before, the longitudinal emittance of each bunch is 0.5 eV-s. **Figures 46** through **56** show the evolution of the bunches from 0 to 90 ms. Here we see that for most of this time the bunches simply move toward one another with no mixing of particles. Mixing starts around 72 ms and the end of merge occurs at 90 ms as before. One sees again that the merge proceeds slowly enough to keep the oscillation amplitude layers intact and the gross longitudinal emittance unchanged. Note, however, that the mixing time is just 18 ms, which is 5 times smaller than the corresponding 90 ms time in the previous section. The effect of this is counteracted by the five-fold decrease in the synchrotron period as explained in Section 22. At 120 ms the merge is complete with the bunches sitting in harmonic 6 buckets as shown in **Figure 57**. The oscillation amplitude layers are shown individually in **Figures 58** through **63**. Here one again sees slight filamentation in the blue layer and more pronounced filamentation in the pink layer.

Figures 63 through **71** show the evolution of the bunches during the second 2 to 1 merge. One sees again that the merge proceeds slowly enough to keep the oscillation amplitude layers intact and the gross longitudinal emittance unchanged.

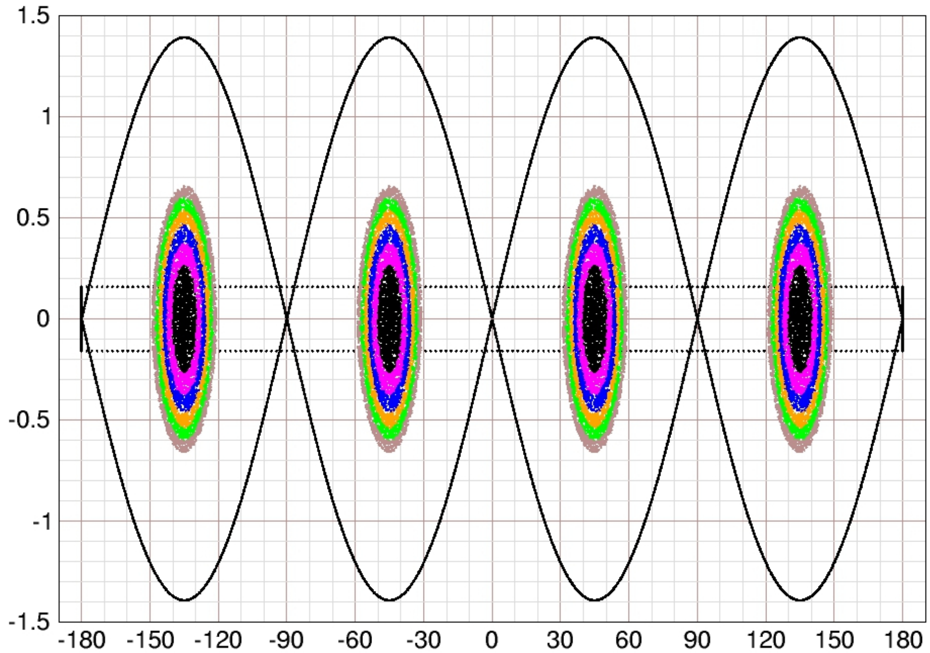
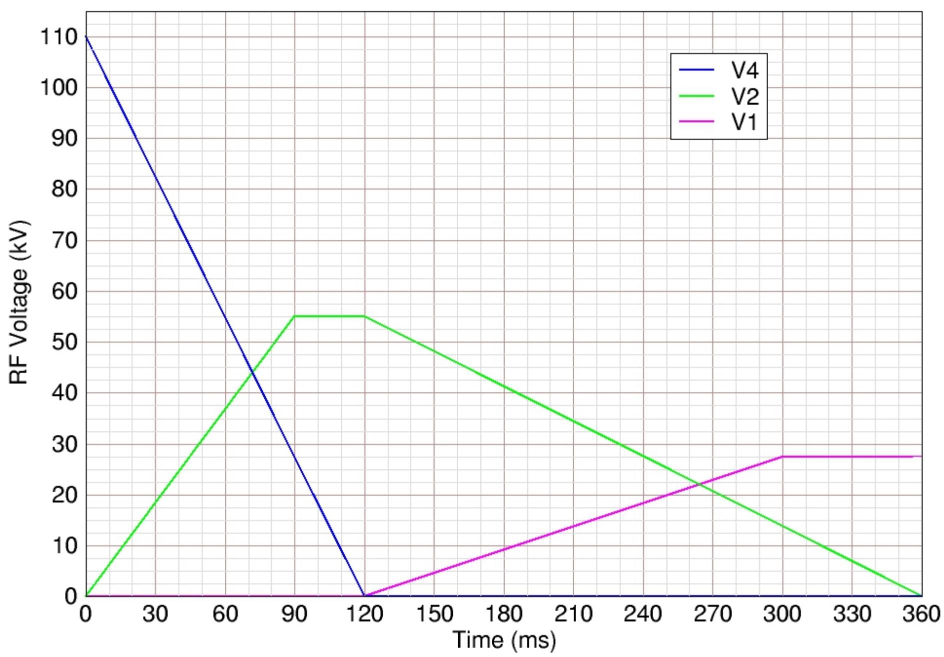


Figure 46: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



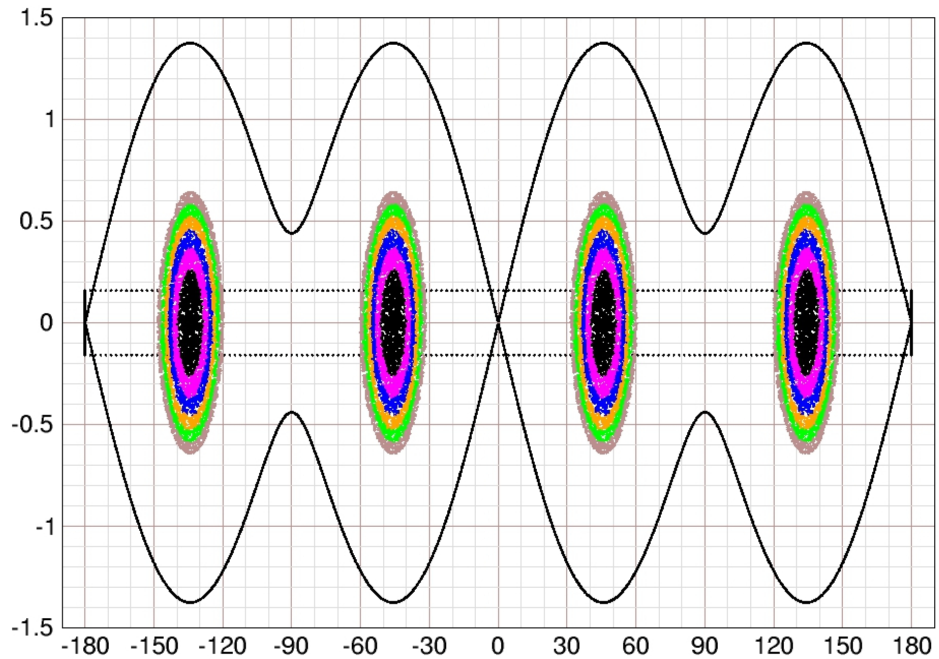
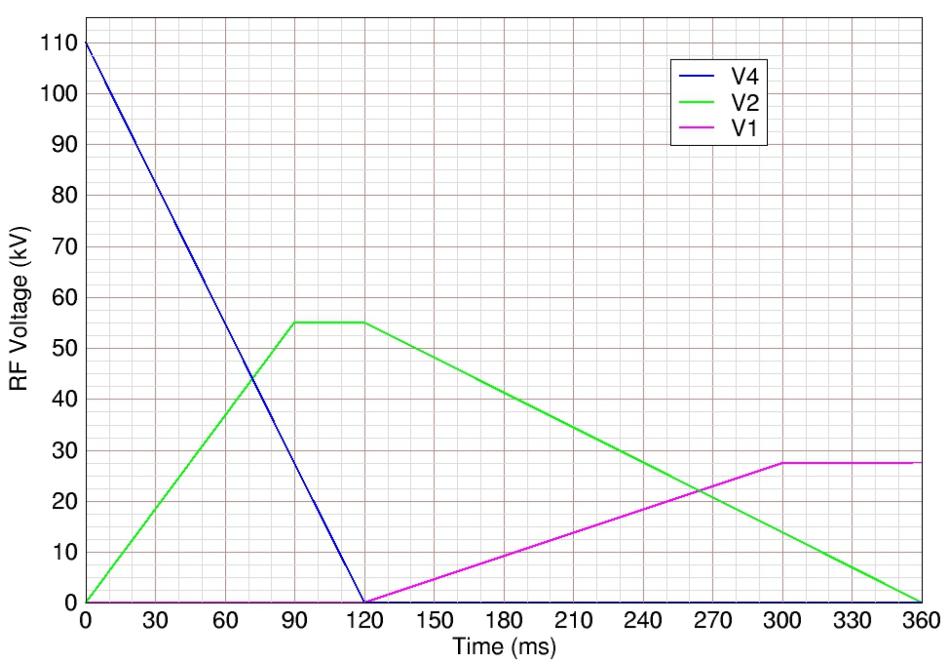


Figure 47: 4 to 2 to 1 merge at time $\underline{t = 9 \text{ ms}}$.



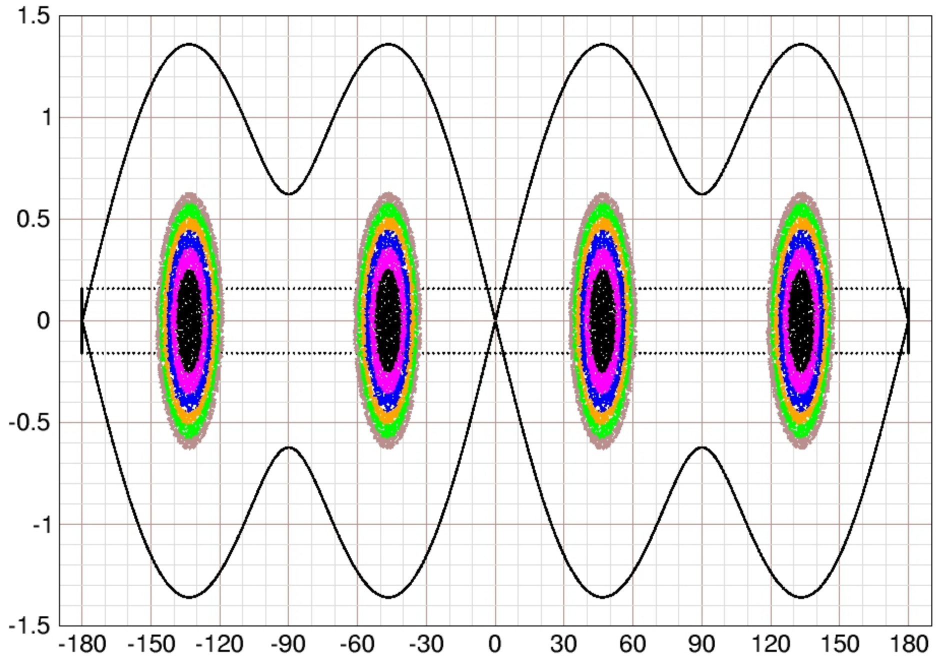
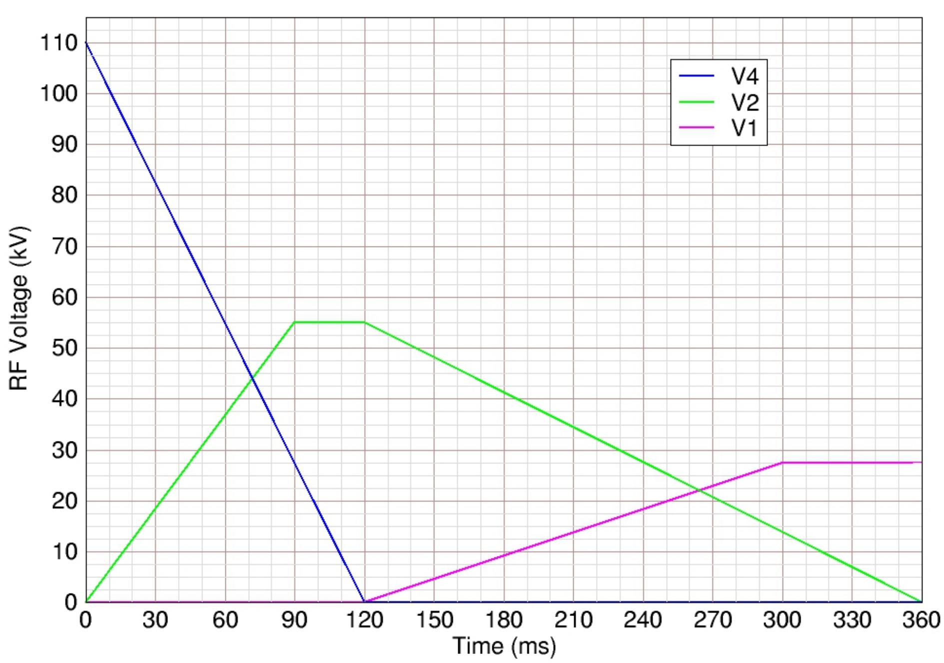


Figure 48: 4 to 2 to 1 merge at time $\underline{t = 18\ \text{ms}}$.



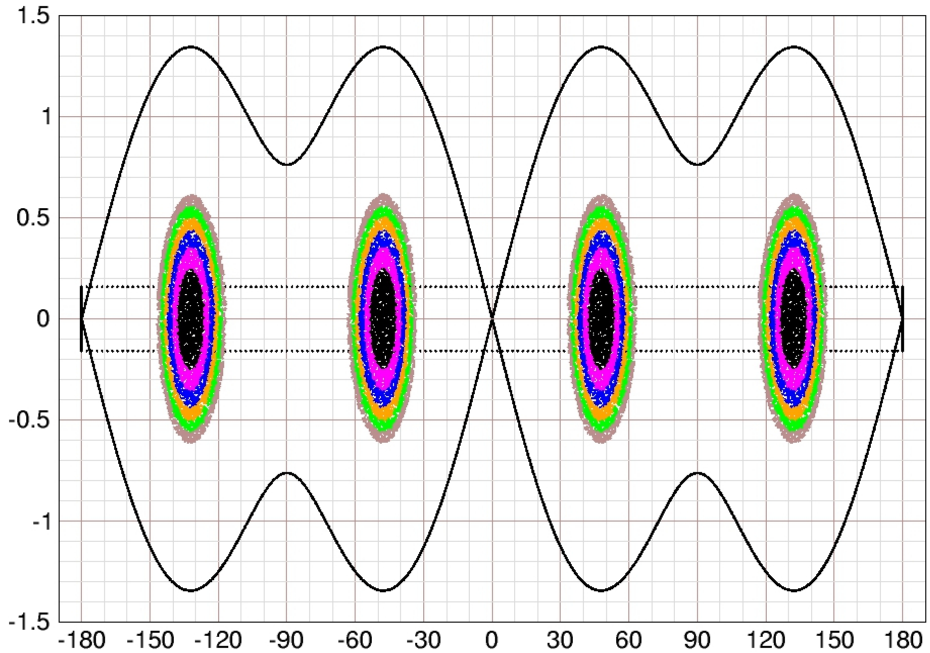
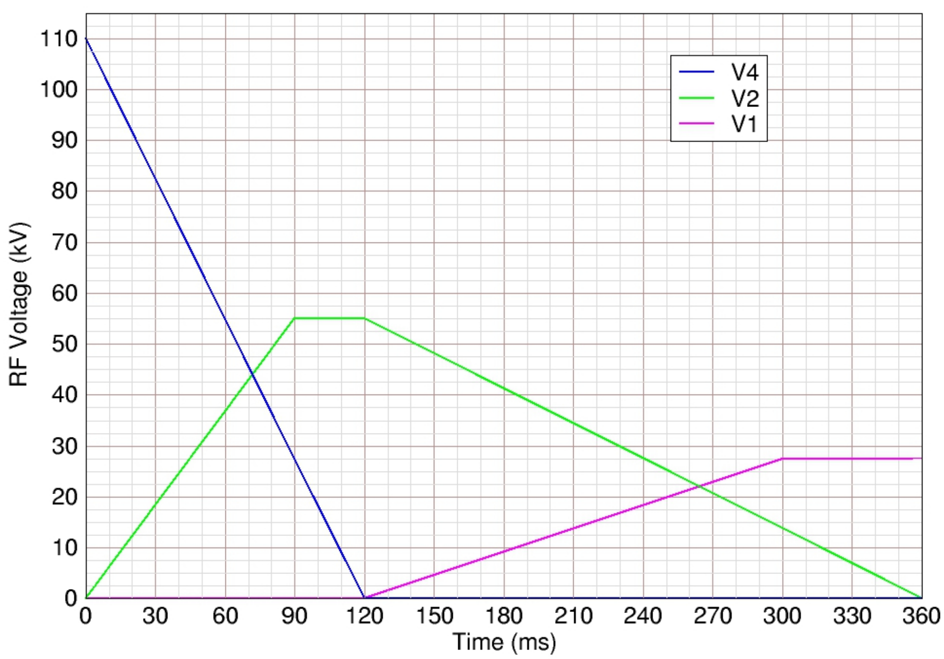


Figure 49: 4 to 2 to 1 merge at time $\underline{t = 27\text{ ms}}$.



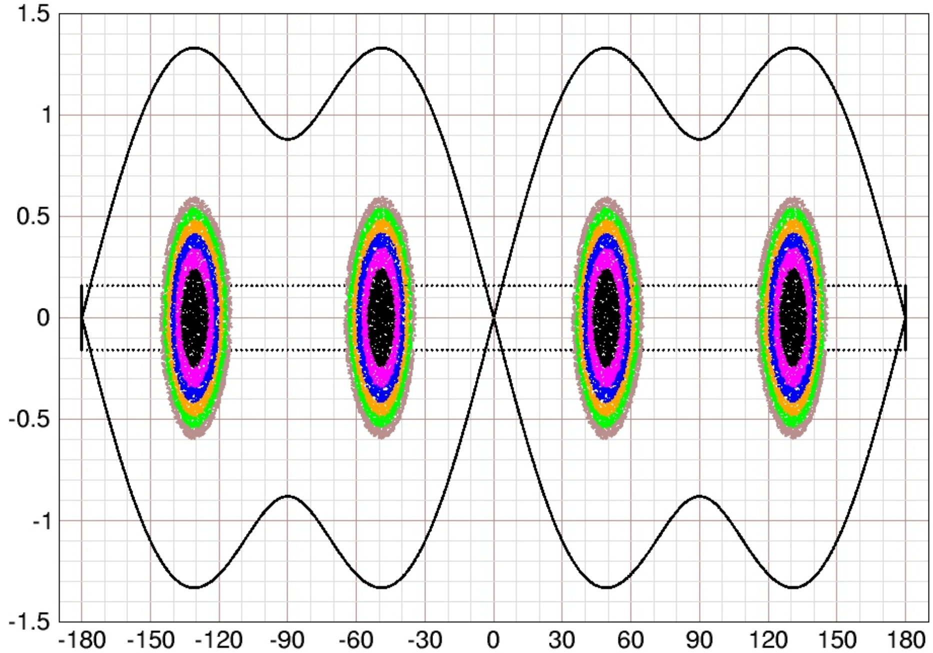
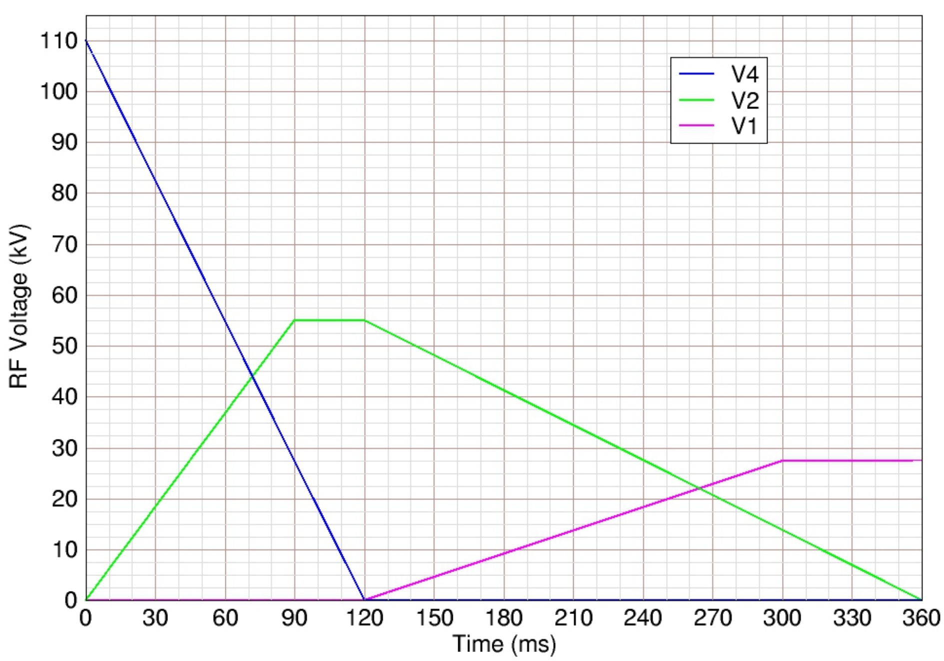


Figure 50: 4 to 2 to 1 merge at time $\underline{t = 36\ \text{ms}}$.



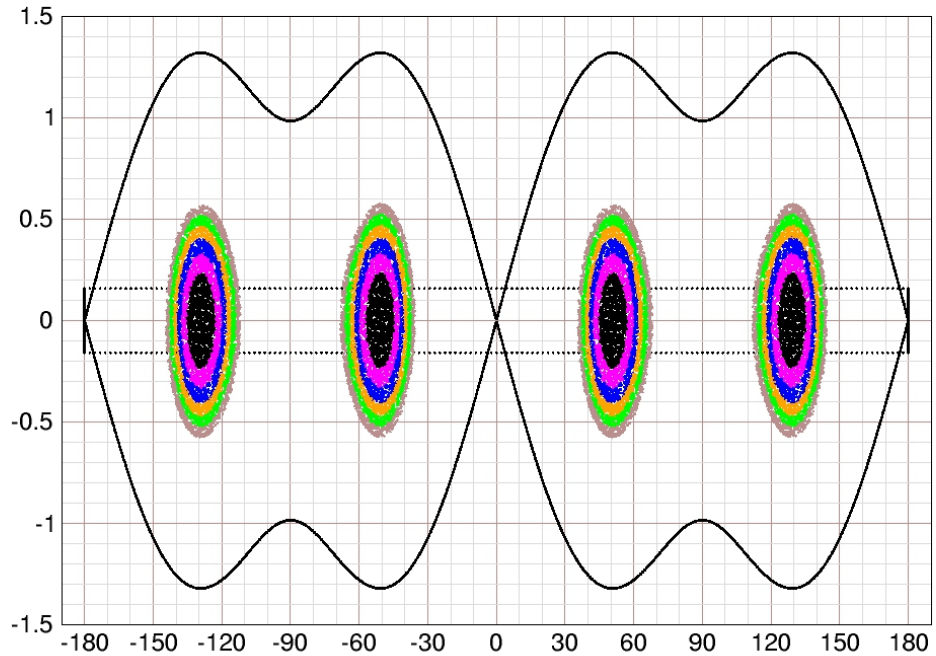
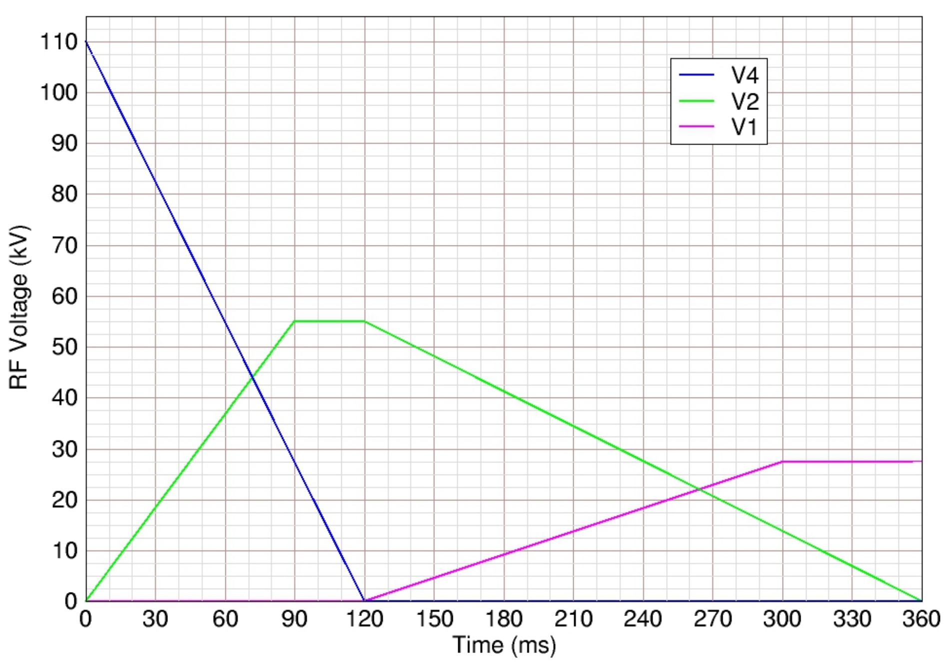


Figure 51: 4 to 2 to 1 merge at time <u>$t = 45$ ms</u>.



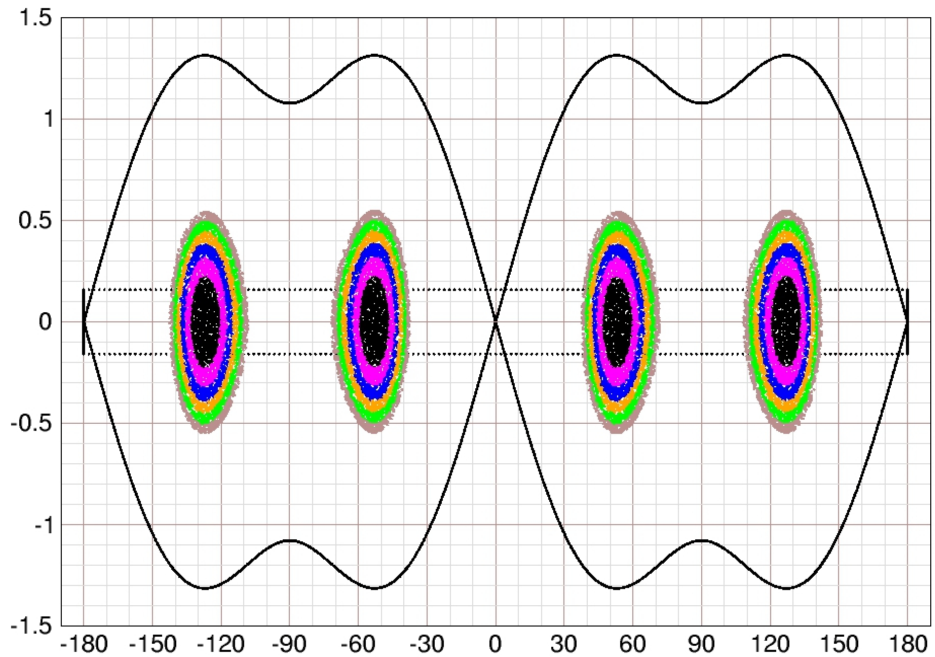
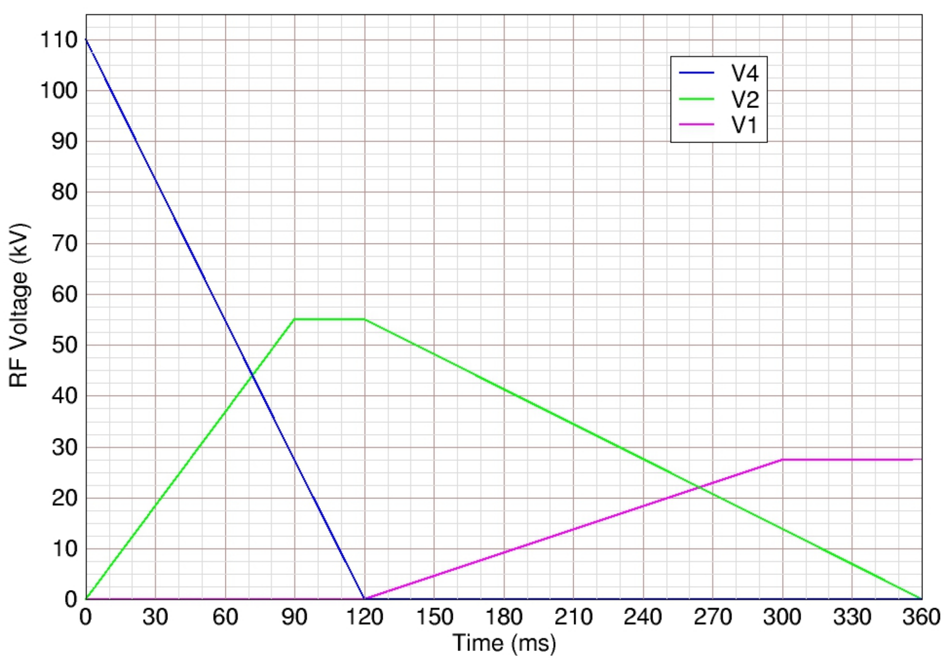


Figure 52: 4 to 2 to 1 merge at time $\underline{t = 54 \text{ ms}}$.



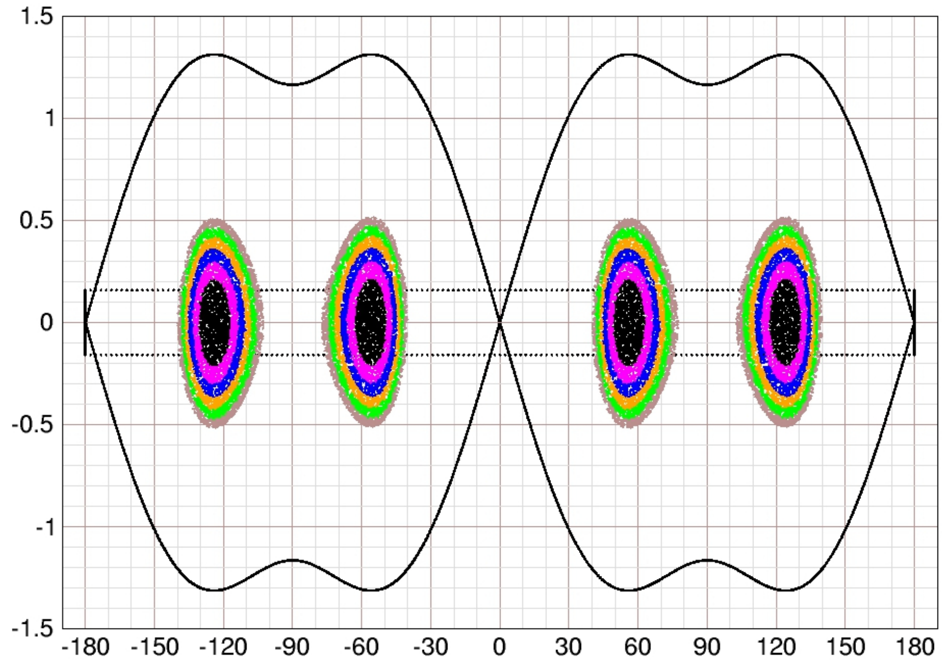
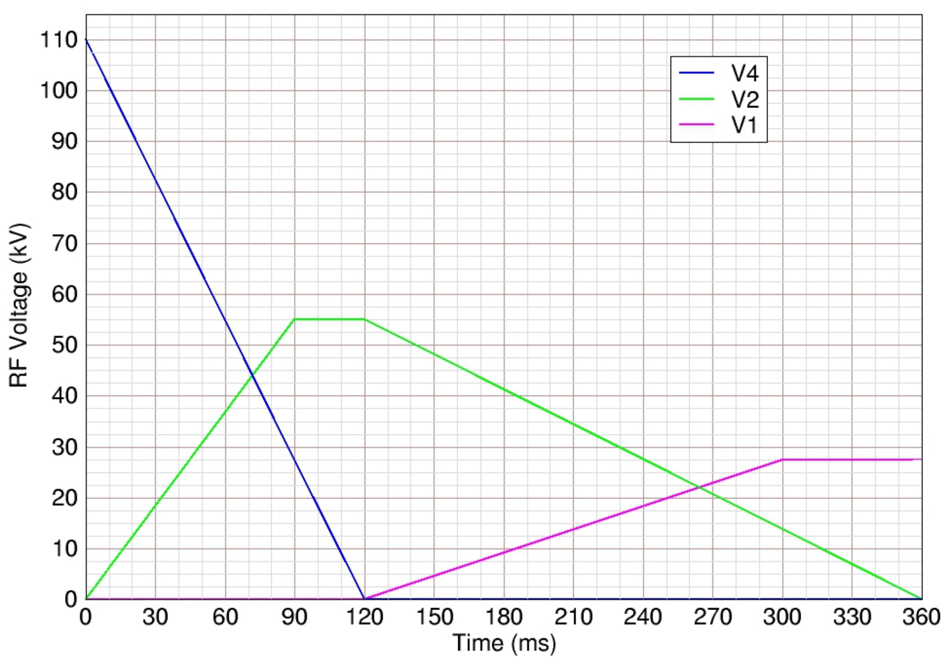


Figure 53: 4 to 2 to 1 merge at time $\underline{t = 63 \text{ ms}}$.



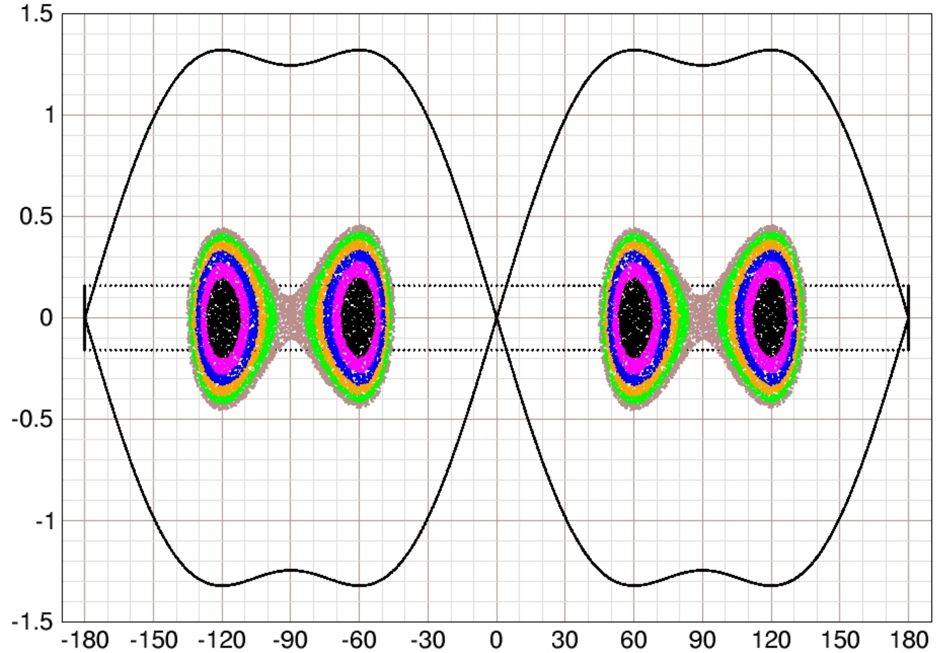
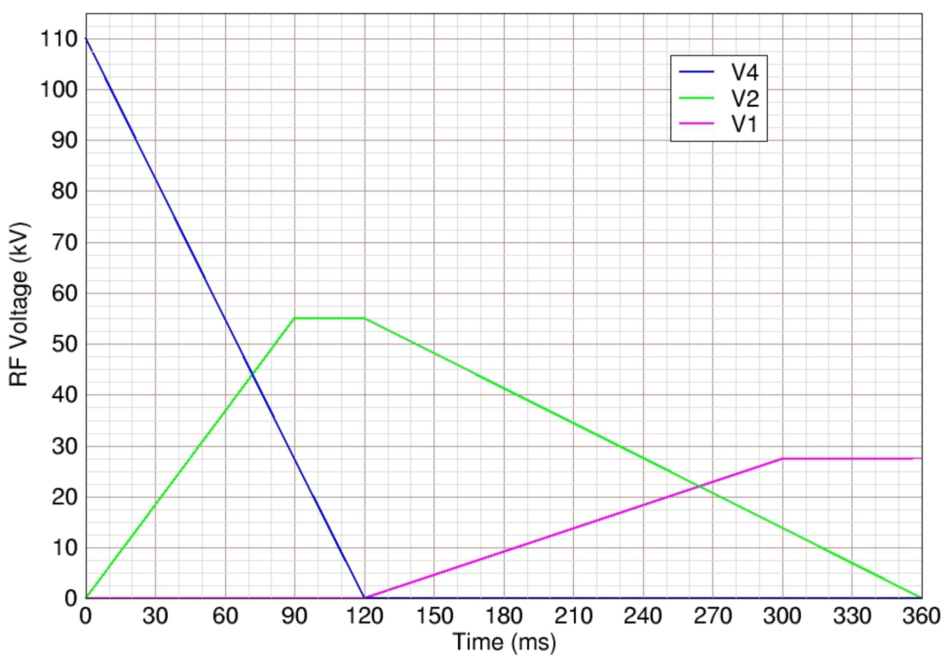


Figure 54: 4 to 2 to 1 merge at time $\underline{t = 72 \text{ ms}}$.



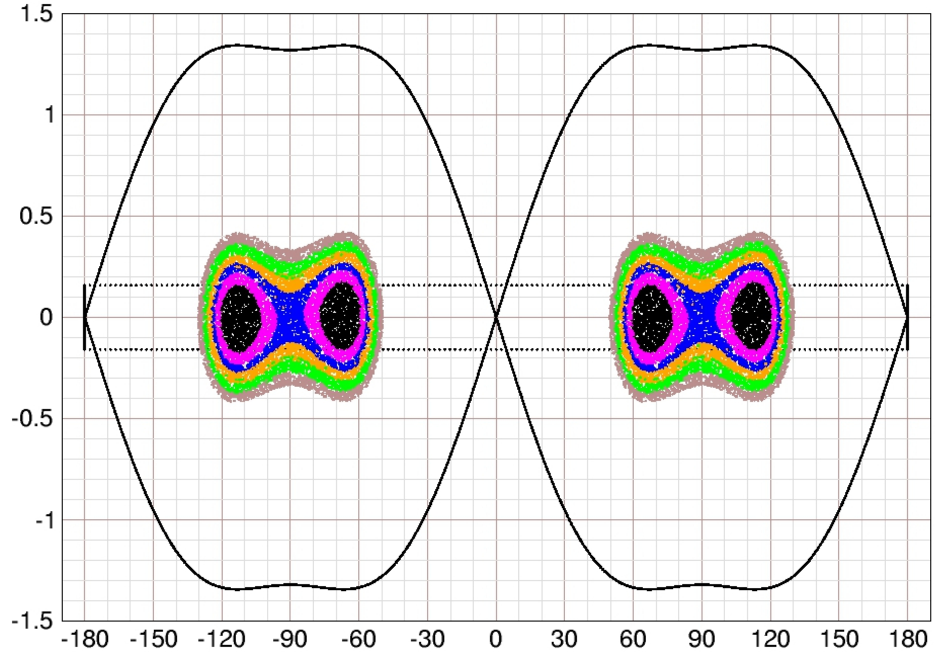
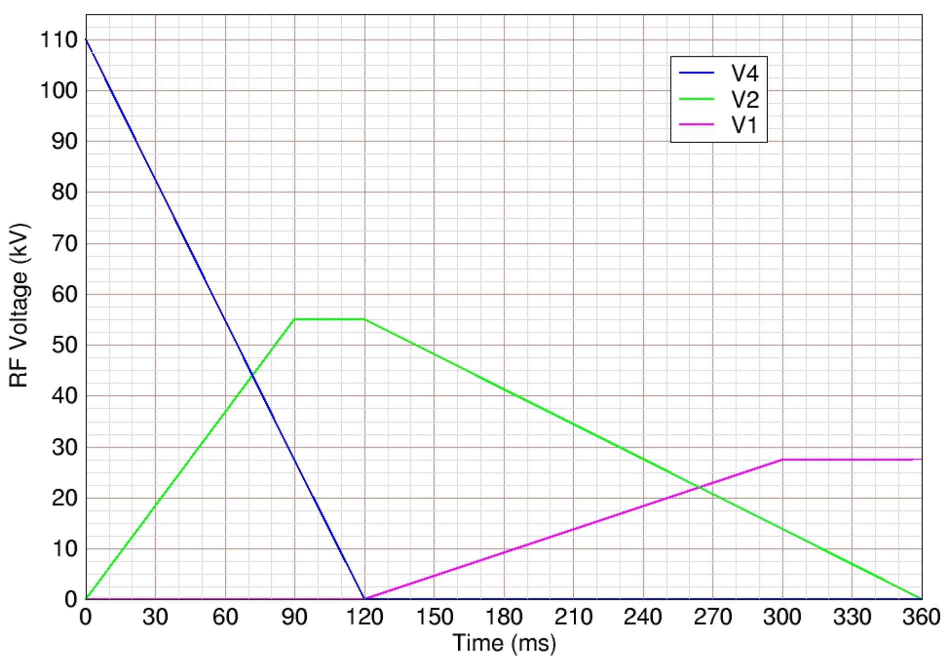


Figure 55: 4 to 2 to 1 merge at time $\underline{t = 81\ \text{ms}}$.



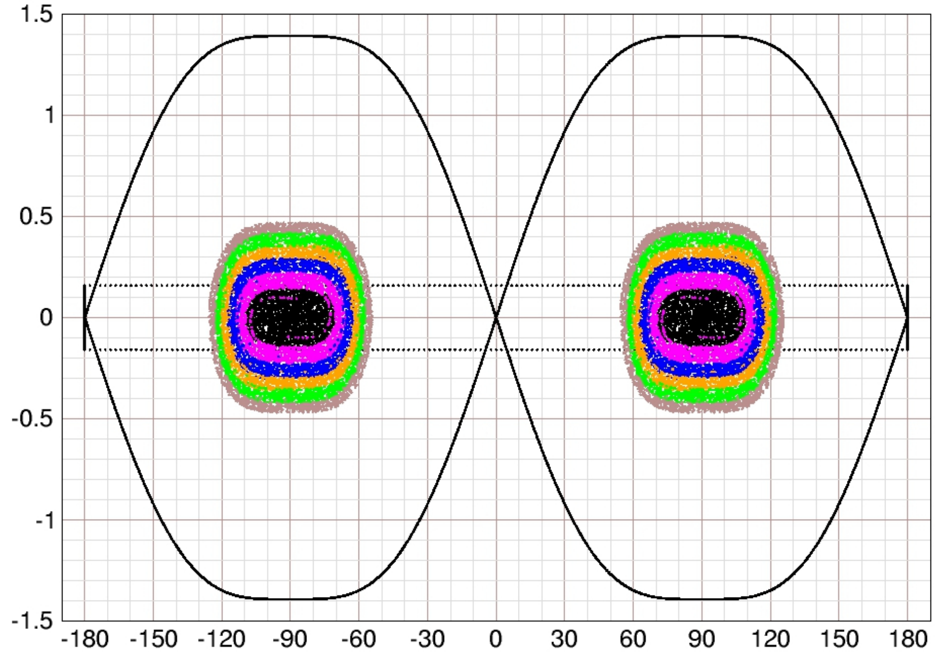
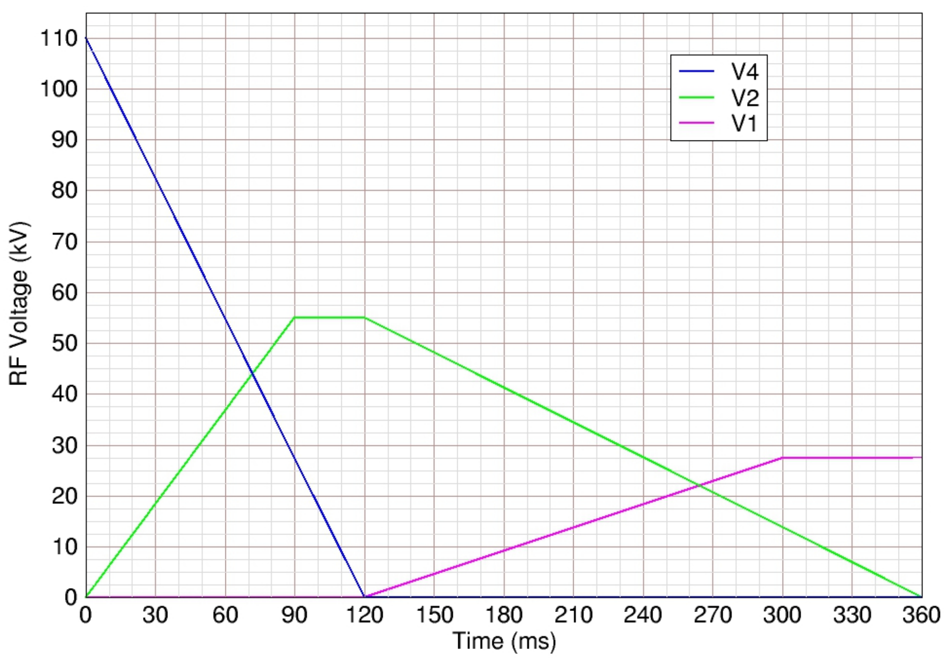


Figure 56: 4 to 2 to 1 merge at time $\underline{t = 90\ \text{ms}}$.



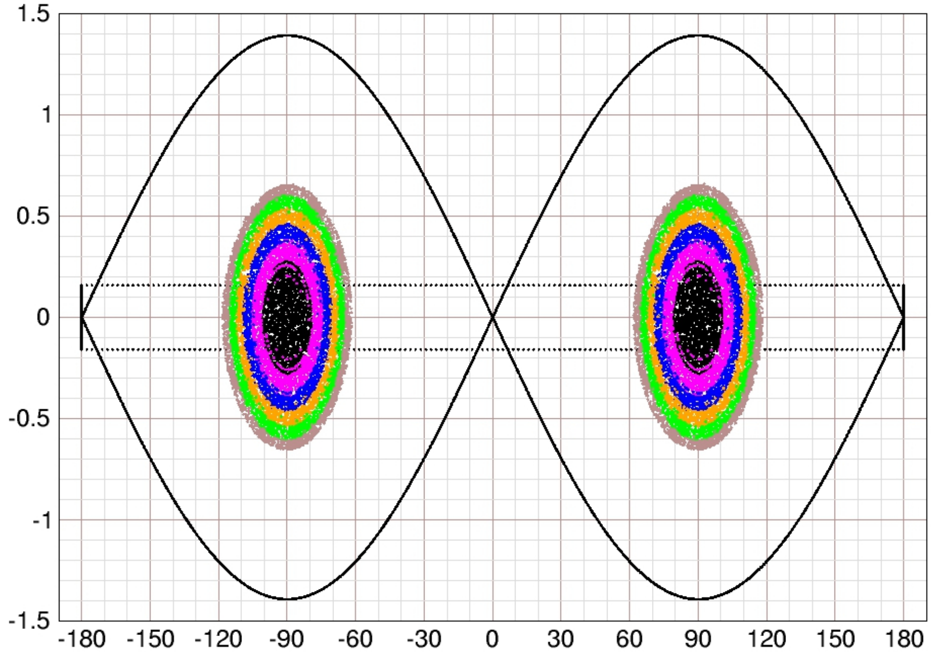
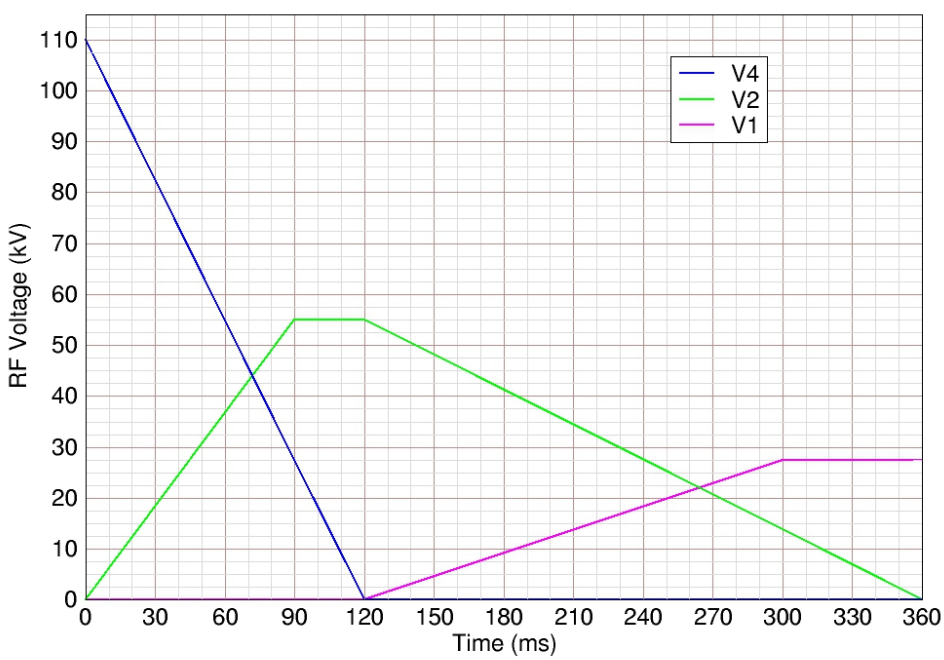


Figure 57: 4 to 2 to 1 merge at time $\underline{t = 120 \text{ ms}}$.



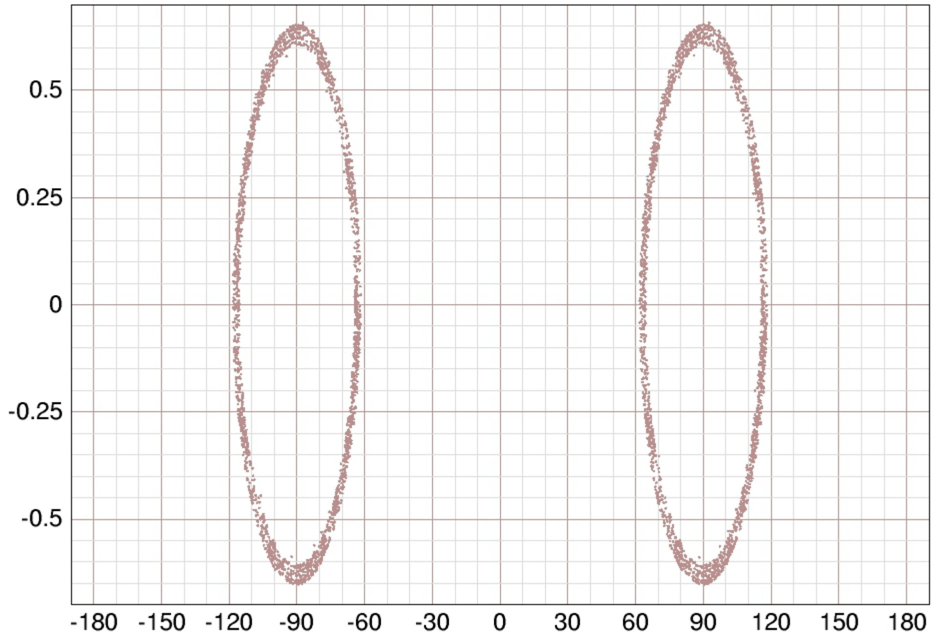
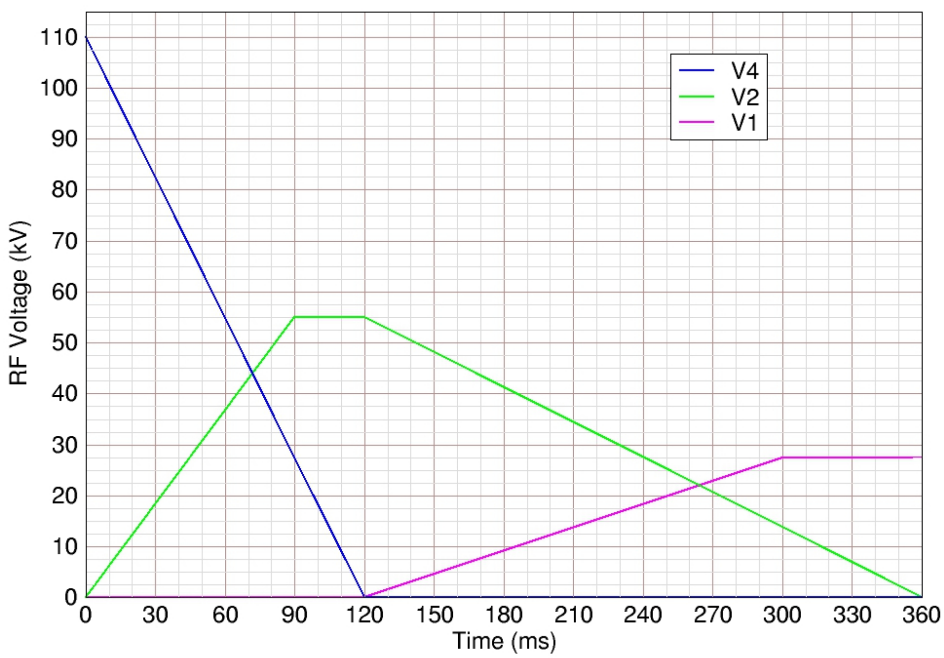


Figure 58: First layer deposited in harmonic 6 bucket.



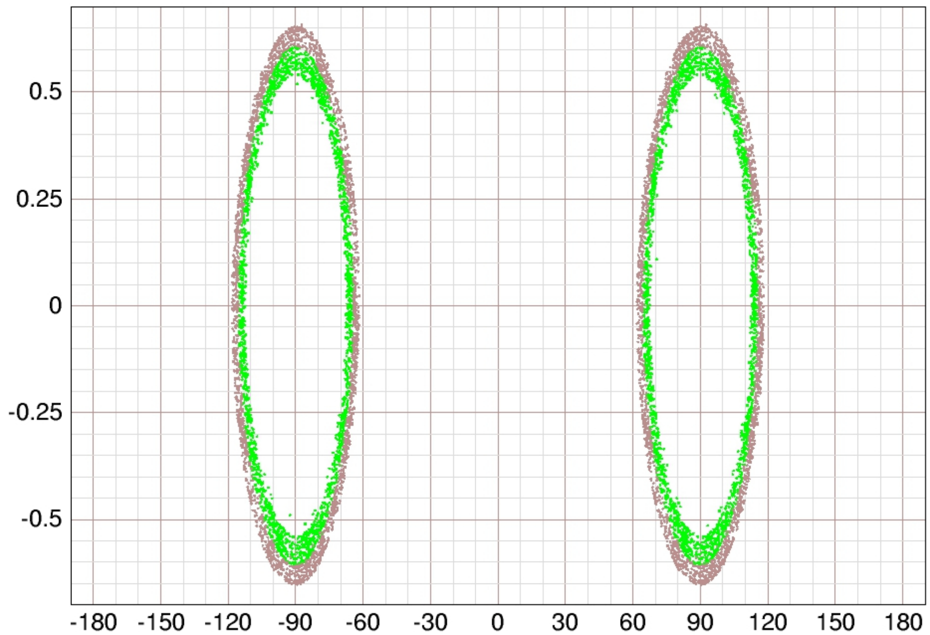
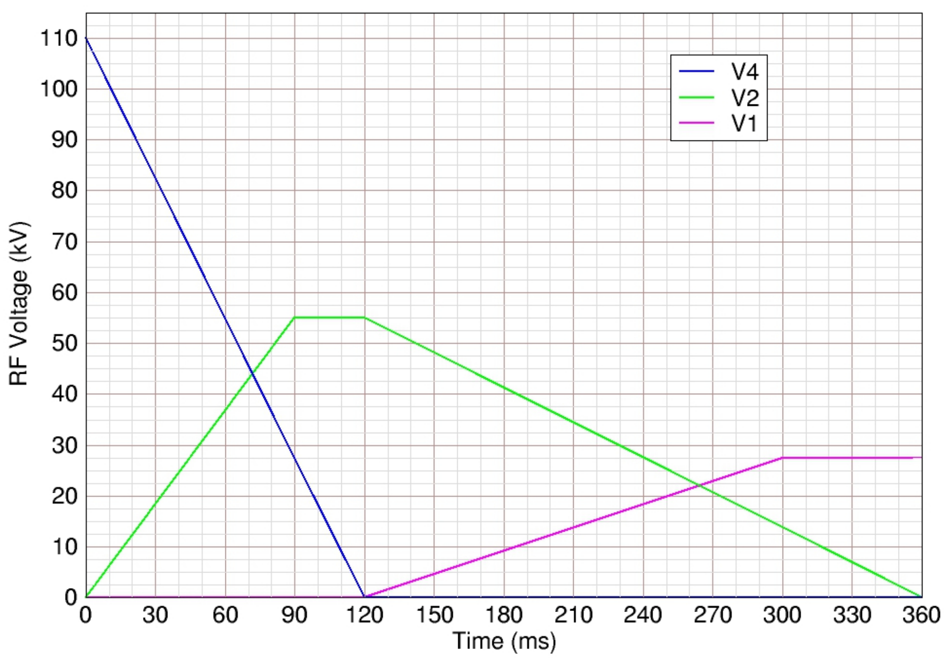


Figure 59: Second layer deposited in harmonic 6 bucket.



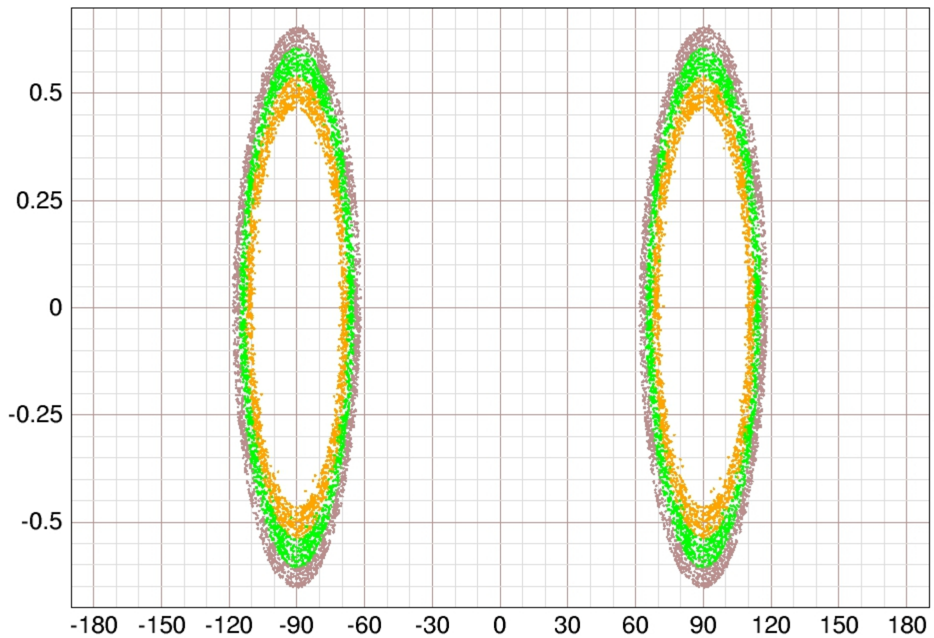
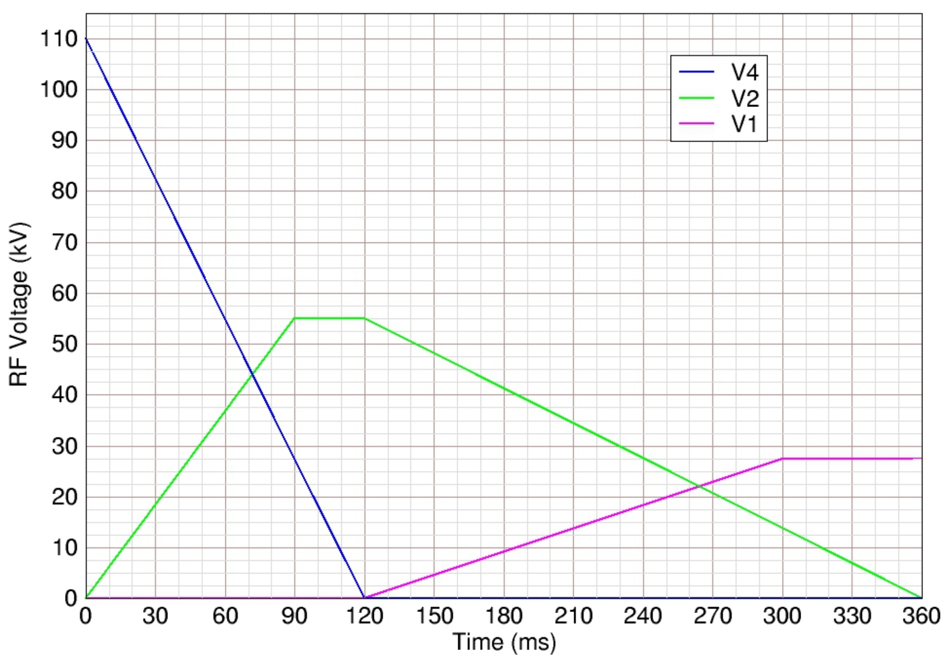


Figure 60: Third layer deposited in harmonic 6 bucket.



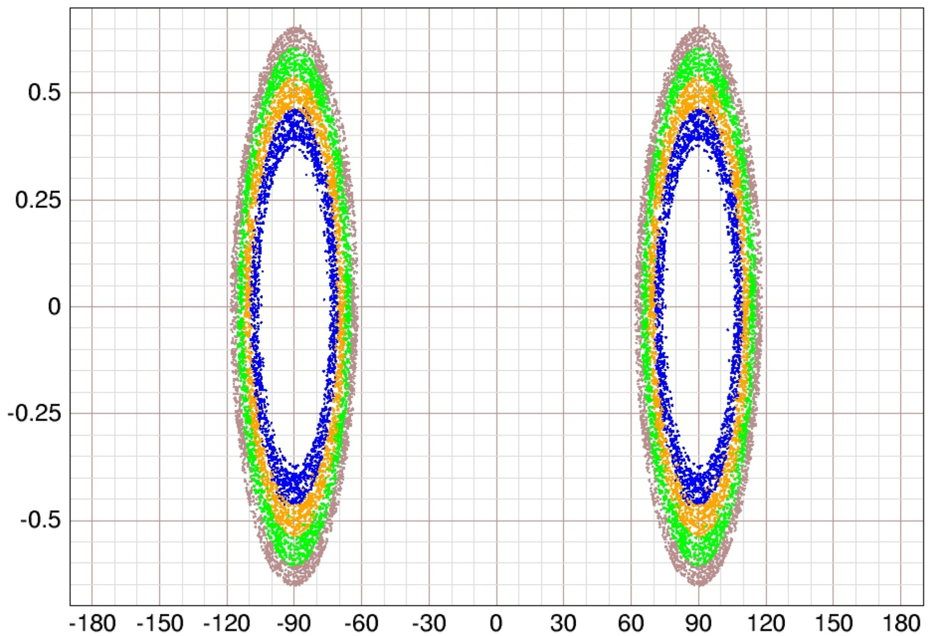
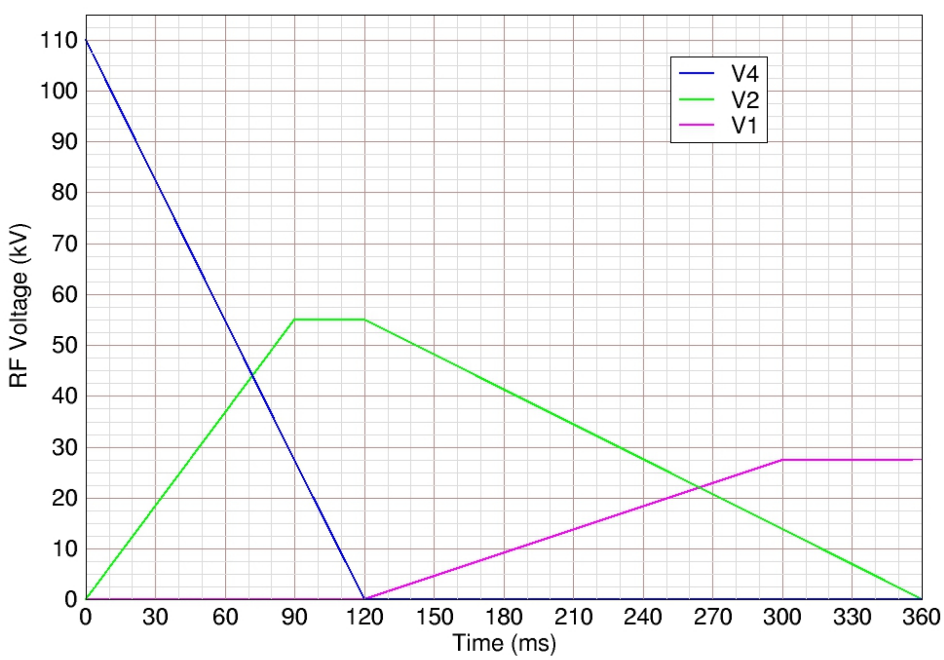


Figure 61: Fourth layer deposited in harmonic 6 bucket.



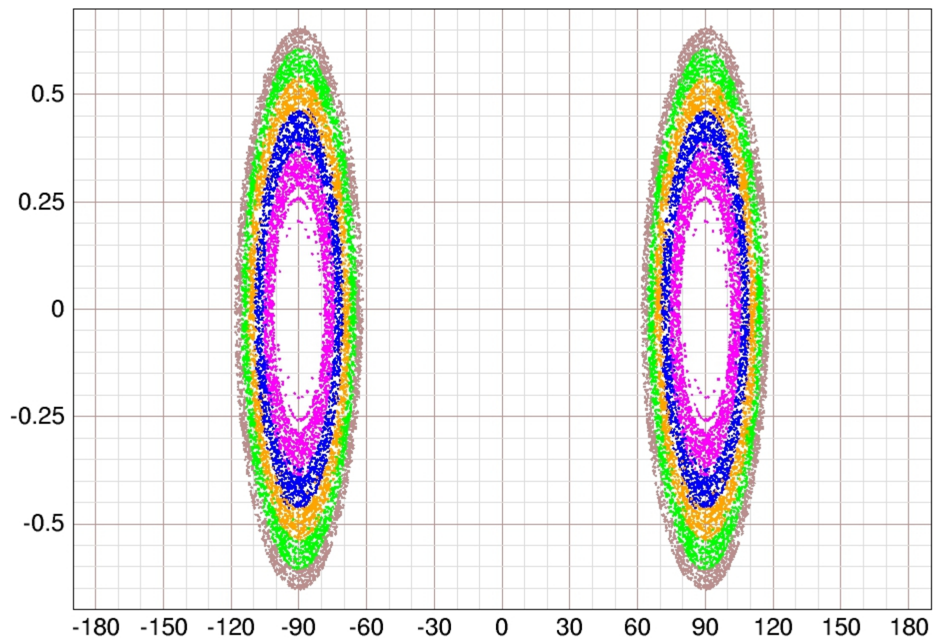
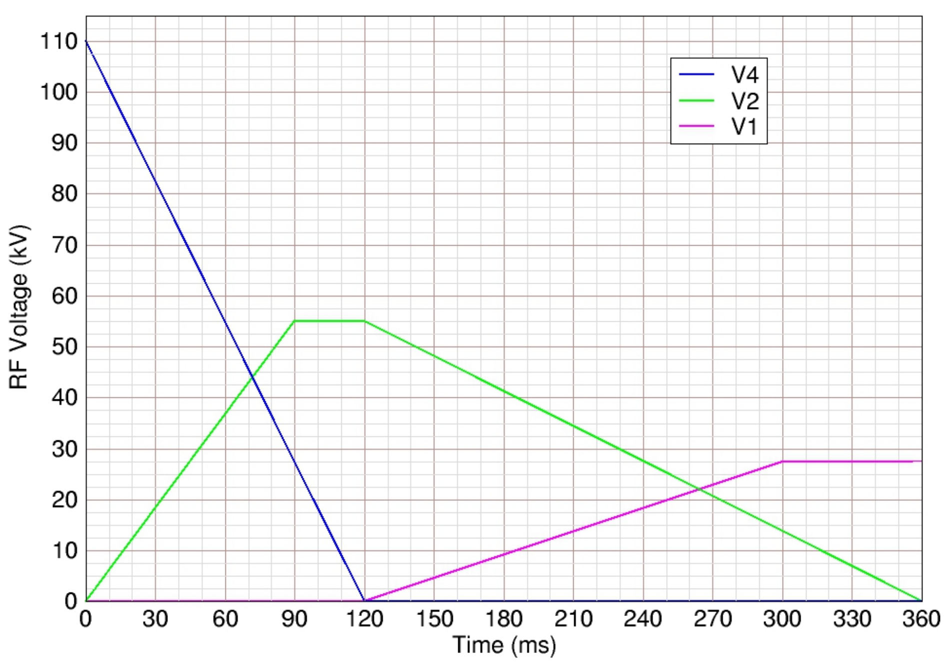


Figure 62: Fifth layer deposited in harmonic 6 bucket.



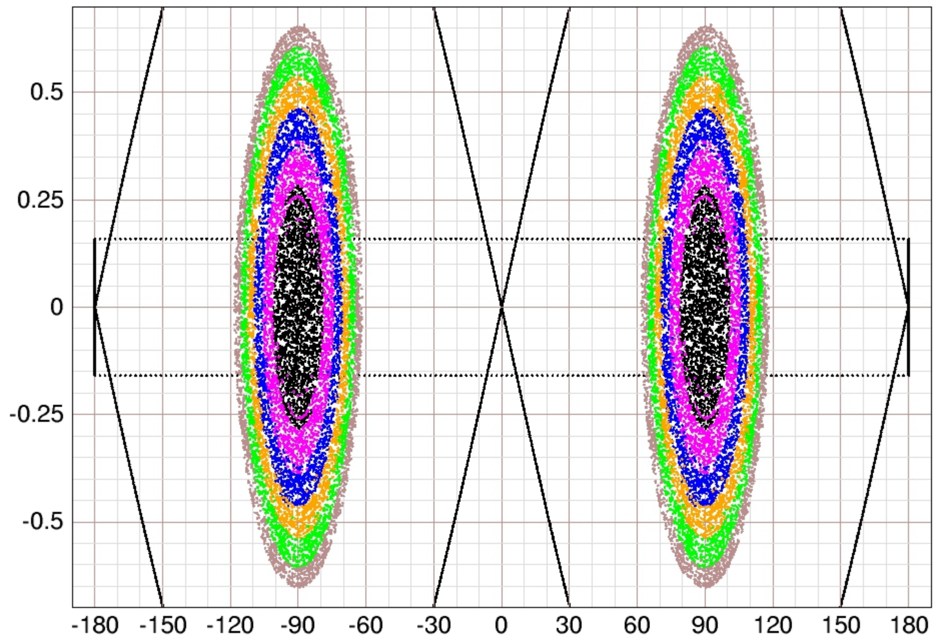
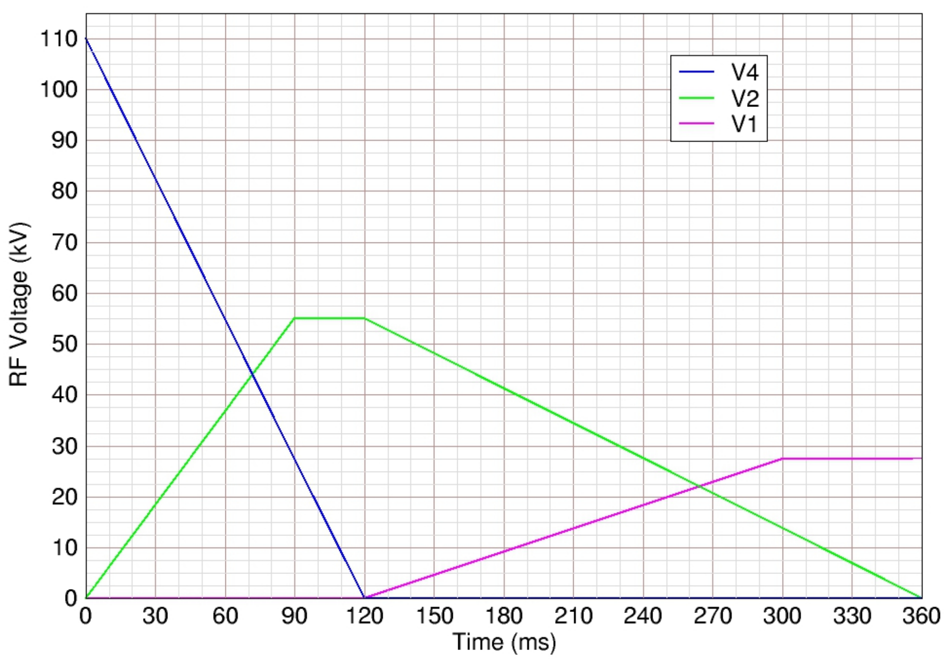


Figure 63: Final layer deposited in harmonic 6 bucket.



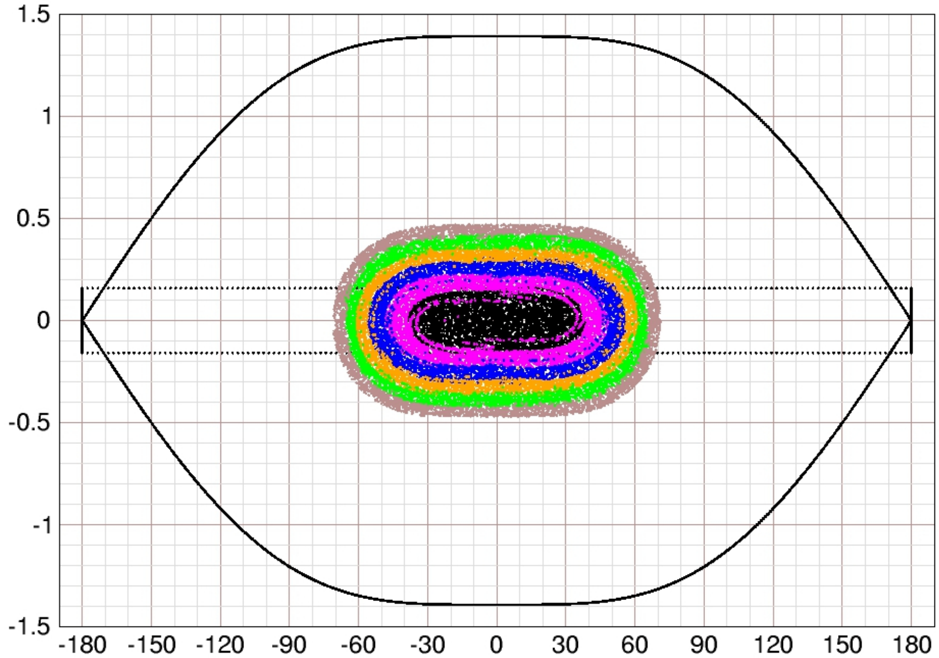
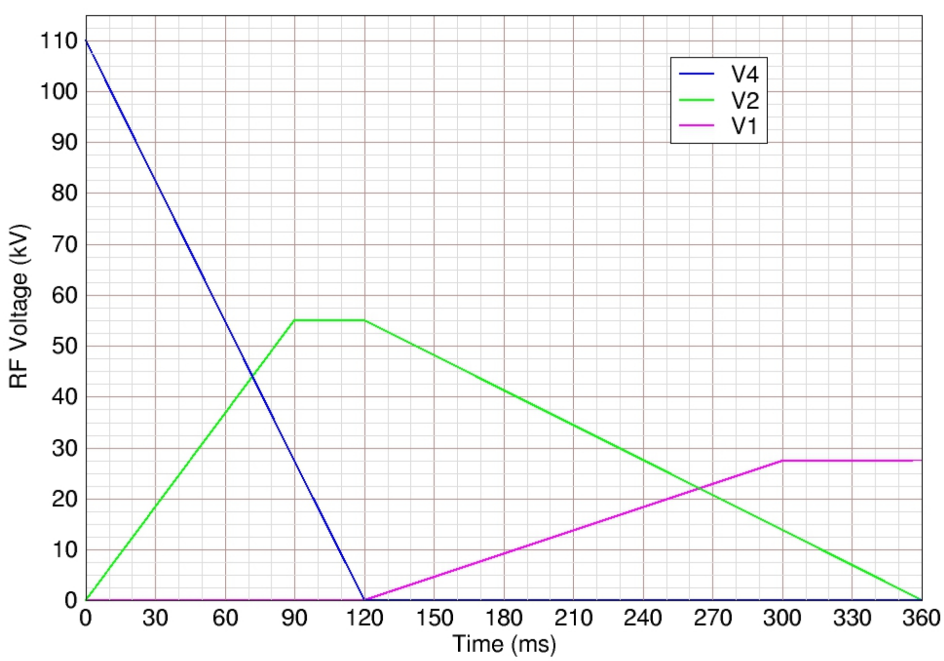


Figure 64: 4 to 2 to 1 merge at time $\underline{t = 300 \text{ ms}}$.



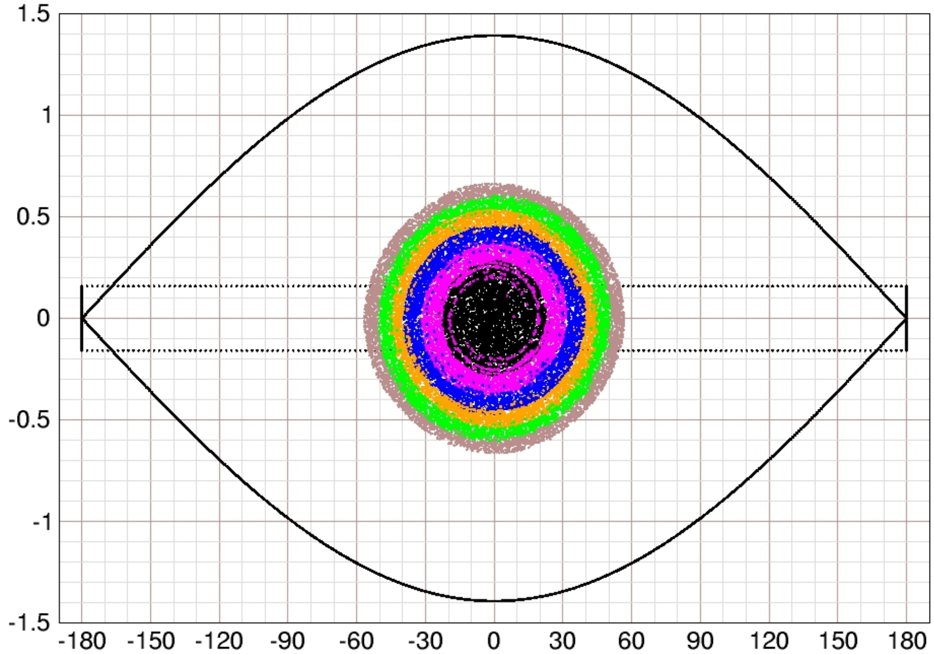
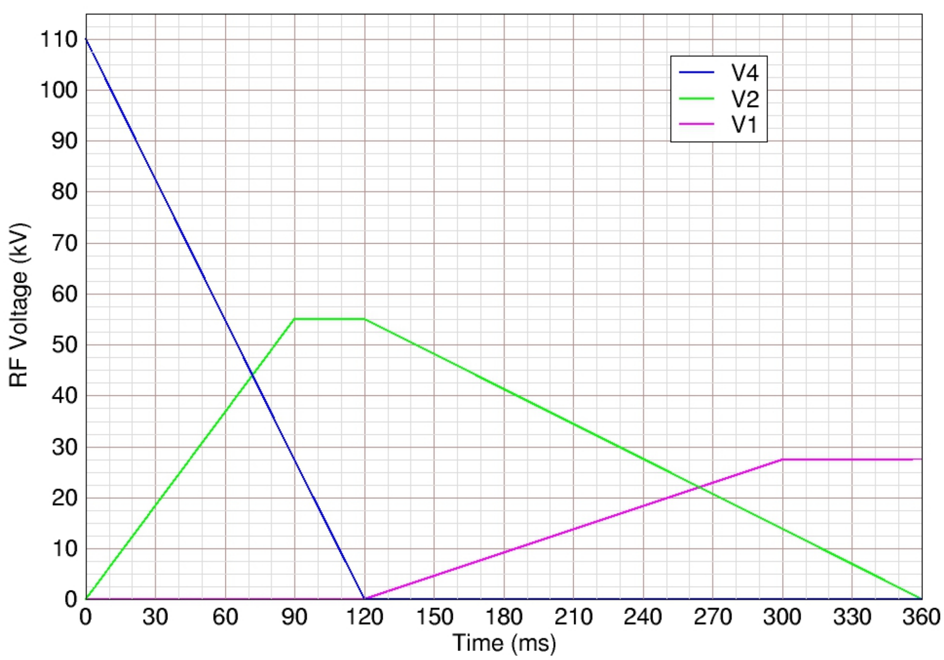


Figure 65: 4 to 2 to 1 merge at time $\underline{t = 360 \text{ ms}}$.



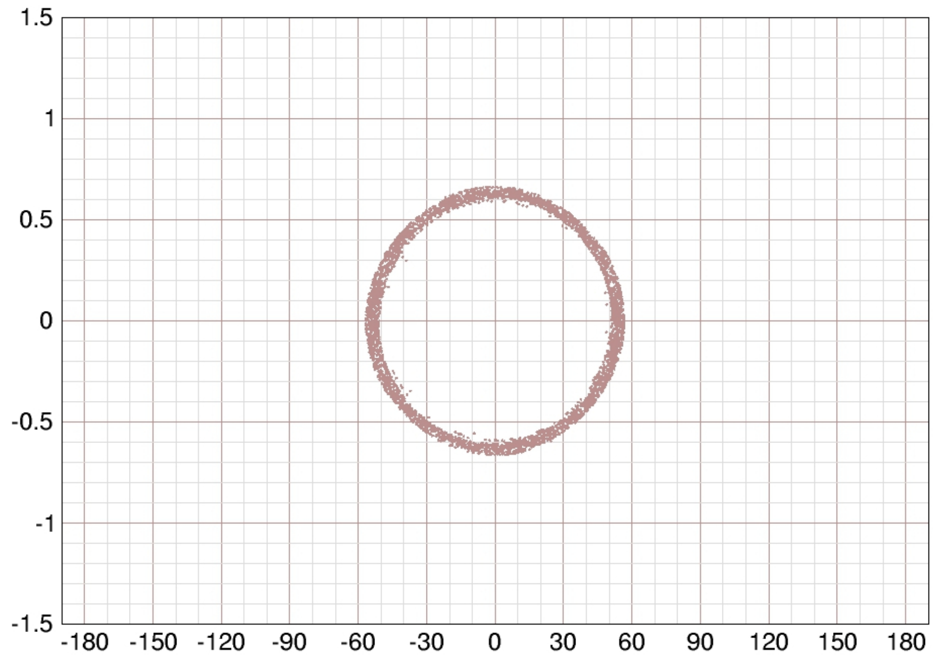
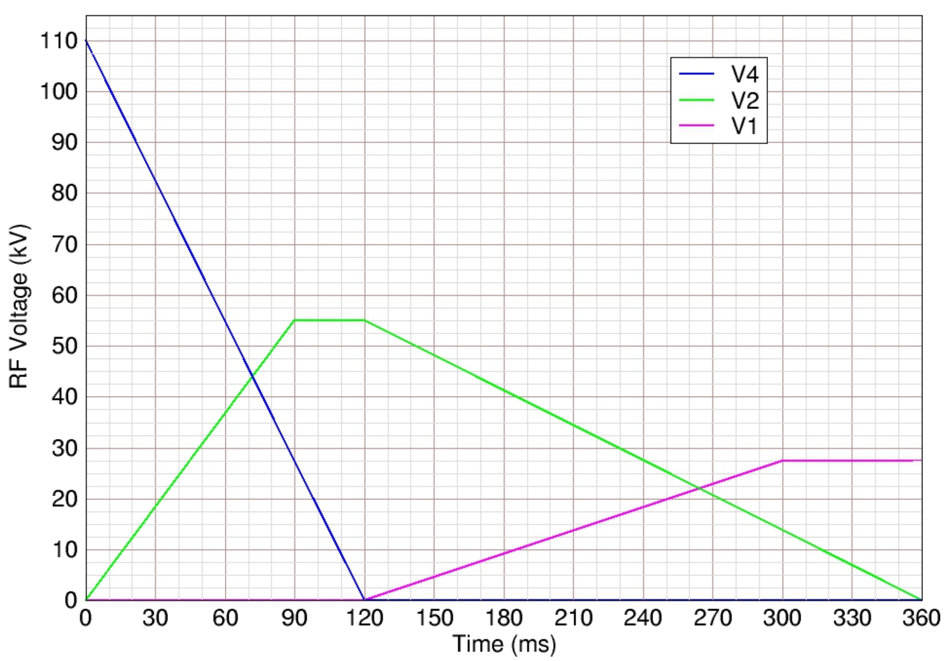


Figure 66: First layer deposited in harmonic 3 bucket.



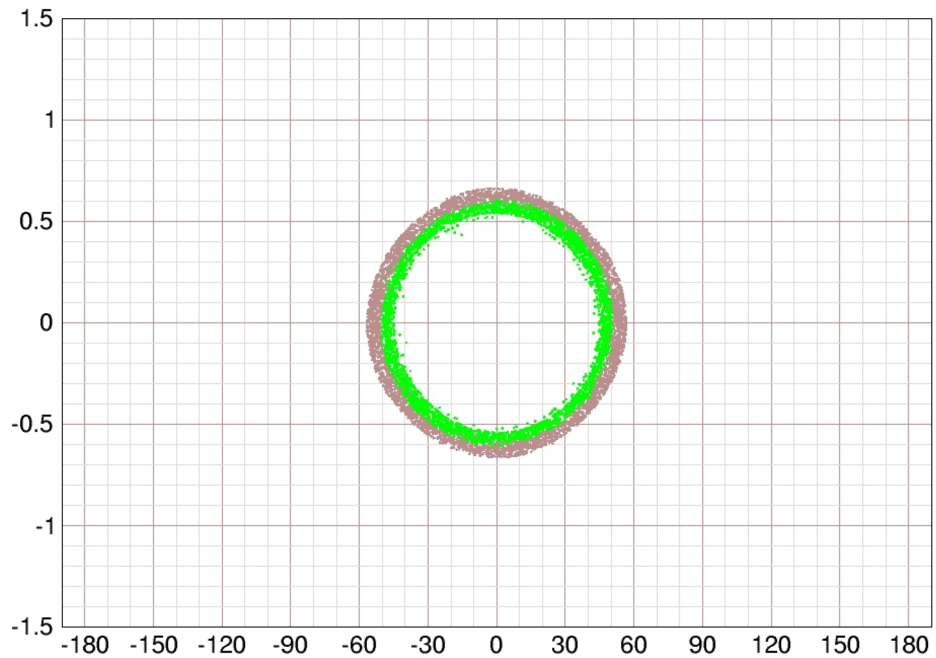
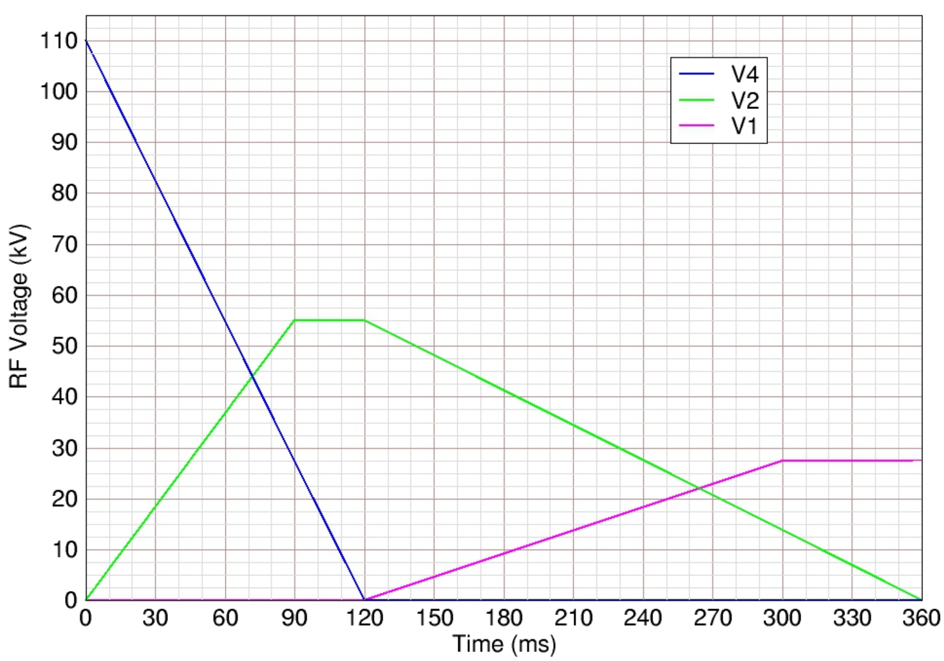


Figure 67: Seoond layer deposited in harmonic 6 bucket.



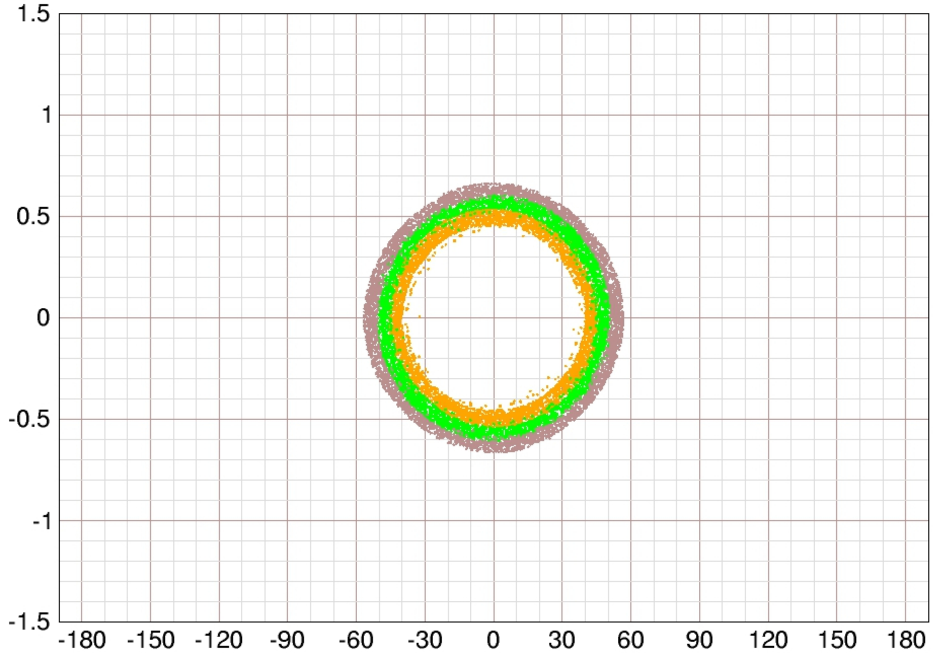
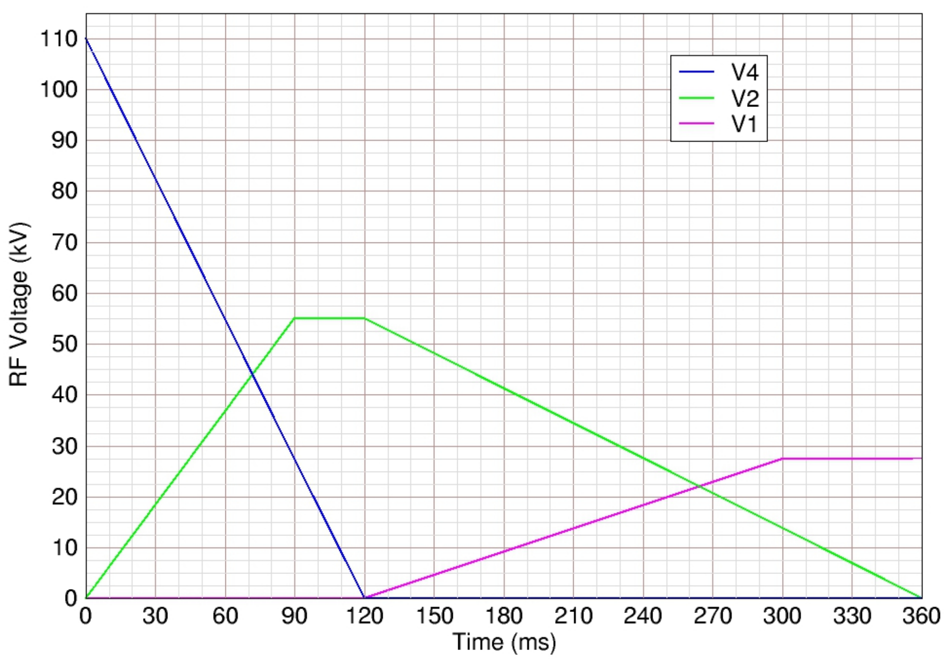


Figure 68: Third layer deposited in harmonic 3 bucket.



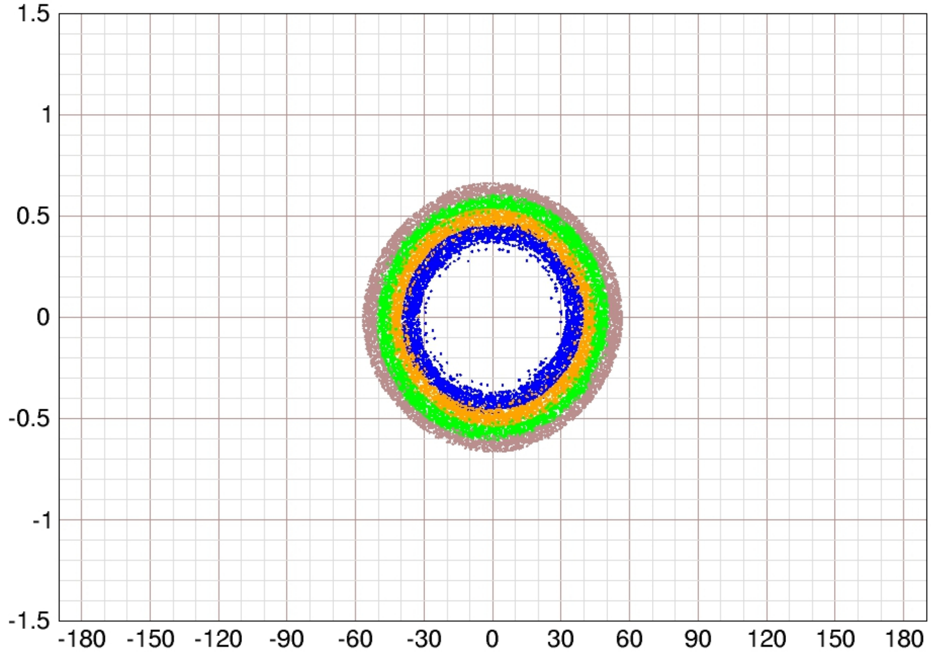
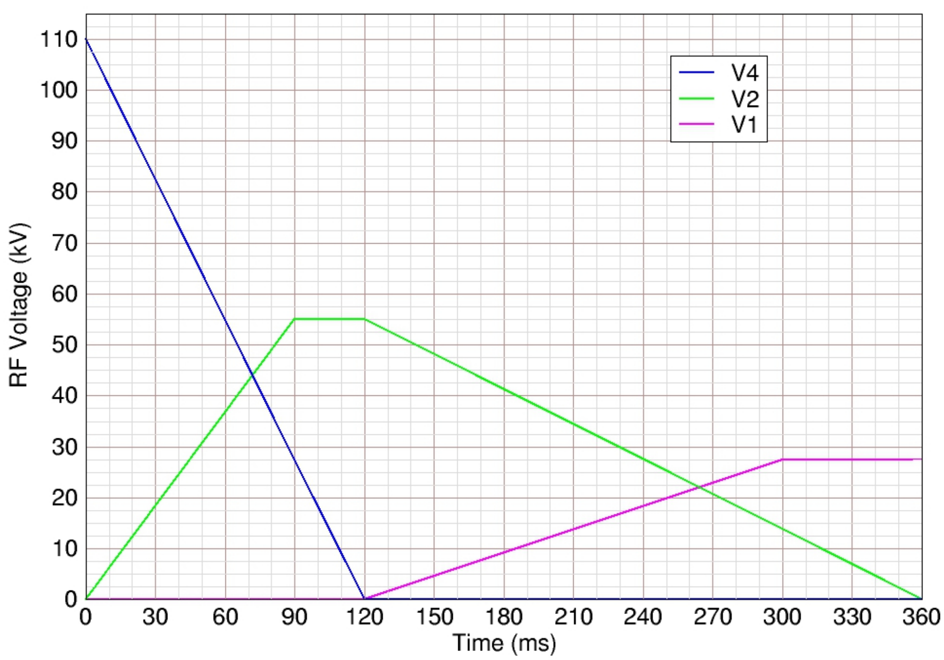


Figure 69: Fourth layer deposited in harmonic 3 bucket.



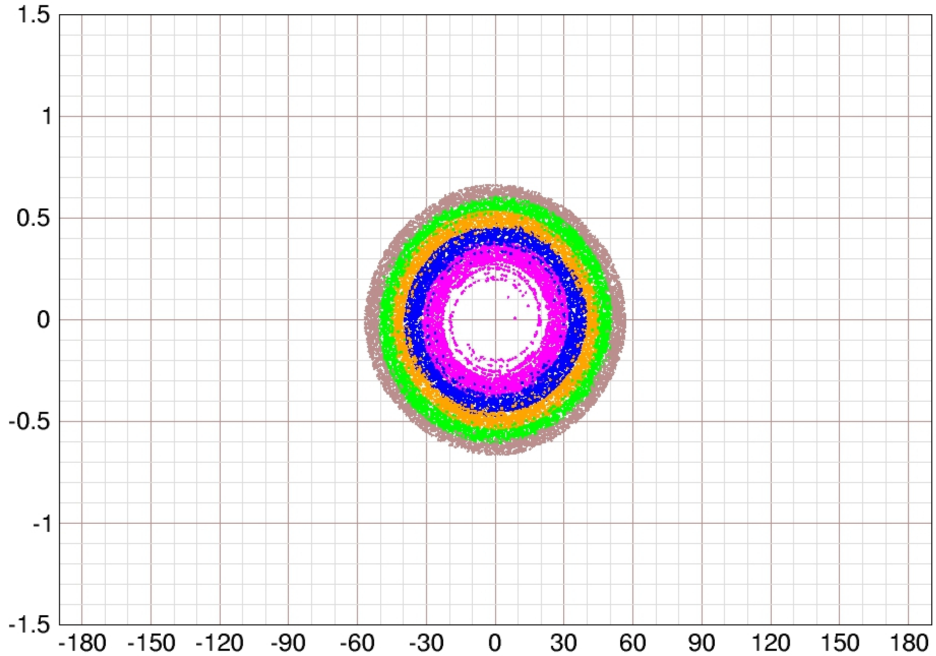
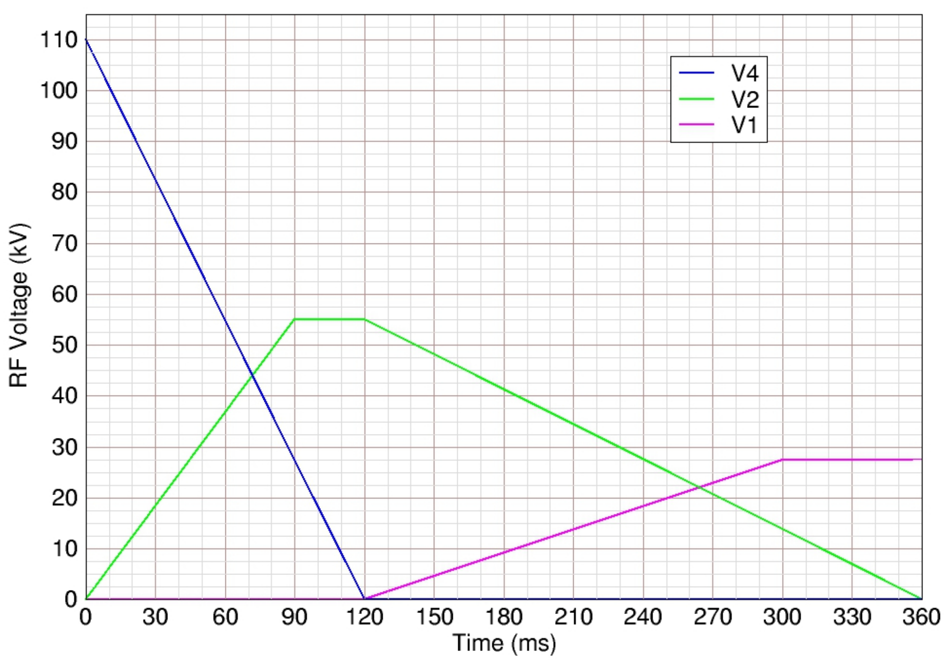


Figure 70: Fifth layer deposited in harmonic 3 bucket.



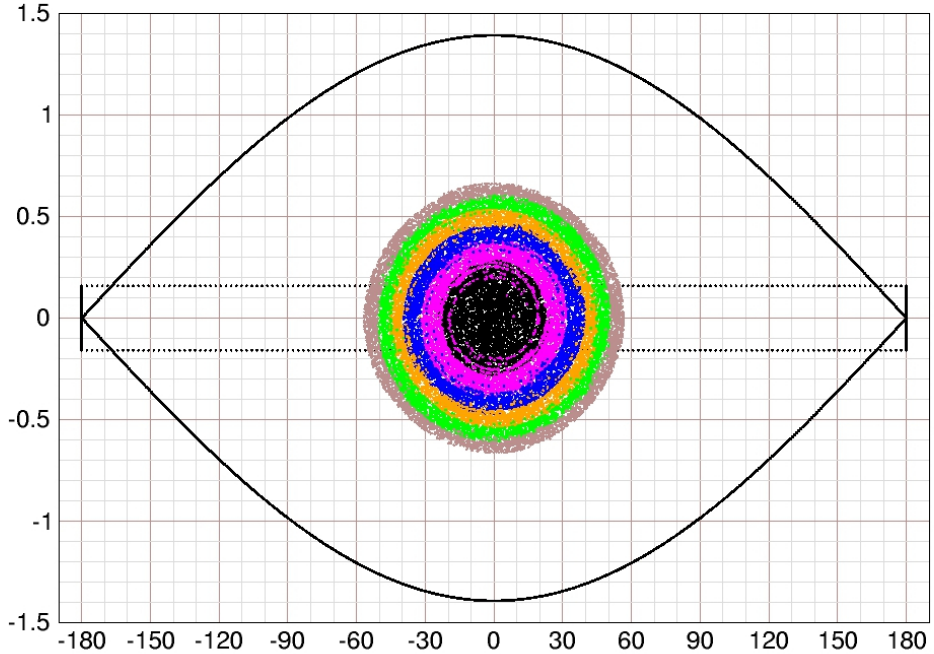
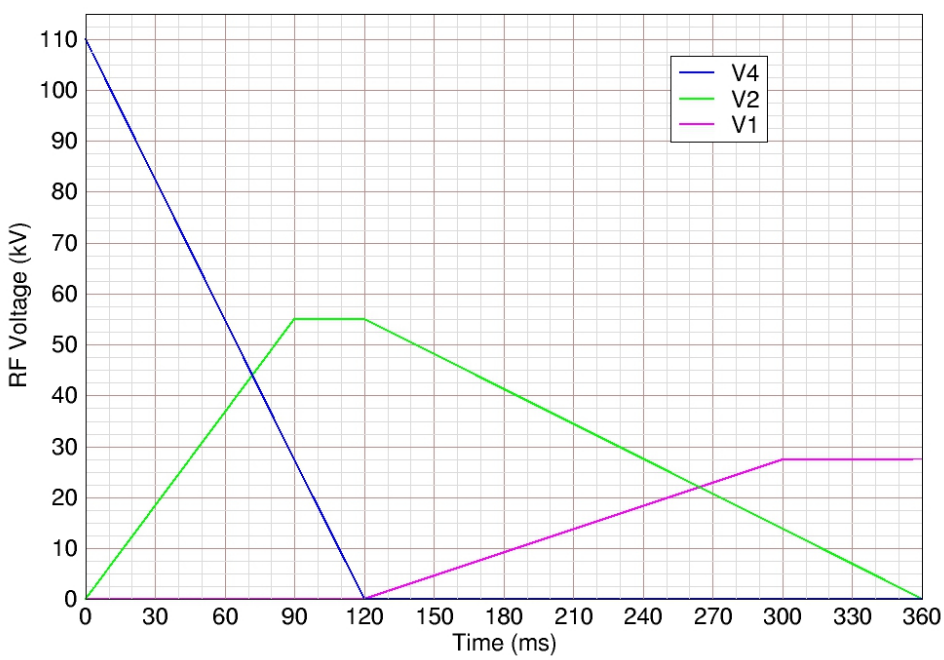


Figure 71: Final layer deposited in harmonic 3 bucket.



25 4000 ms merge in AGS at $G\gamma = 45.5$

The upper half of **Figure 72** shows the start of a simulation in which polarized proton bunches undergo two 2 to 1 merges on the AGS flattop at $G\gamma = 45.5$. The bunches initially are sitting in four adjacent harmonic 12 buckets as shown in the figure. The longitudinal emittance of each bunch is 0.5 eV-s. The bunches are formed by adiabatic capture of a uniform distribution of unbunched particles as explained in [6]. The particles in the bunches are color coded according to their initial longitudinal oscillation amplitudes. In order of increasing amplitude, the colors are black, pink, blue, orange, green, and brown.

The first 2 to 1 merge is done over a period of 4000 ms. The harmonic 12 and 6 voltages that bring about the merge are programmed in accordance with equations (198) through (204) and are shown by the blue and green lines (labeled V4 and V2 respectively) in the lower half of **Figure 72**. The harmonic 12 voltage falls from 0.05 to 0 kV in 4000 ms. The harmonic 6 voltage rises from 0 to 0.025 kV in the first 3000 ms and then is held constant from 3000 to 4000 ms. The initial area of each harmonic 12 bucket is 0.648 eV-s. The synchrotron period, 384 ms, is quite long due to the low RF voltages and high gamma on the flattop. **Figures 72** and **73** show the evolution of the bunches from 0 to 3000 ms. Here one sees that even with a merge time of 4000 ms the merge does not proceed slowly enough to keep all of the oscillation amplitude layers intact. Only the higher amplitude orange, green, and brown layers remain intact. The lower amplitude blue, pink, and black layers show progressively more filamentation as one goes from blue to black. If the black particles alone are considered, one sees a significant increase in the gross emittance of those particles. However, if the initial layers all have the same particle density then one sees that the gross emittance of the entire bunch is essentially unchanged. At 3000 ms the harmonic 12 voltage is half the harmonic 6 voltage and the bunches are contained in a single RF bucket. This is the end-of-merge point as explained in Section 12. The merge is complete when the harmonic 12 voltage has been brought down to zero and the merged bunch is sitting in a harmonic 6 bucket as shown in **Figure 74**.

The second 2 to 1 merge is done over a period of 4000 ms starting at 4000 ms as shown in the lower half of **Figure 75**. Here the harmonic 6 voltage (labeled V2) falls from 0.025 to 0 kV in 4000 ms. The harmonic 3 voltage (labeled V1) rises from 0 to 0.0125 kV in 3000 ms and then is held constant for 1000 ms, all in accordance with equations (198) through

(204). **Figures 75** and **76** show the merged bunch at times 7000 and 8000 ms. Here one sees considerably more mixing of the black, pink, and blue oscillation amplitude layers. This is because the merge should have been done with a total time twice that of the first 2 to 1 merge as explained in Section 17.

Figures 77 through **79** give another view of the first 2 to 1 merge, this time with the bunch particles color coded according to the initial harmonic 12 bucket they occupy. Here one sees tightly wound colored layers indicating a merge that leaves the gross longitudinal emittance essentially unchanged.

In **Figures 80** through **82** the merge times are the same as those in the previous three figures, but the initial harmonic 12 voltage has been raised by a factor of 100 from 0.05 to 5.0 kV. This means that the synchrotron period in the 5.0 kV bucket will be 10 times shorter, and, as explained in Section 22 the mixing time will be 10 times shorter. The expectation is that the effects of these two changes will cancel one another. That this is indeed the case can be seen by comparing **Figures 82** and **79**, where one sees similarly wound colored layers.

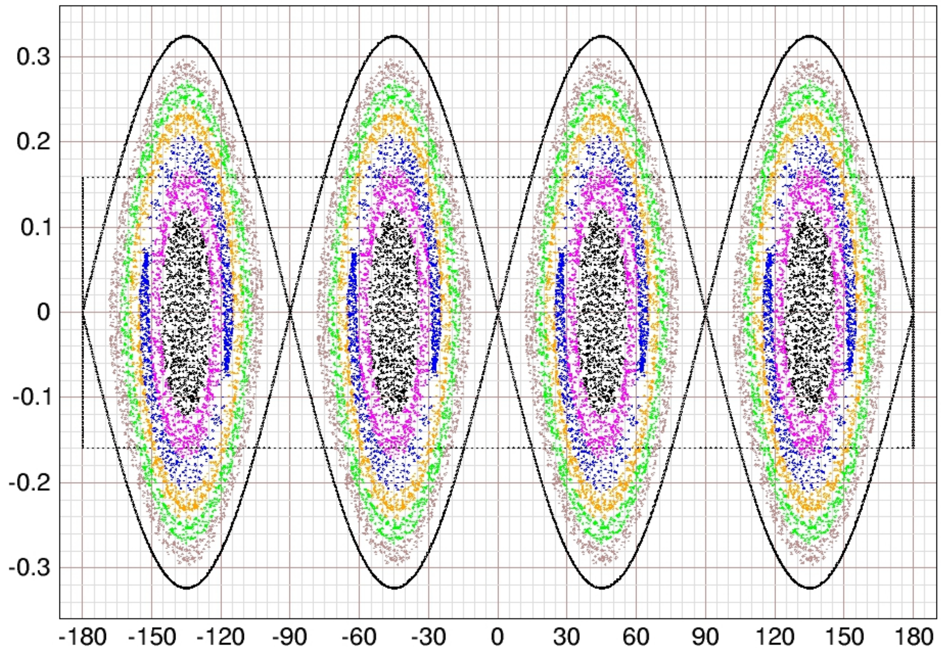
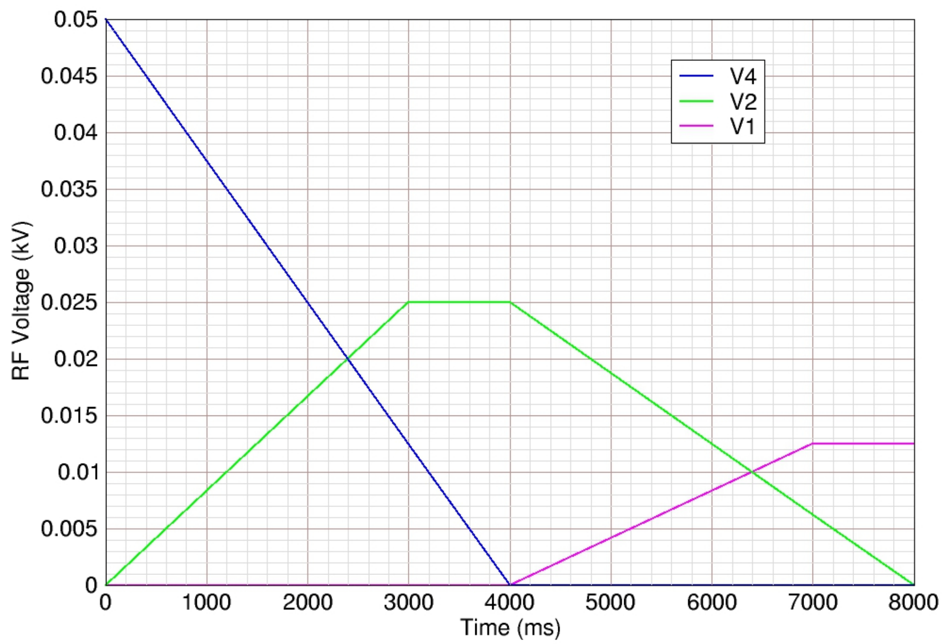


Figure 72: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



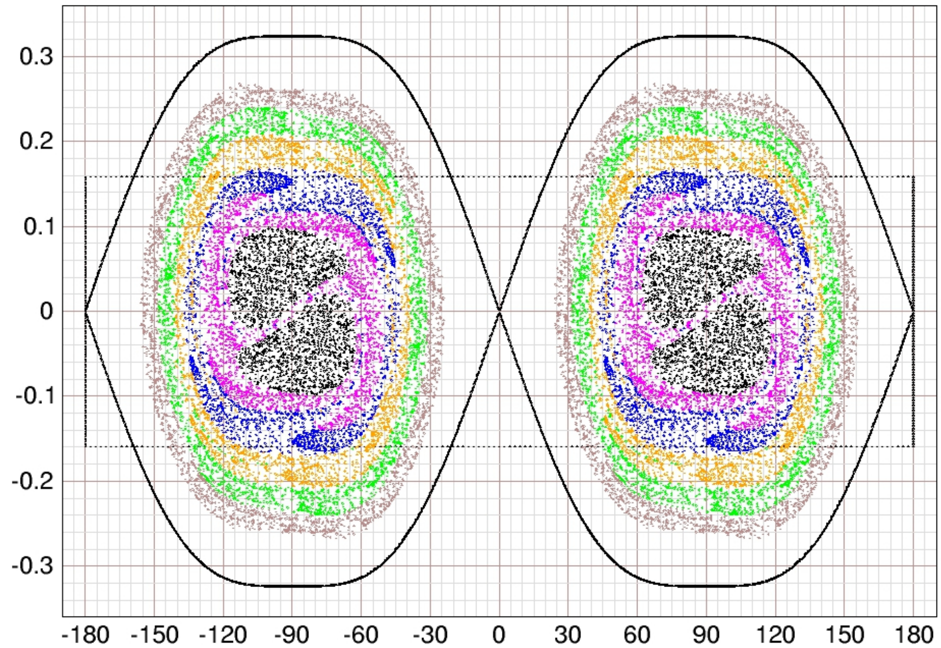
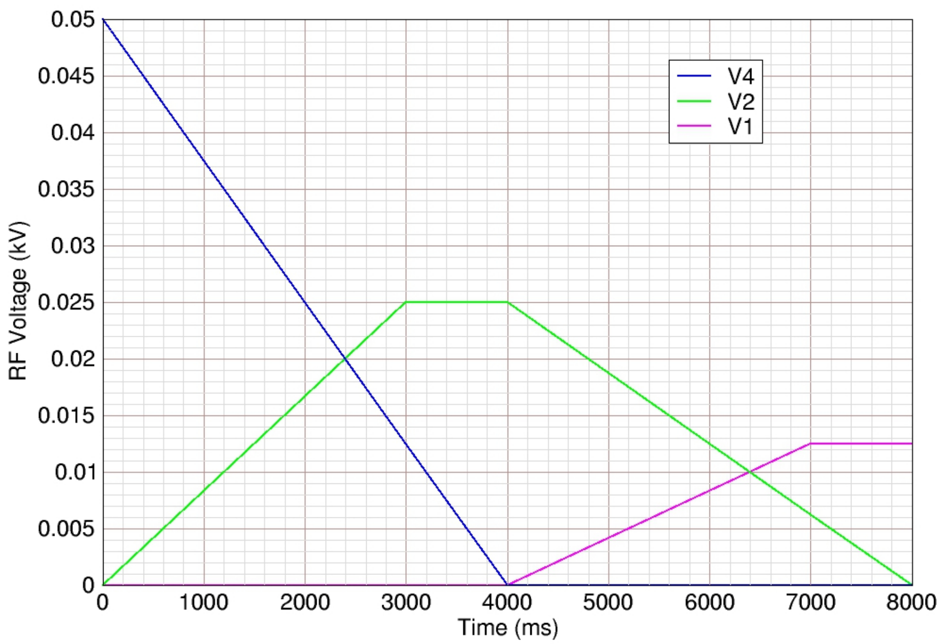


Figure 73: 4 to 2 to 1 merge at time $\underline{t = 3000\text{ ms}}$.



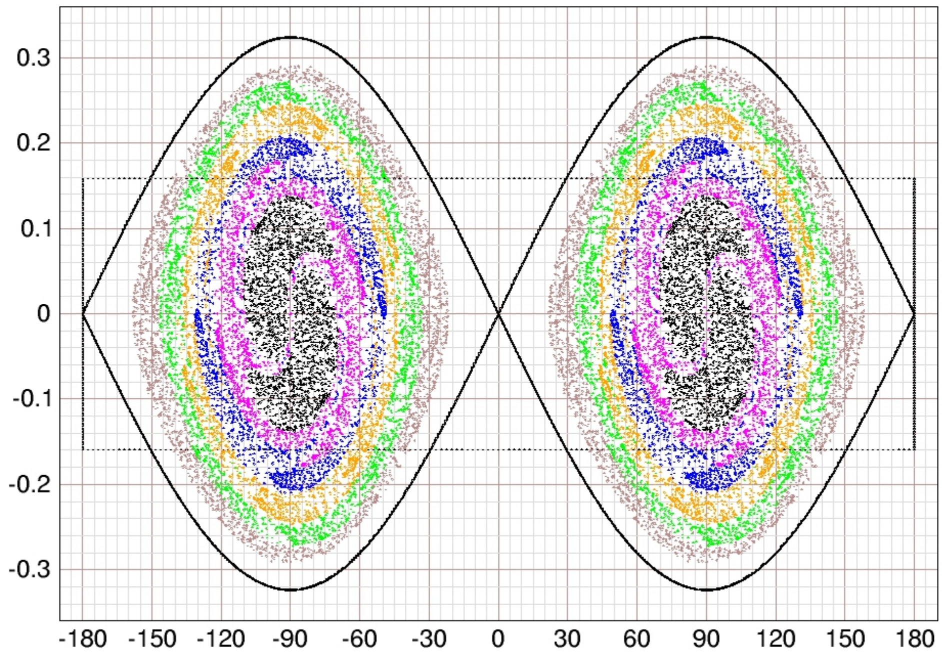
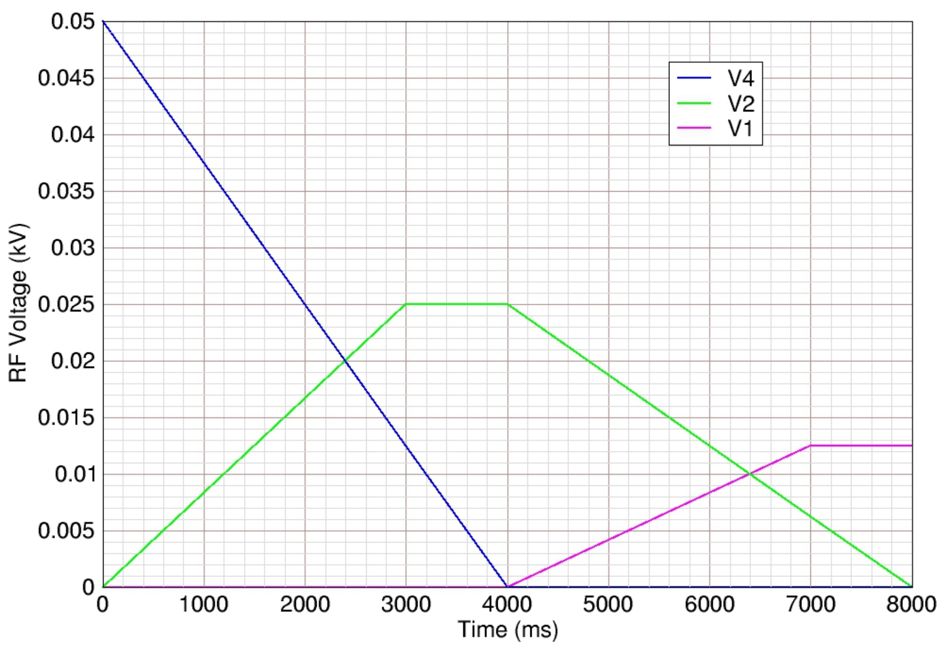


Figure 74: 4 to 2 to 1 merge at time $\underline{t = 4000 \text{ ms}}$.



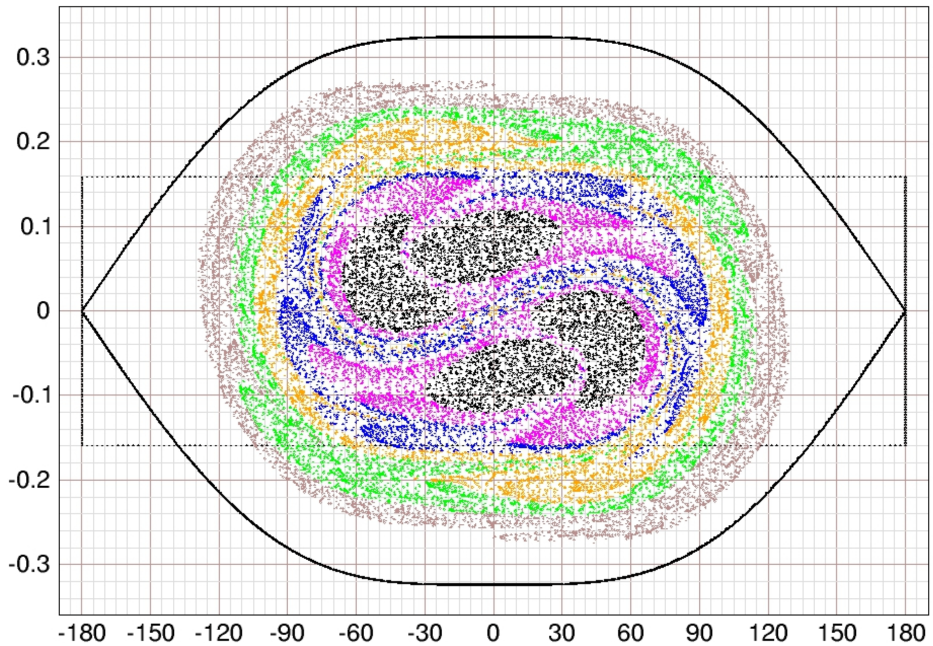
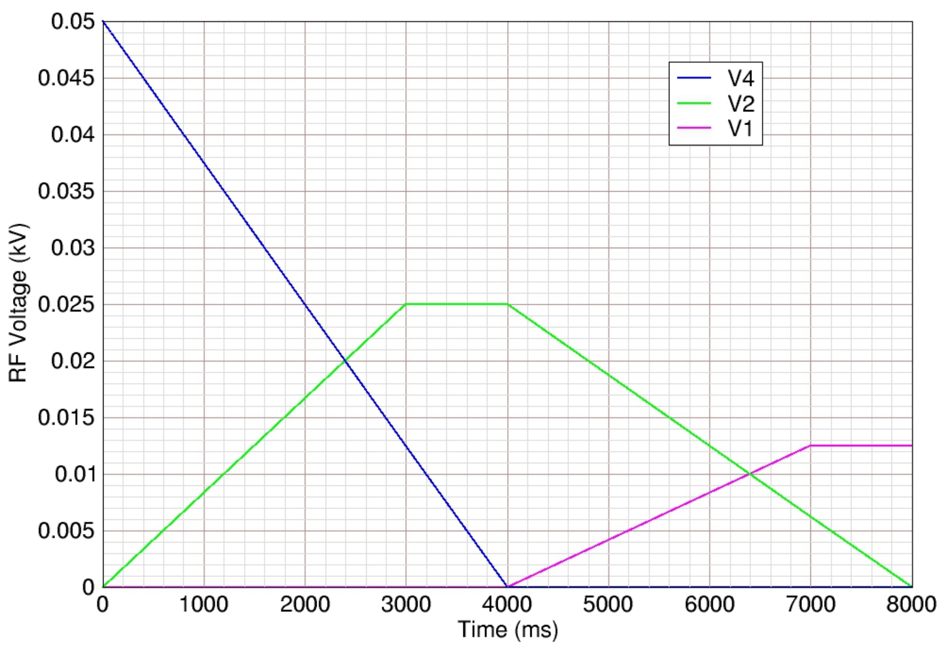


Figure 75: 4 to 2 to 1 merge at time $\underline{t = 7000 \text{ ms}}$.



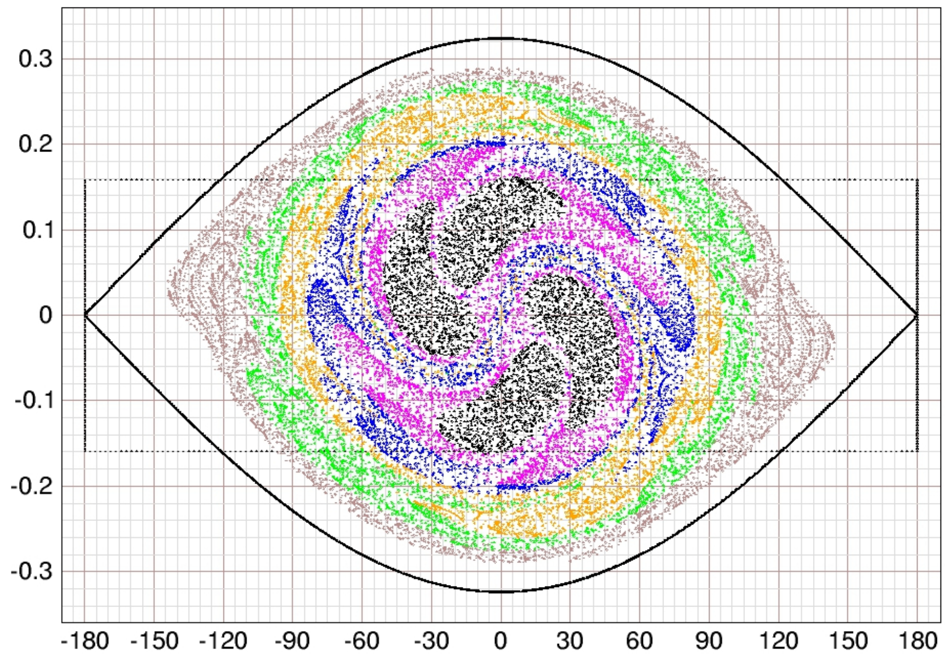
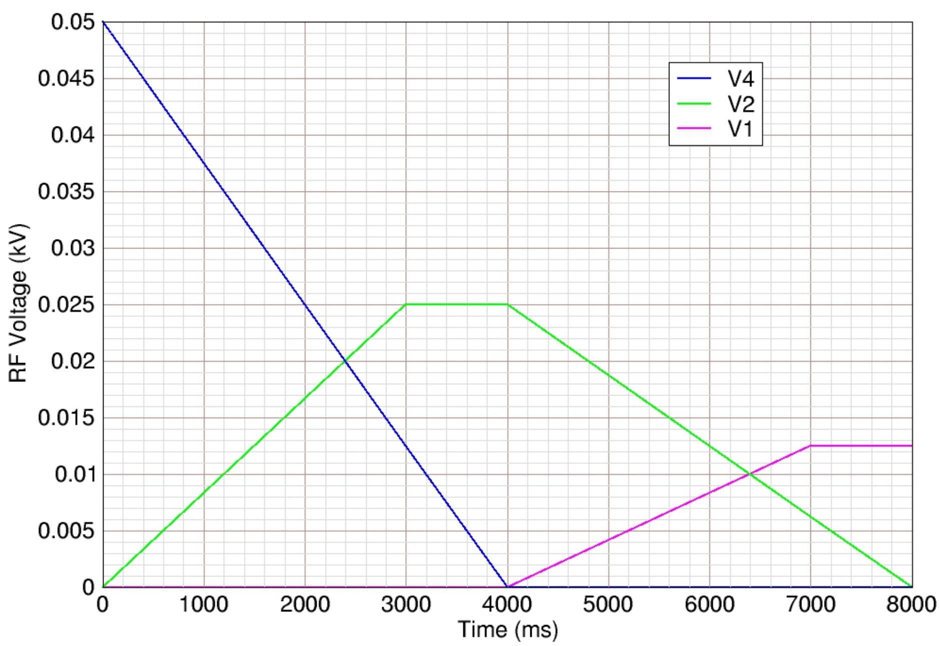


Figure 76: 4 to 2 to 1 merge at time $\underline{t = 8000 \text{ ms}}$.



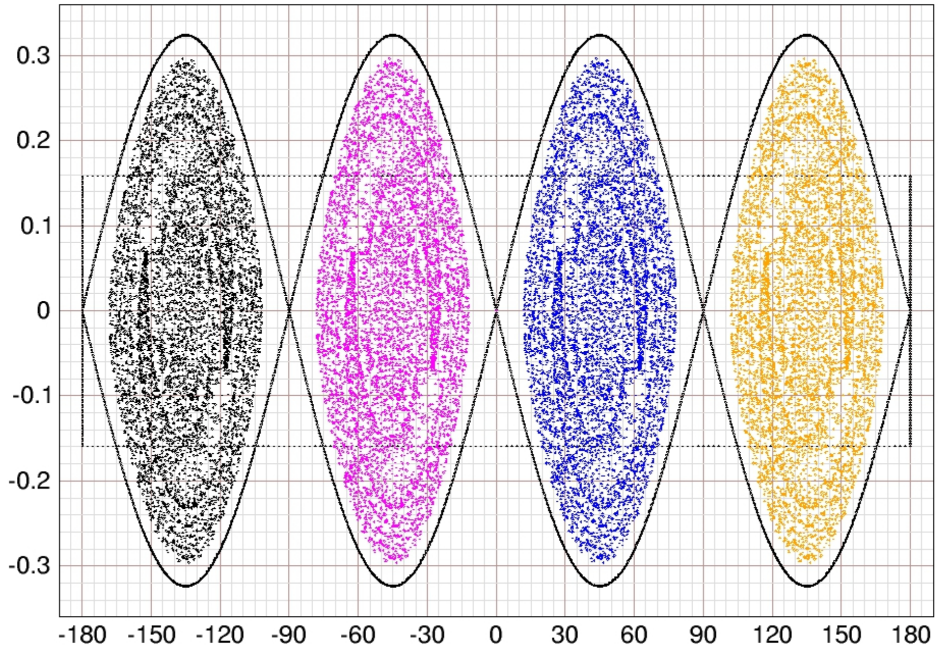
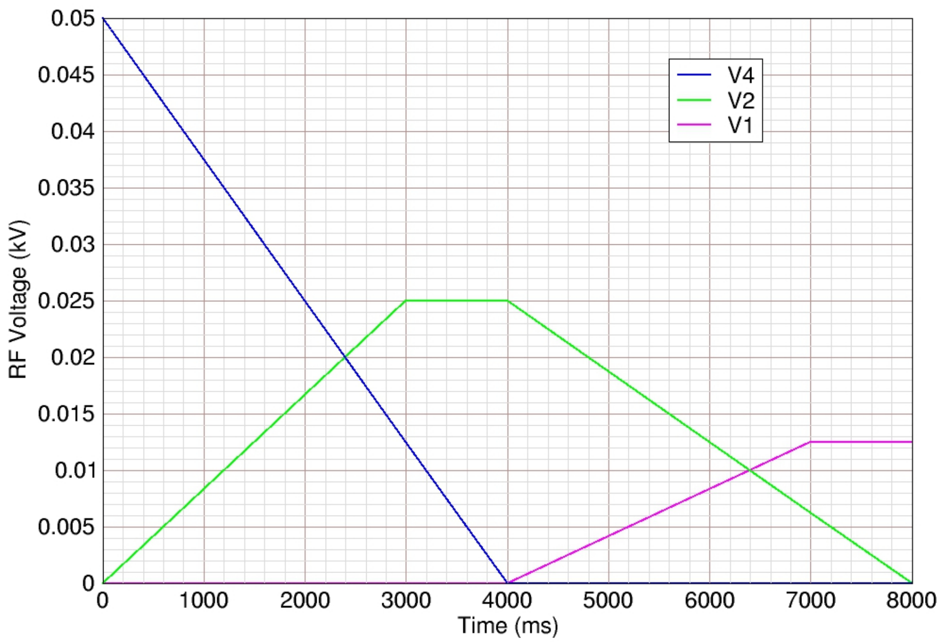


Figure 77: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



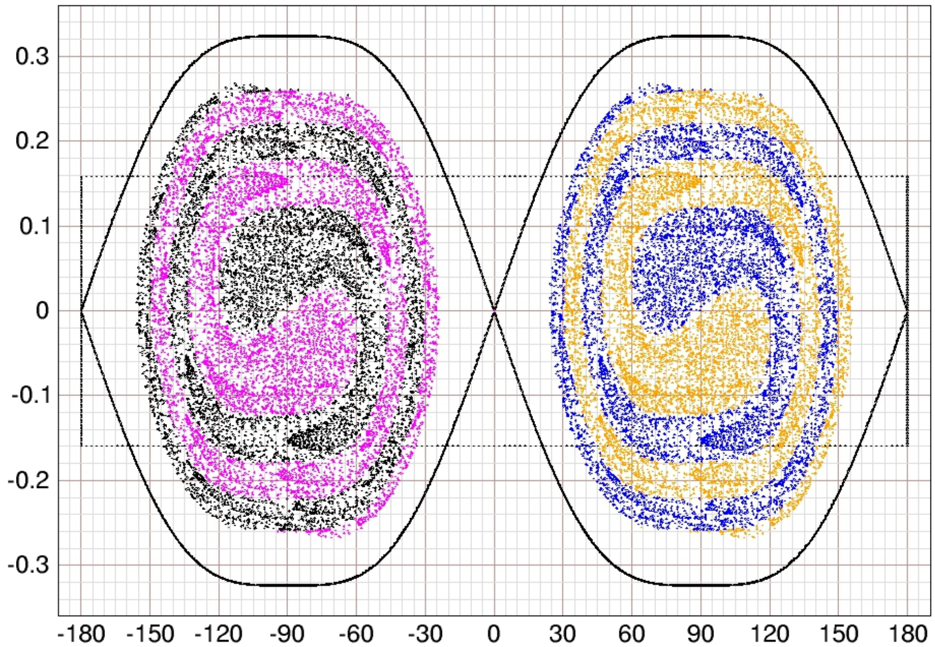
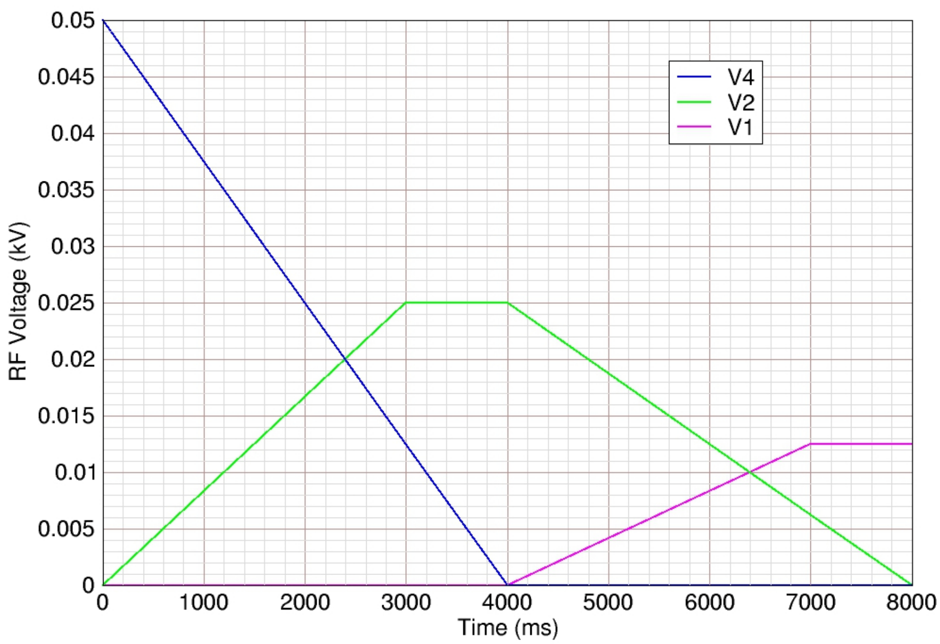


Figure 78: 4 to 2 to 1 merge at time $\underline{t = 3000 \text{ ms}}$.



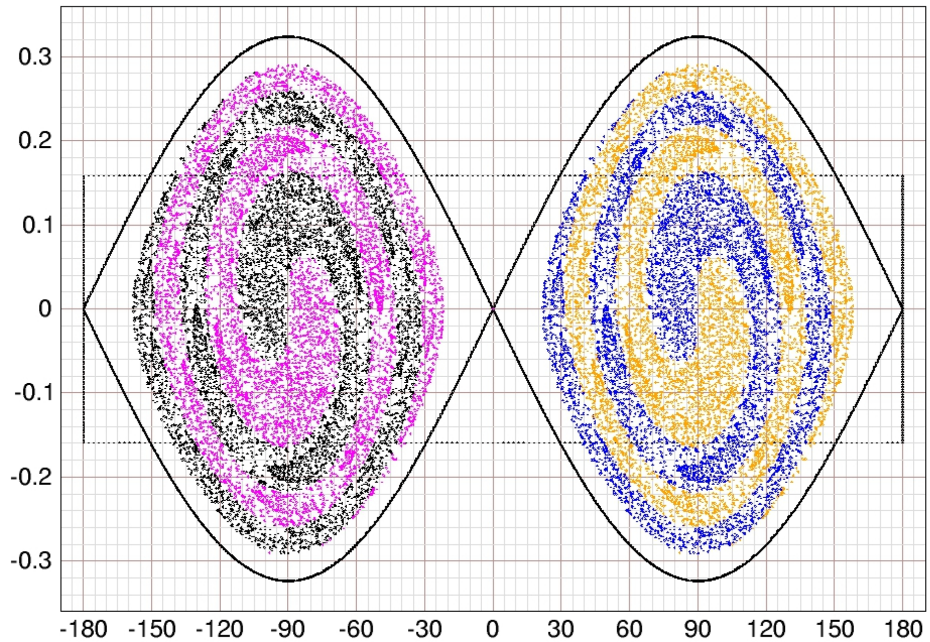
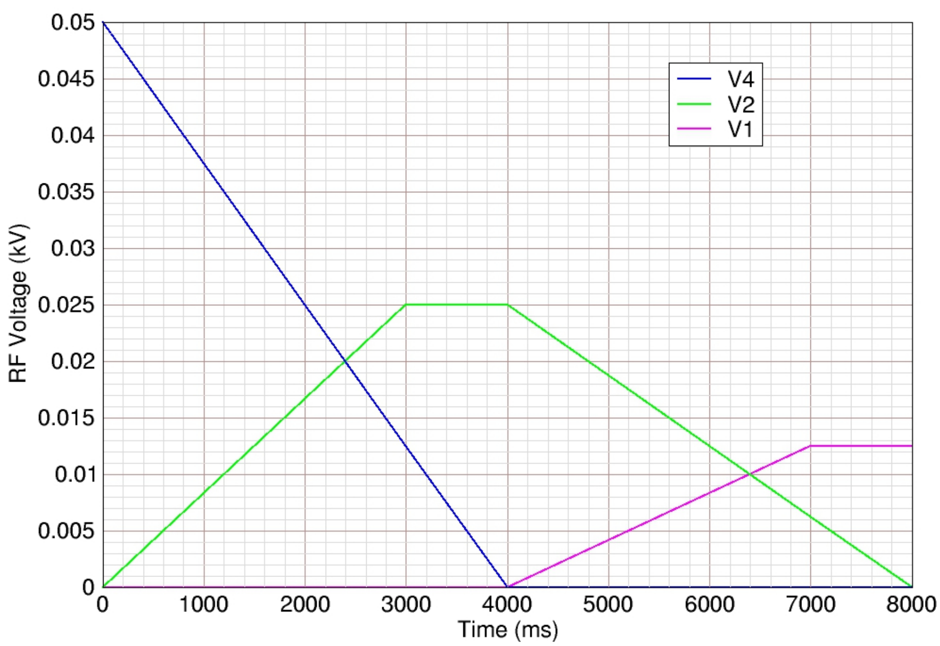


Figure 79: 4 to 2 to 1 merge at time $\underline{t = 4000 \text{ ms}}$.



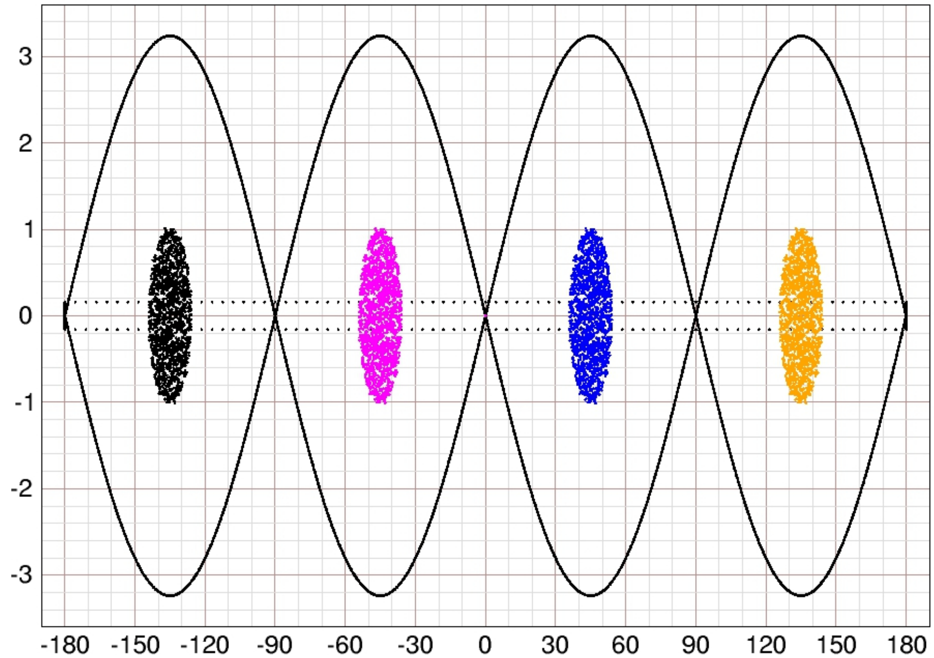
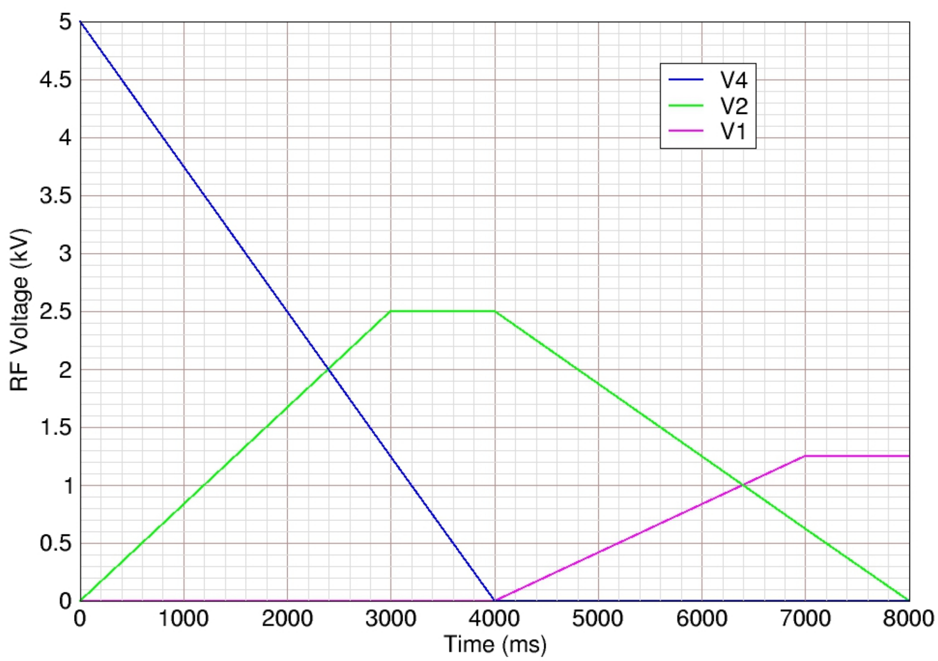


Figure 80: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



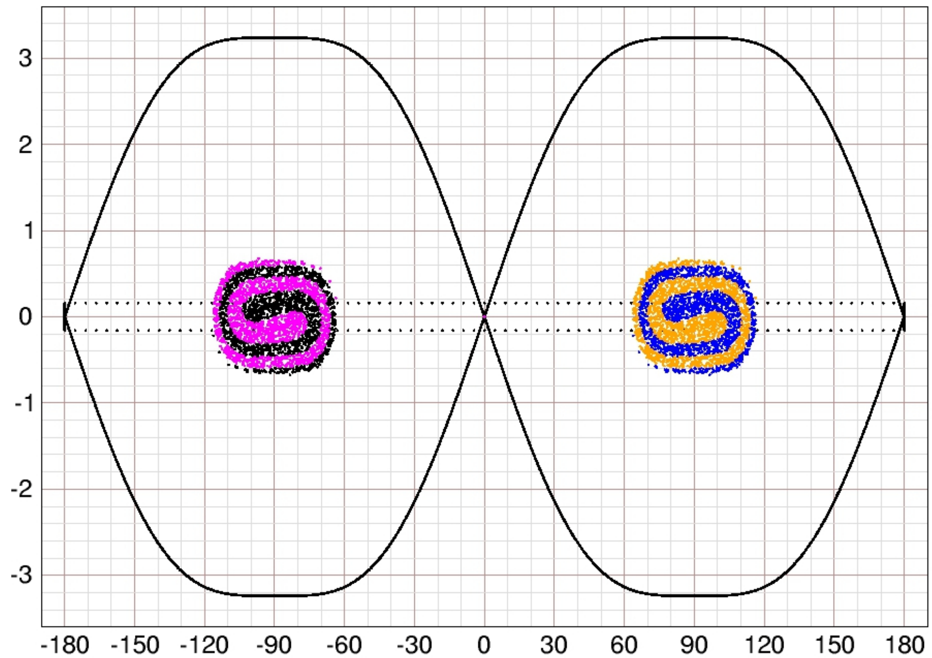
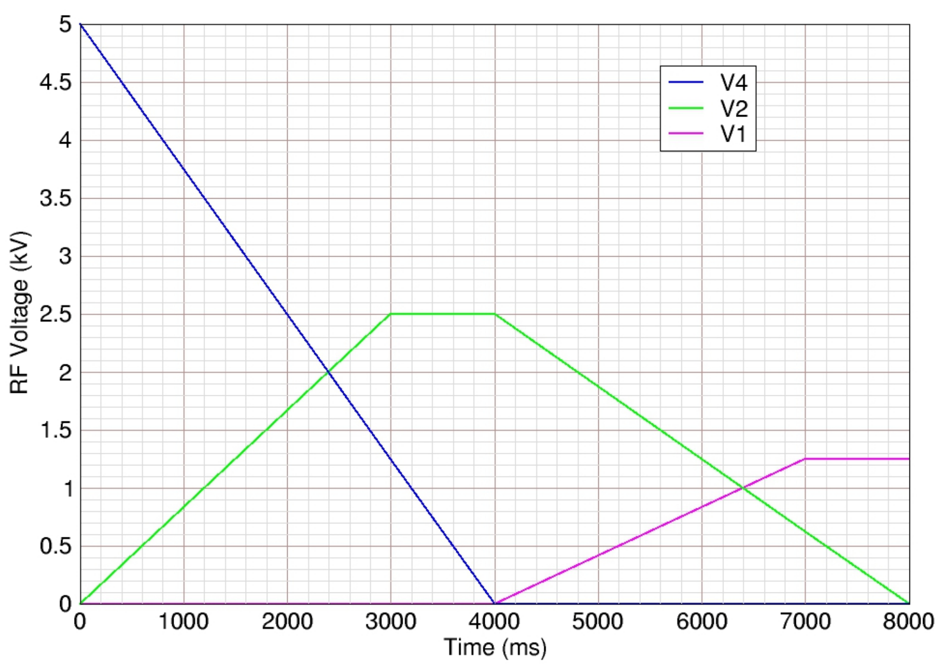


Figure 81: 4 to 2 to 1 merge at time $\underline{t = 3000\text{ ms}}$.



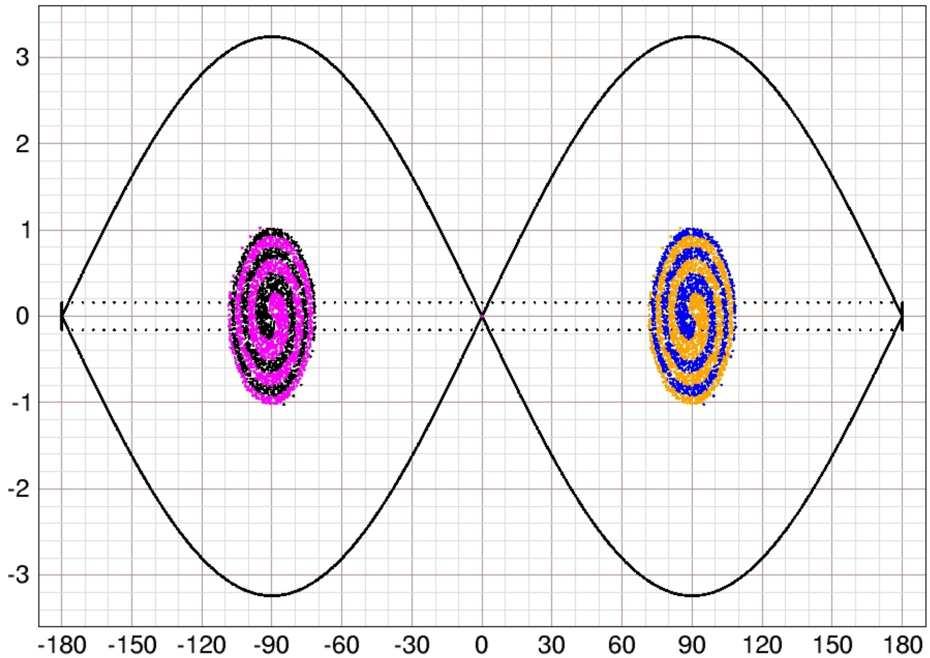
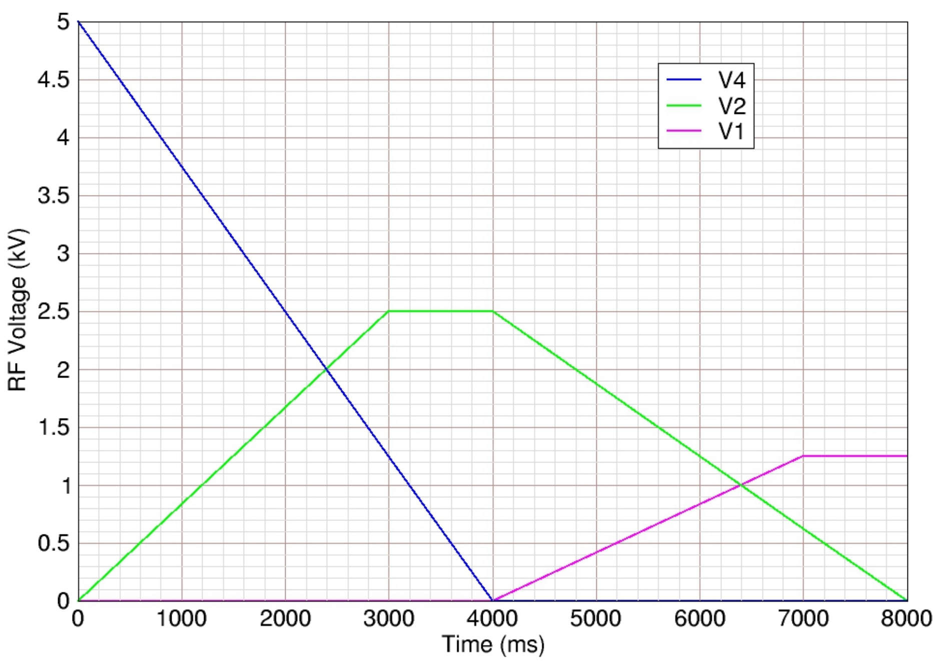


Figure 82: 4 to 2 to 1 merge at time $\underline{t = 4000 \text{ ms}}$.



26 2000 ms merge in AGS at $G\gamma = 45.5$

As in the previous section, the upper half of **Figure 83** shows the start of a simulation in which polarized proton bunches undergo two 2 to 1 merges on the AGS flattop at $G\gamma = 45.5$. The bunches initially are sitting in four adjacent harmonic 12 buckets as shown in the figure. The area of each bucket is 0.648 eV-s. The longitudinal emittance of each bunch is 0.5 eV-s. The bunches are formed by adiabatic capture of a uniform distribution of unbunched particles as explained in [6]. The particles in the bunches are color coded according to the initial harmonic 12 bucket they occupy or according to the initial longitudinal oscillation amplitude. In order of increasing amplitude, the colors are black, pink, blue, orange, green, and brown.

The first 2 to 1 merge is done over a period of 2000 ms. **Figures 83** through **86** show the evolution of the merge. One sees that with the shorter merge time the oscillation amplitude layers are more filamented than those in the previous section. However, comparing **Figures 86** and **79** one sees that gross emittance of the merged bunch is only slightly larger.

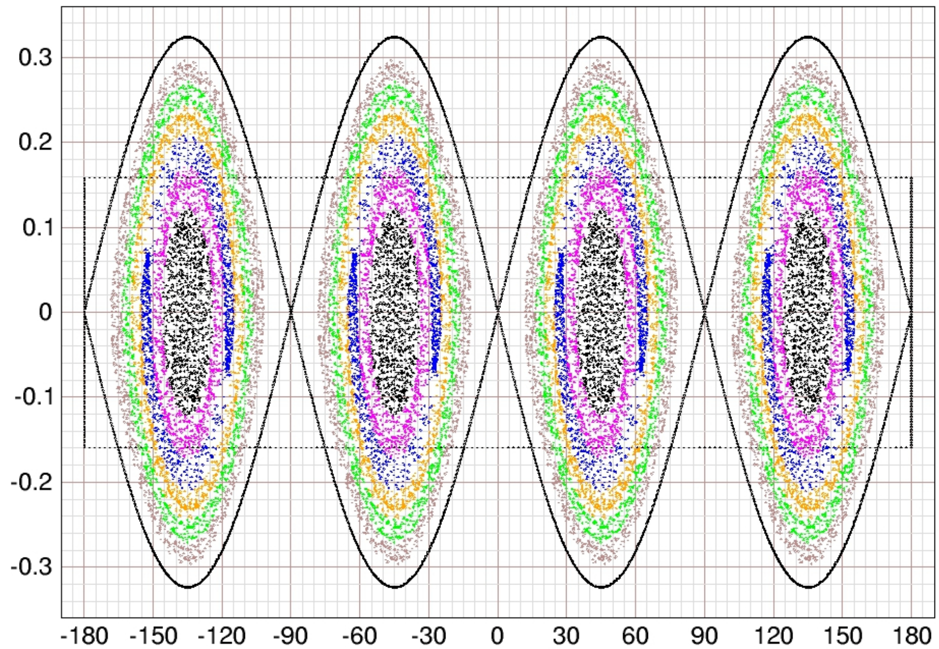
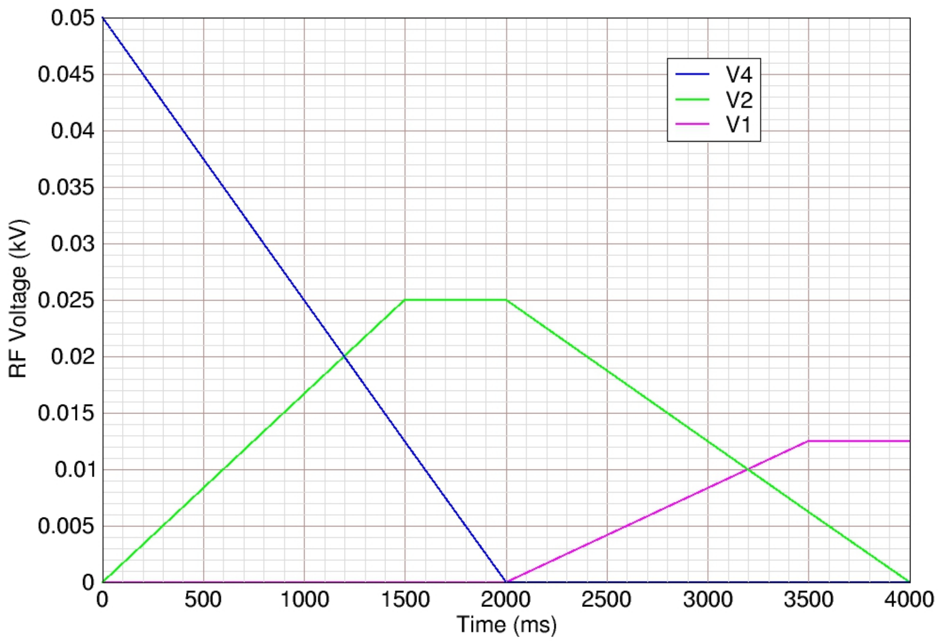


Figure 83: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



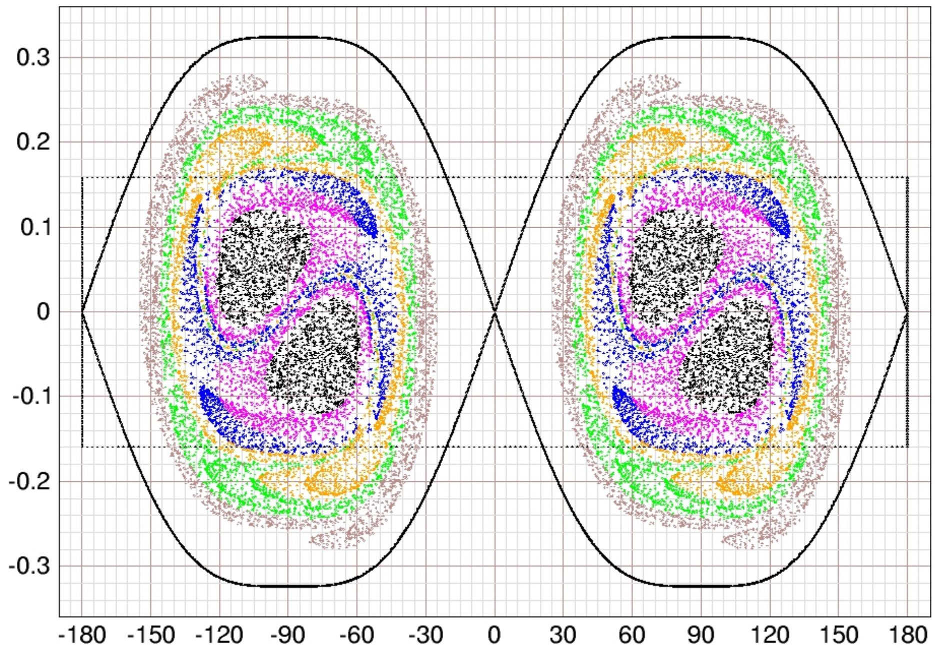
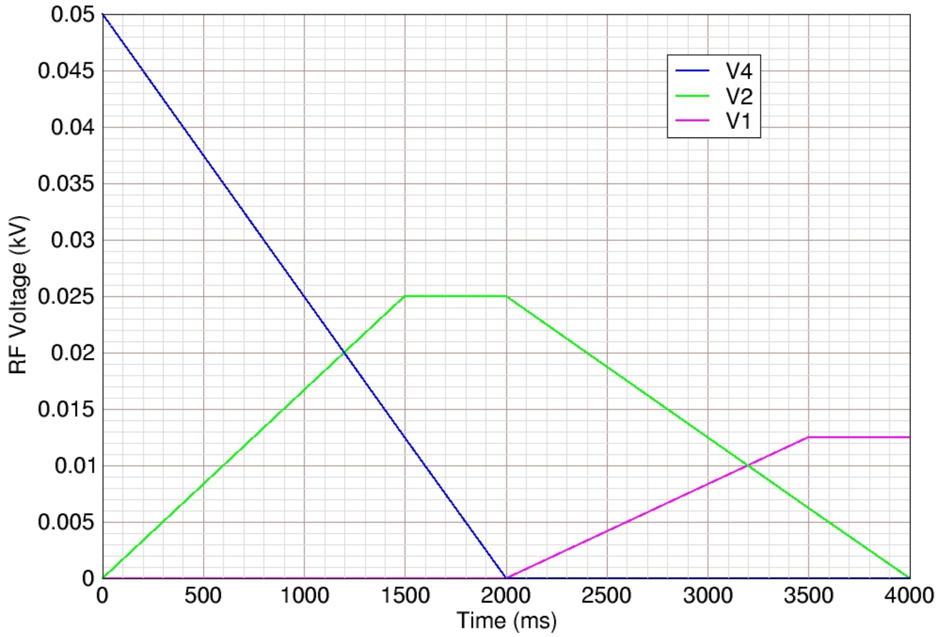


Figure 84: 4 to 2 to 1 merge at time $\underline{t = 1500 \text{ ms}}$.



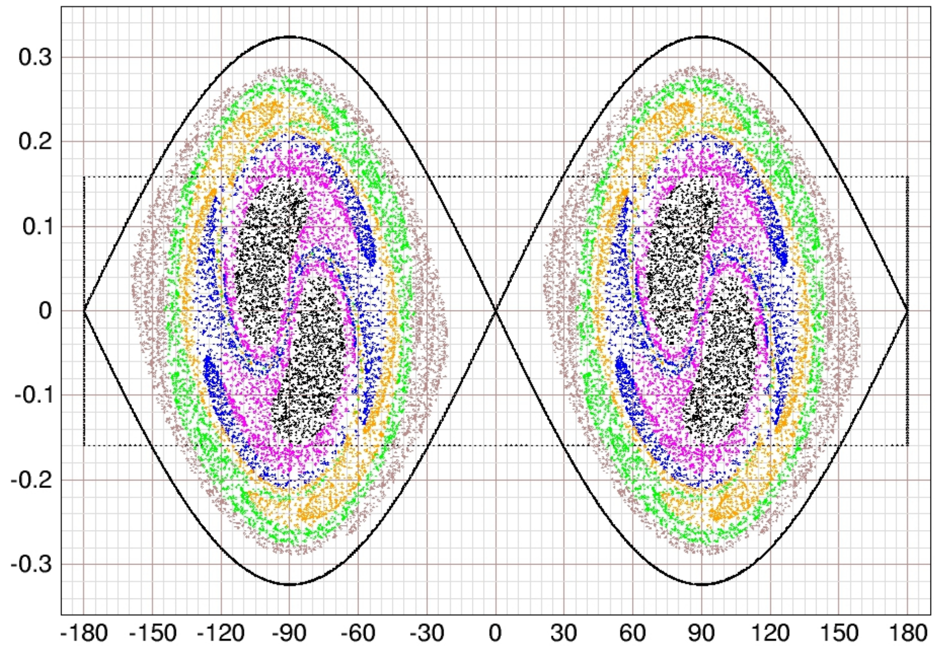
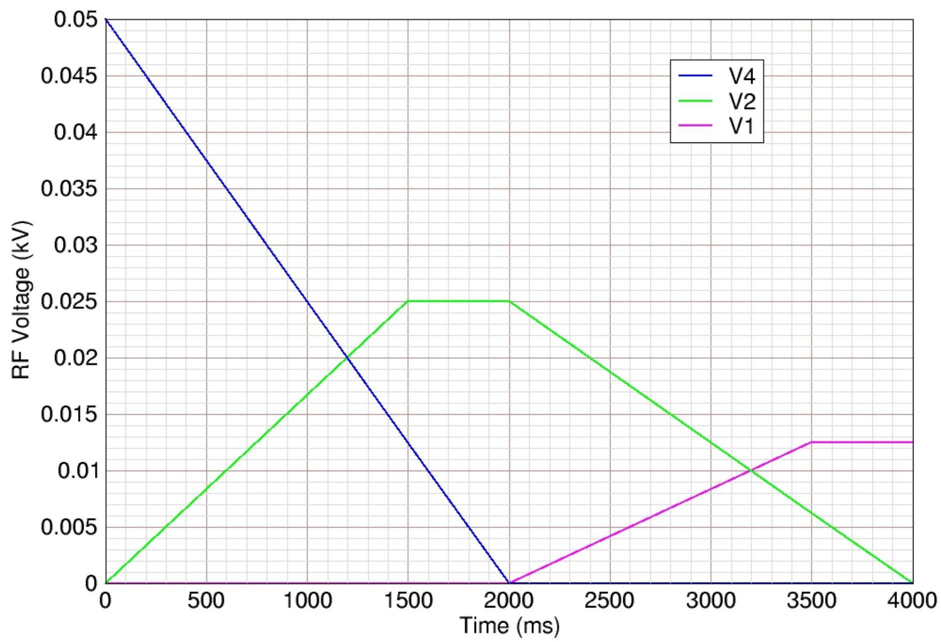


Figure 85: 4 to 2 to 1 merge at time t = 2000 ms.



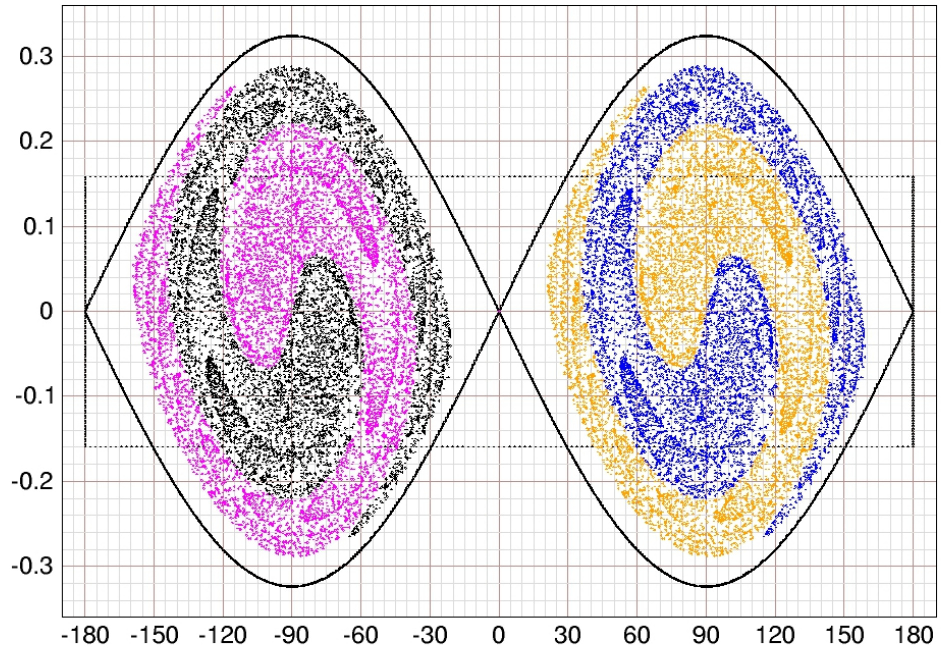
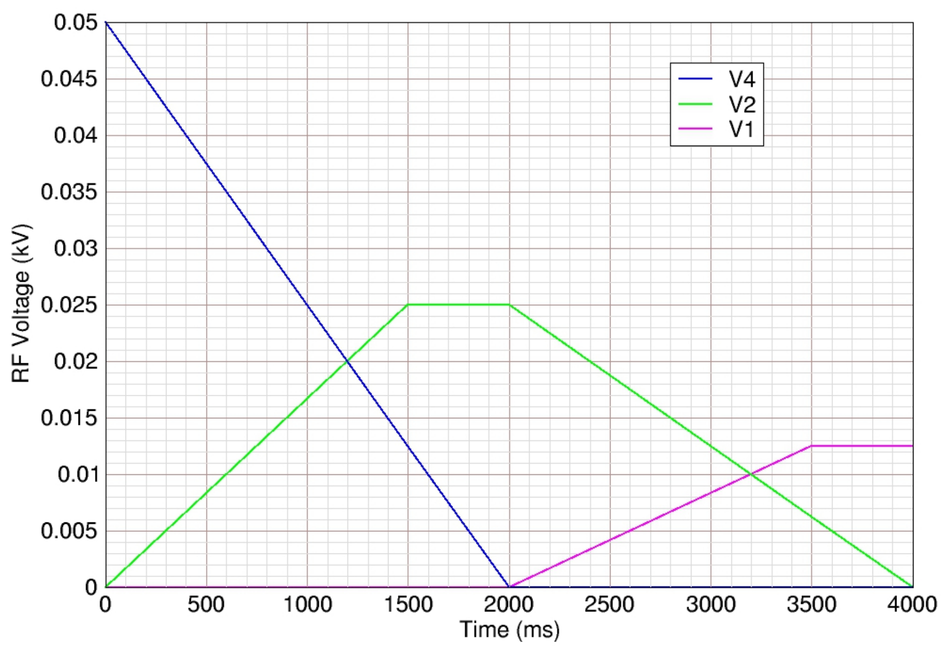


Figure 86: 4 to 2 to 1 merge at time $\underline{t = 2000 \text{ ms}}$.



27 1000 ms merge in AGS at $G\gamma = 45.5$

As in the previous sections, the upper half of **Figure 87** shows the start of a simulation in which polarized proton bunches undergo two 2 to 1 merges on the AGS flattop at $G\gamma = 45.5$. The bunches initially are sitting in four adjacent harmonic 12 buckets as shown in the figure. The area of each bucket is 0.648 eV-s. The longitudinal emittance of each bunch is 0.5 eV-s. The bunches are formed by adiabatic capture of a uniform distribution of unbunched particles as explained in [6]. The particles in the bunches are color coded according to the initial harmonic 12 bucket they occupy or according to the initial longitudinal oscillation amplitude. In order of increasing amplitude, the colors are black, pink, blue, orange, green, and brown.

The first 2 to 1 merge is done over a period of 1000 ms. **Figures 87** through **99** show the evolution of the merge. Here one sees further disruption and filamentation of the oscillation amplitude layers. One also sees that the bunches come right up against the border of the harmonic 6 buckets in **Figures 98** and **99**, showing that as time progresses the bunches will undergo further filamentation and essentially fill the buckets. These buckets have twice the area of the initial harmonic 12 buckets, so the gross emittance of the merged bunches will be $2 \times 0.648 = 1.296$ eV-s. Since the initial emittance of the 2 bunches to be merged is $2 \times 0.5 = 1.0$ eV-s we see that the emittance growth factor is 1.296.

In **Figures 100** through **102** the initial harmonic 12 voltage has been raised from 0.05 to 0.5 kV. In **Figures 103** through **106** the voltage has been raised to 5.0 kV. Finally, in **Figures 107** through **109** the voltage has been raised to 125 kV, a factor of 2500 greater than 0.05 kV.

Comparing **Figures 99**, **102**, **105**, and **109** one sees similarly wound colored layers for the entire range of initial harmonic 12 voltages. The same is true for **Figures 98** and **106** where the particles are color coded according to their initial longitudinal oscillation amplitude. These results are consistent with the discussion in Section 22.

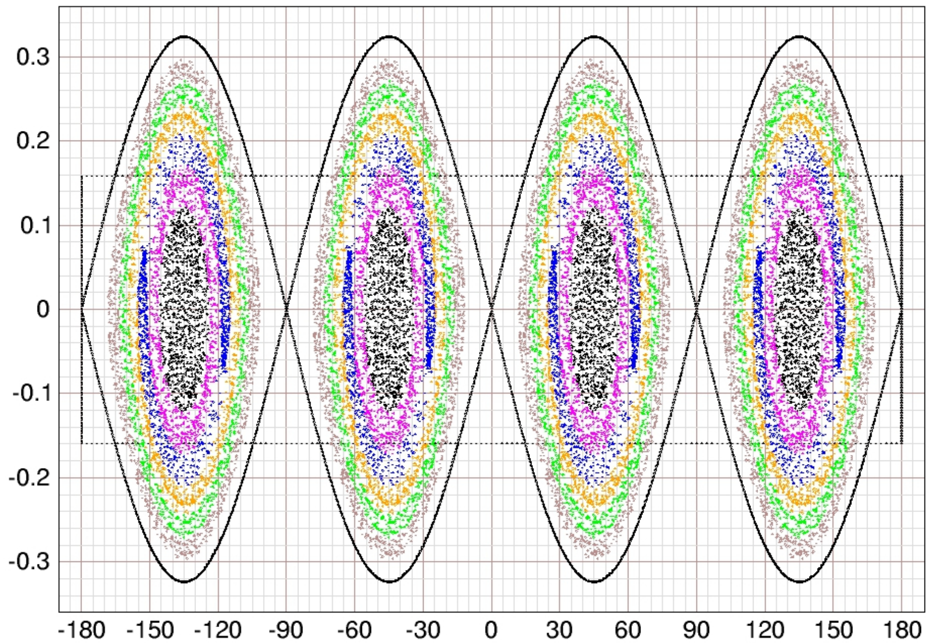
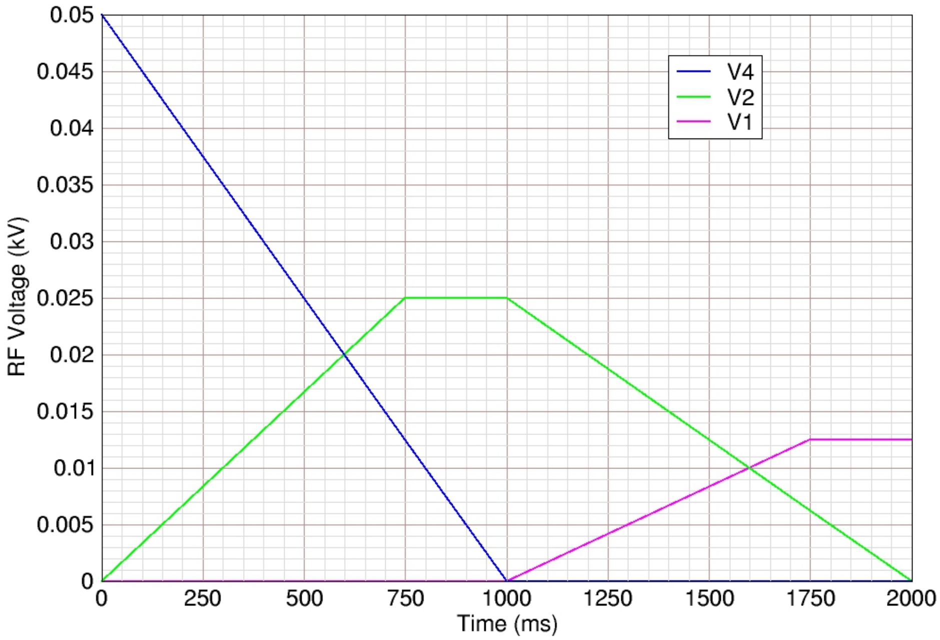


Figure 87: 4 to 2 to 1 merge at time $\underline{t = 0\ \text{ms}}$.



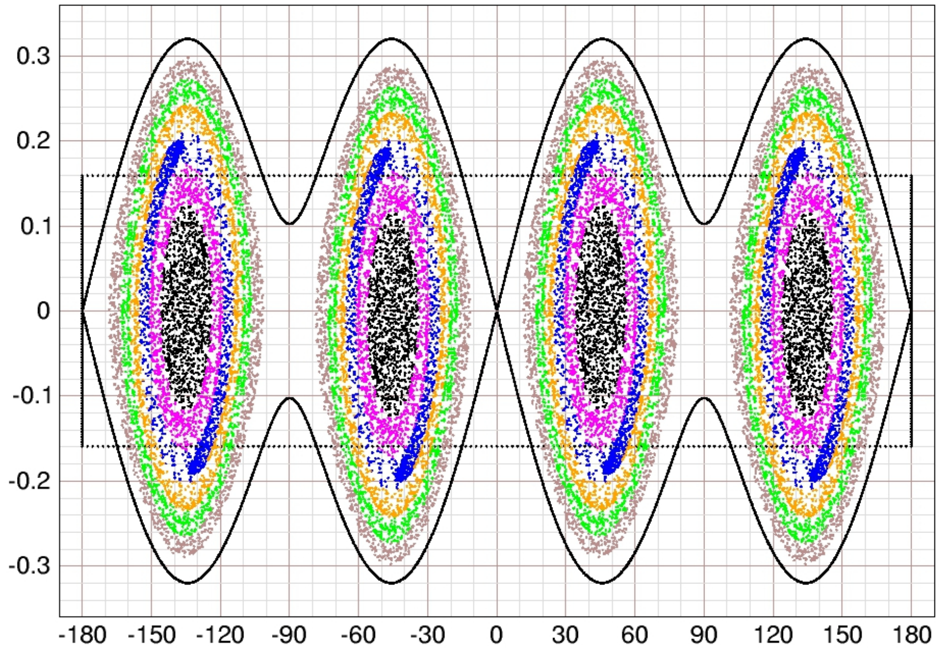
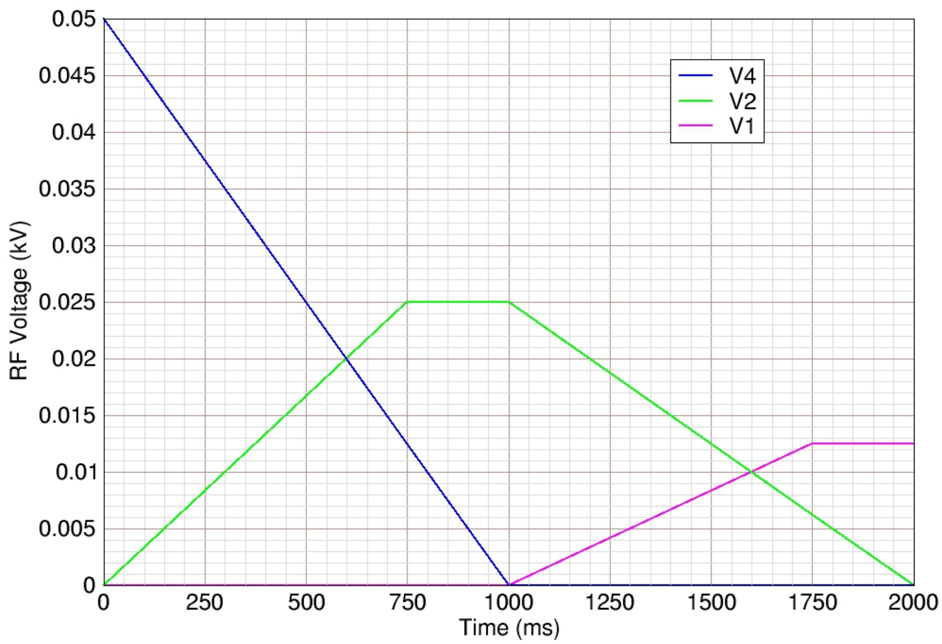


Figure 88: 4 to 2 to 1 merge at time $\underline{t = 75 \text{ ms}}$.



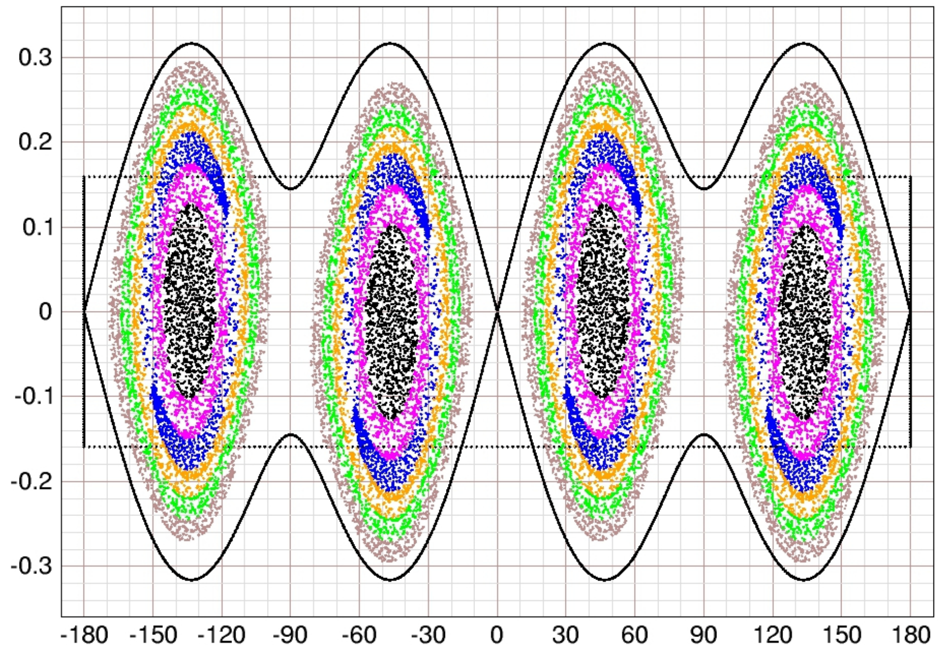
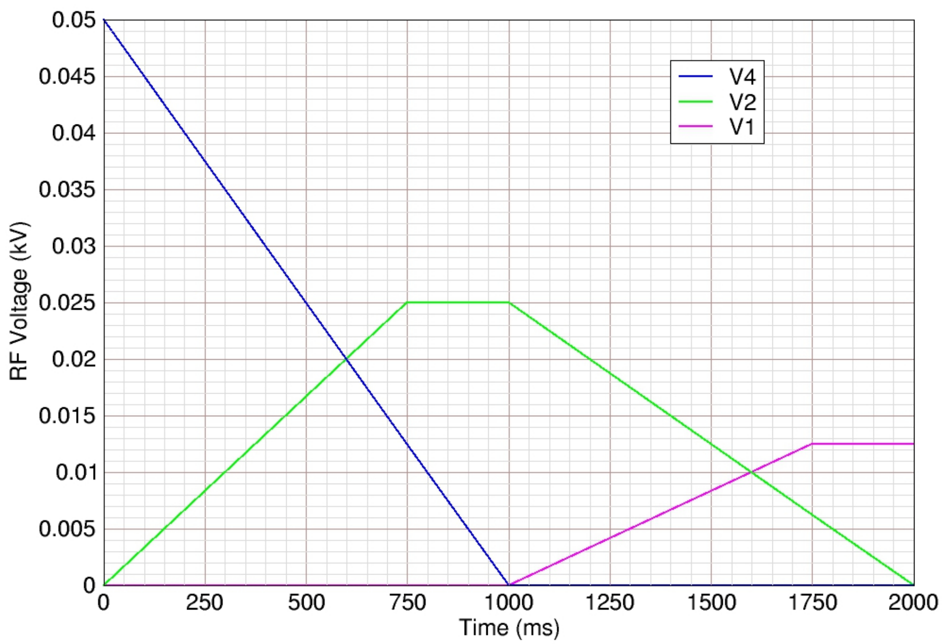


Figure 89: 4 to 2 to 1 merge at time $\underline{t = 150\ \text{ms}}$.



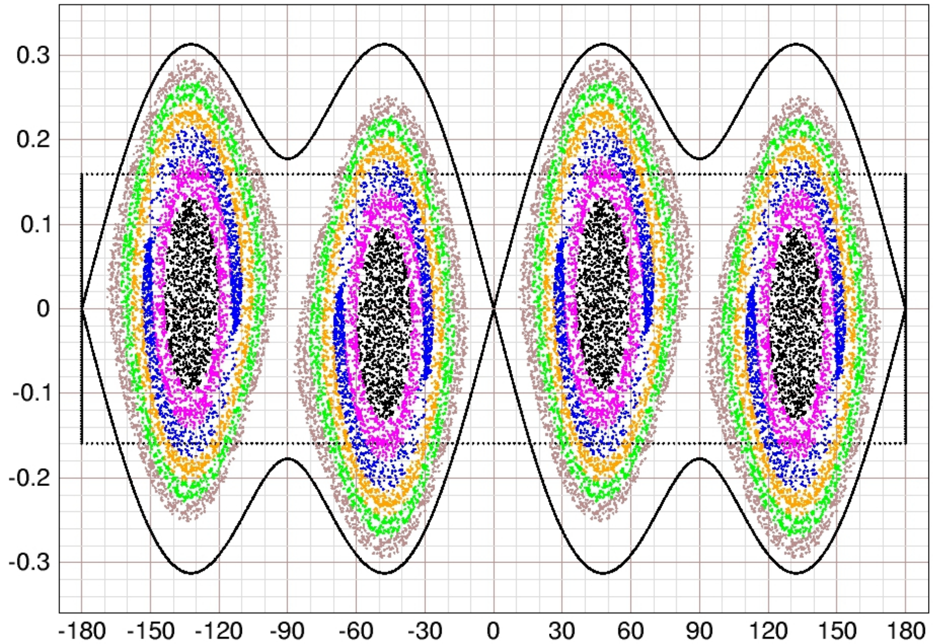
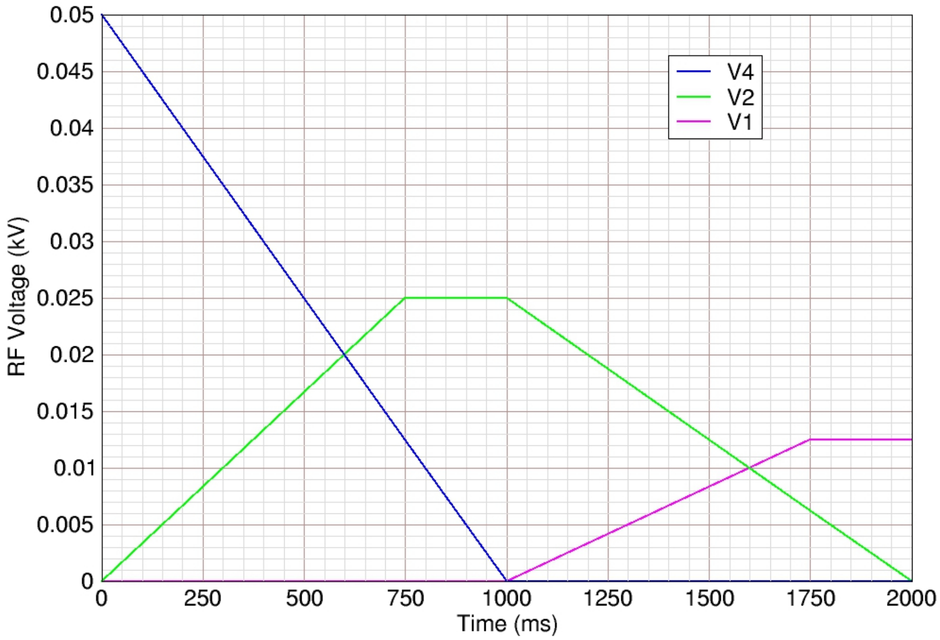


Figure 90: 4 to 2 to 1 merge at time $\underline{t = 225 \text{ ms}}$.



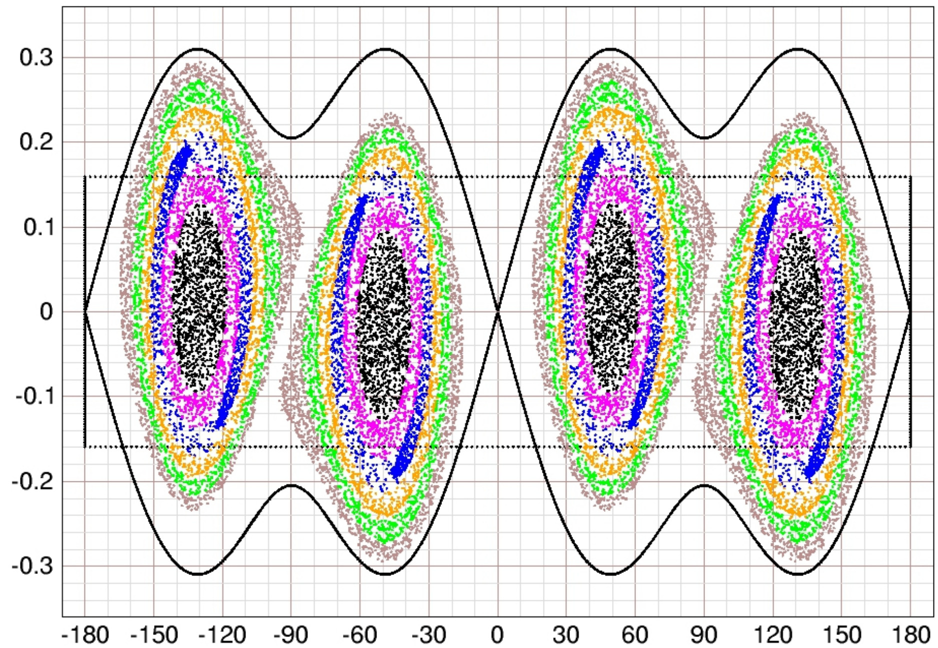
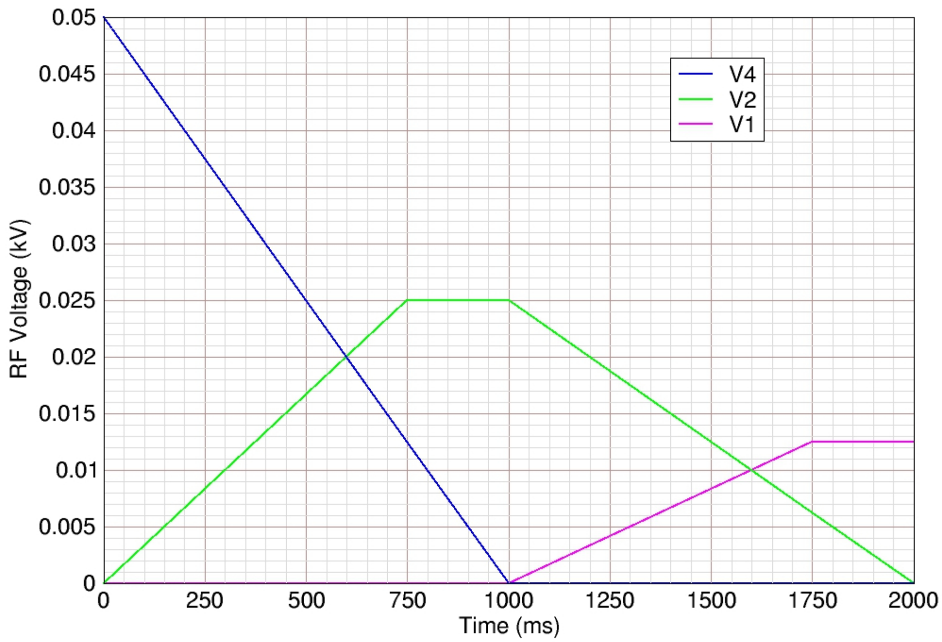


Figure 91: 4 to 2 to 1 merge at time $\underline{t = 300\text{ ms}}$.



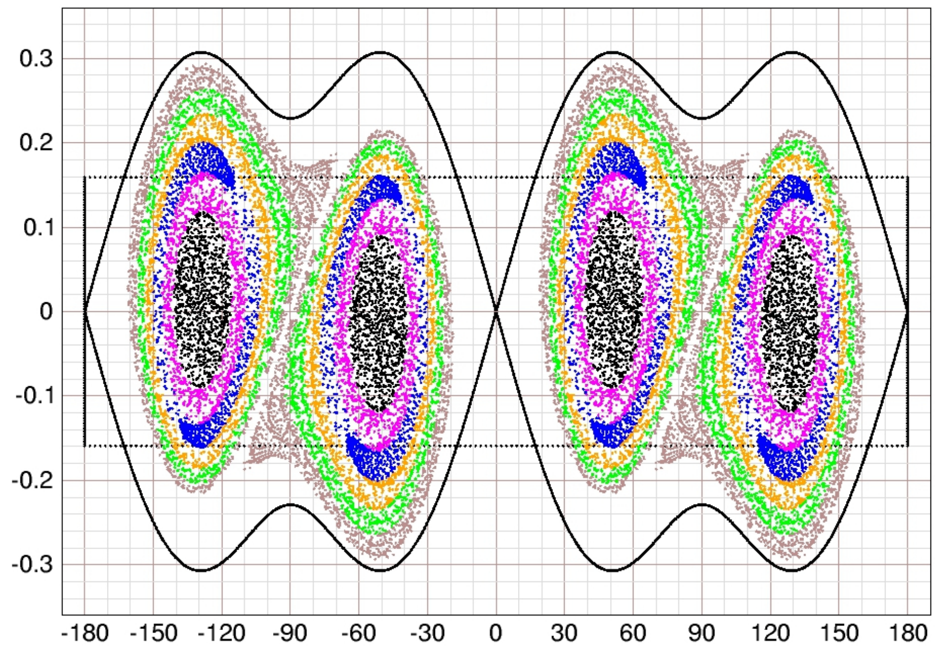
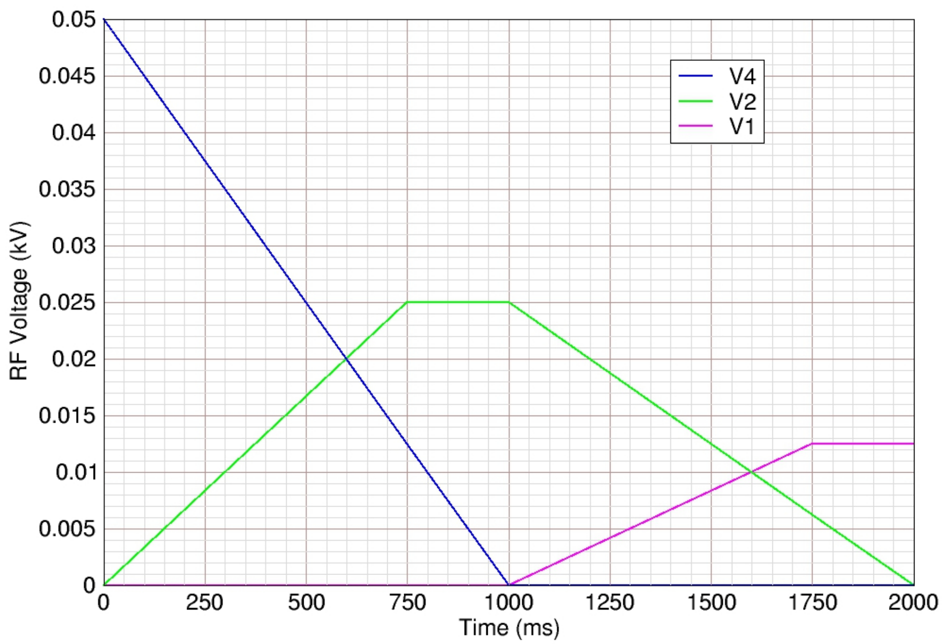


Figure 92: 4 to 2 to 1 merge at time $\underline{t = 375 \text{ ms}}$.



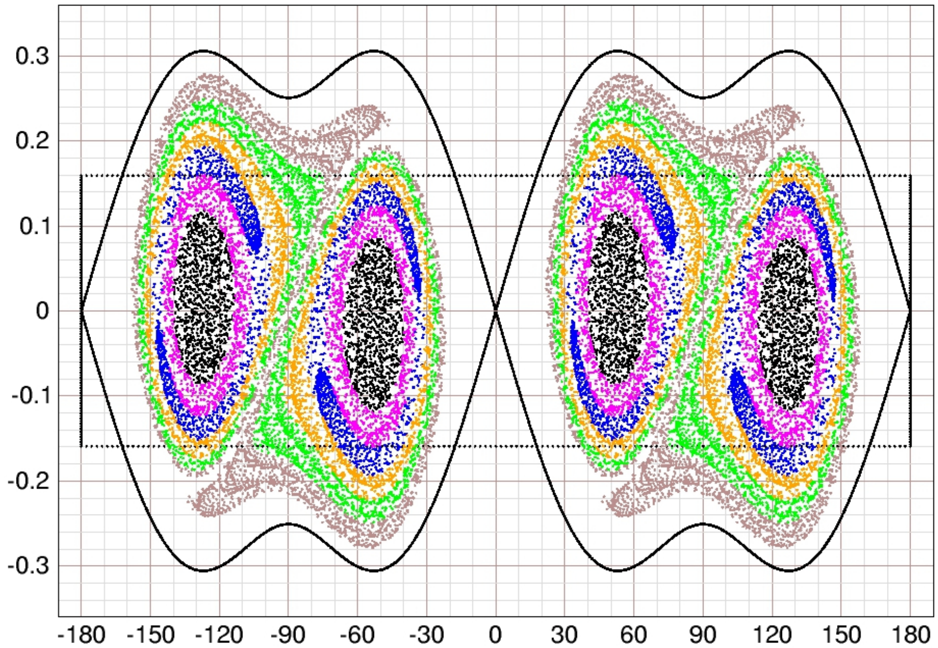
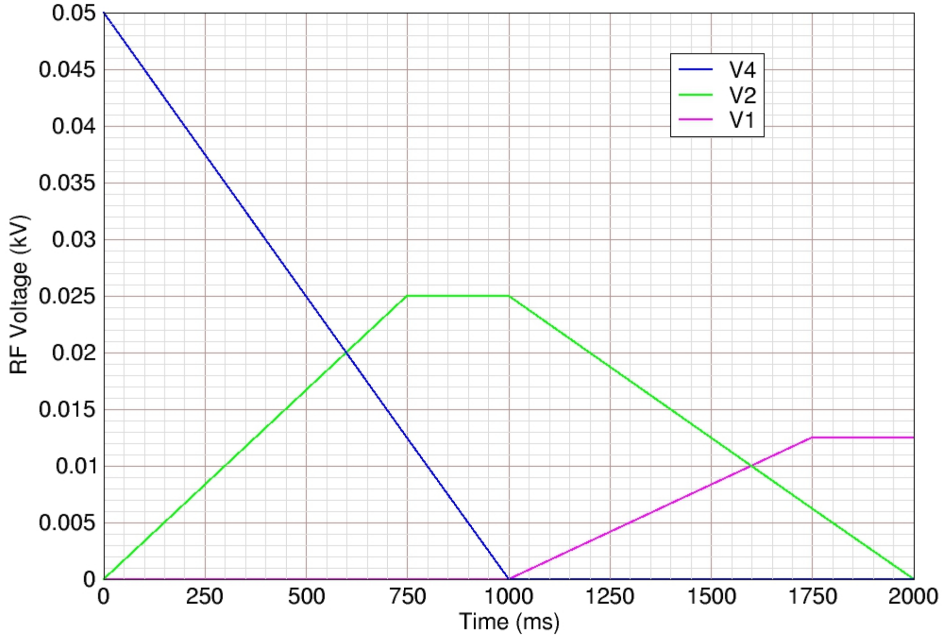


Figure 93: 4 to 2 to 1 merge at time $\underline{t = 450 \text{ ms}}$.



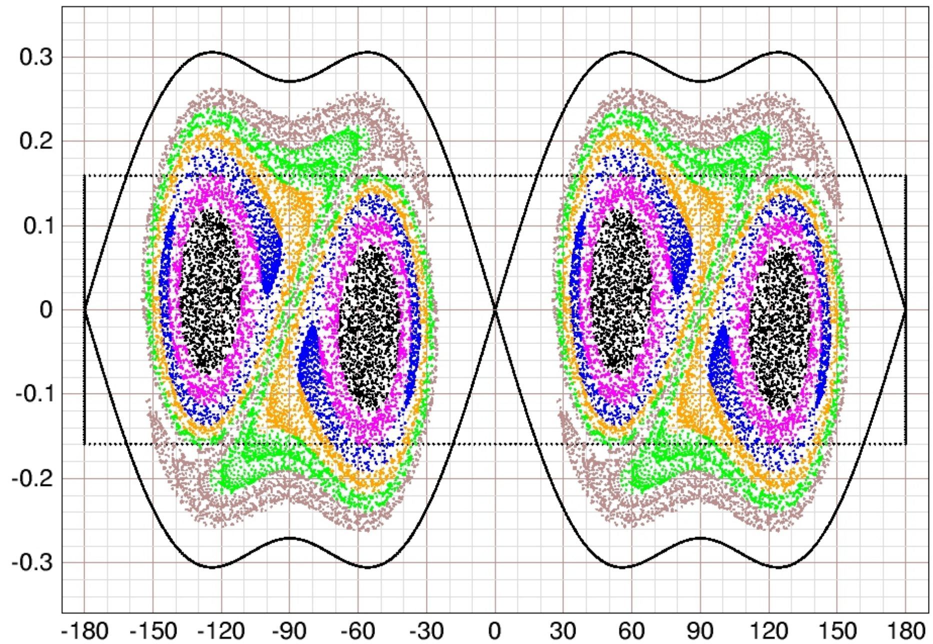
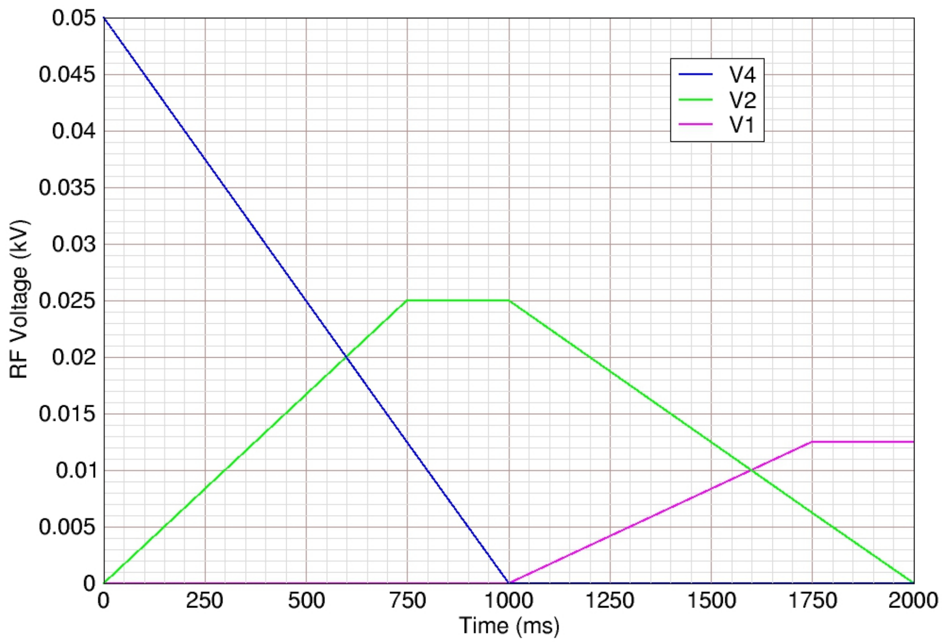


Figure 94: 4 to 2 to 1 merge at time $\underline{t = 525 \text{ ms}}$.



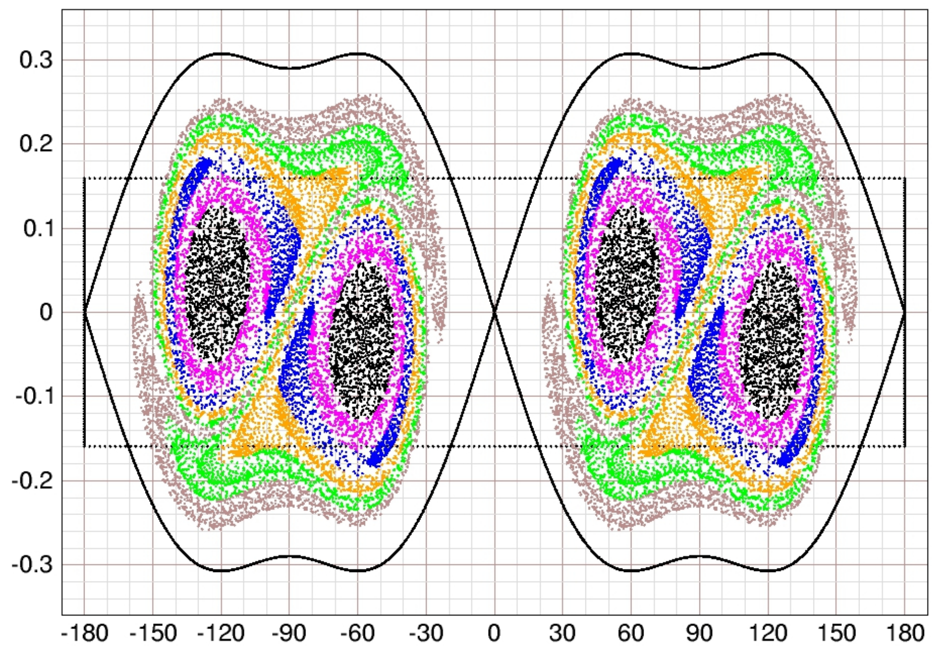
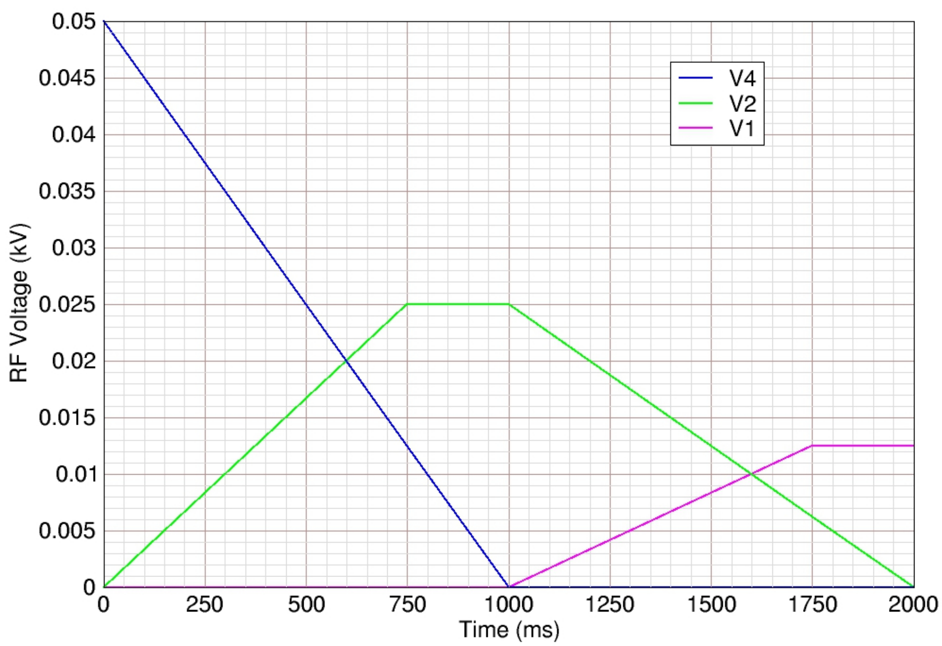


Figure 95: 4 to 2 to 1 merge at time $\underline{t = 600 \text{ ms}}$.



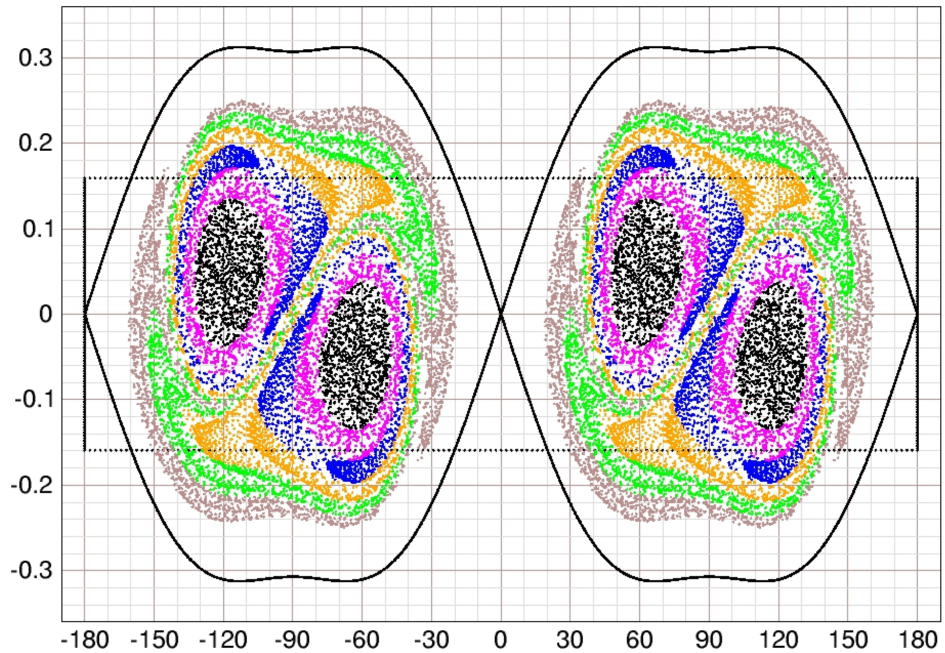
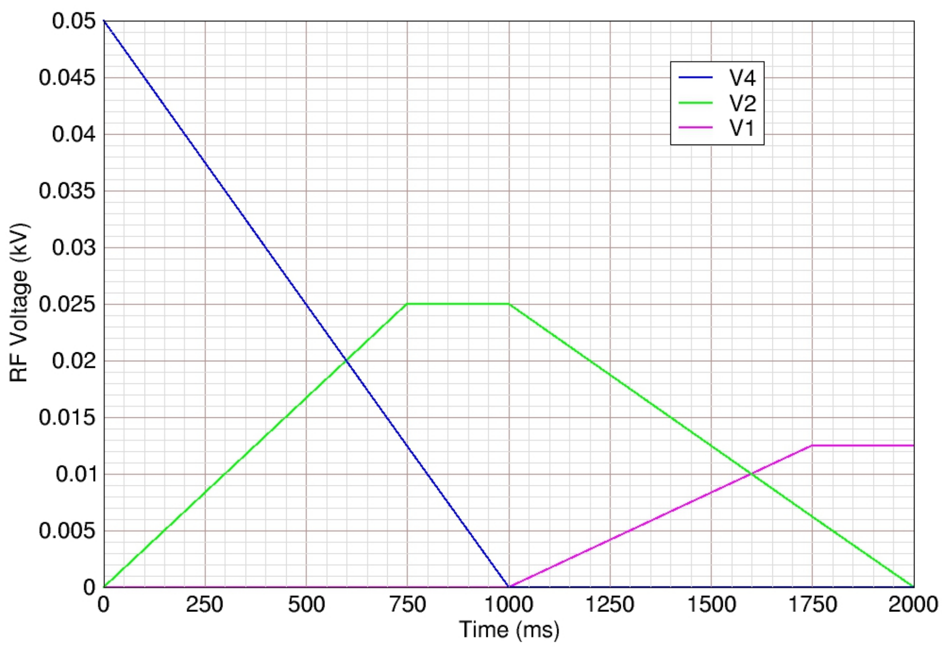


Figure 96: 4 to 2 to 1 merge at time $\underline{t = 675\ \text{ms}}$.



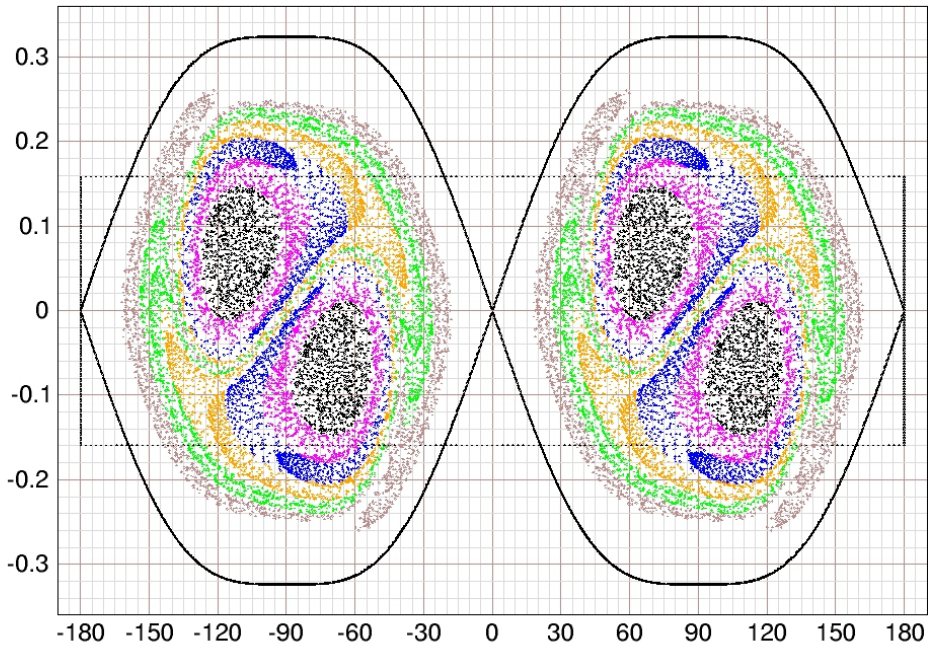
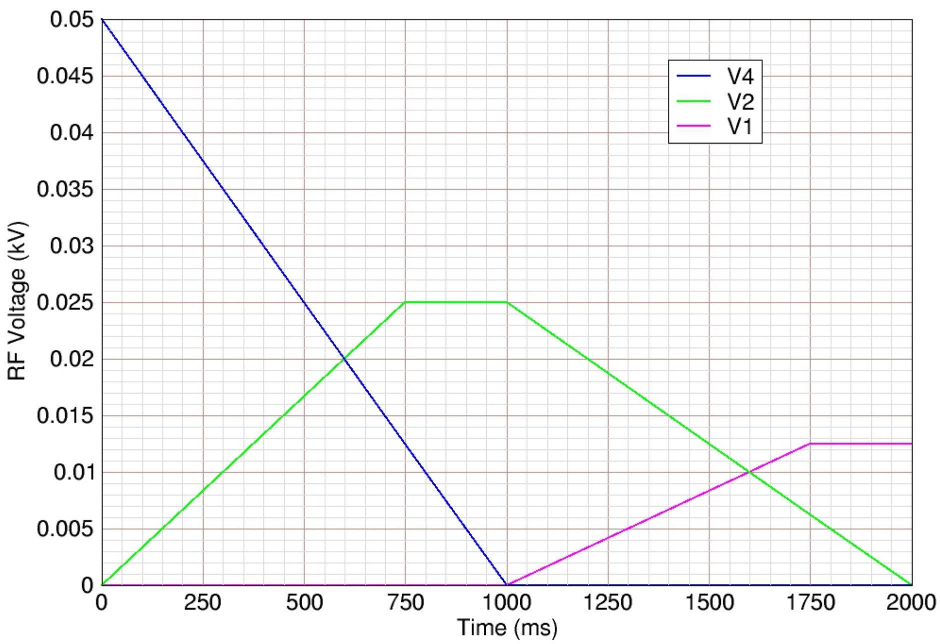


Figure 97: 4 to 2 to 1 merge at time $\underline{t = 750\text{ ms}}$.



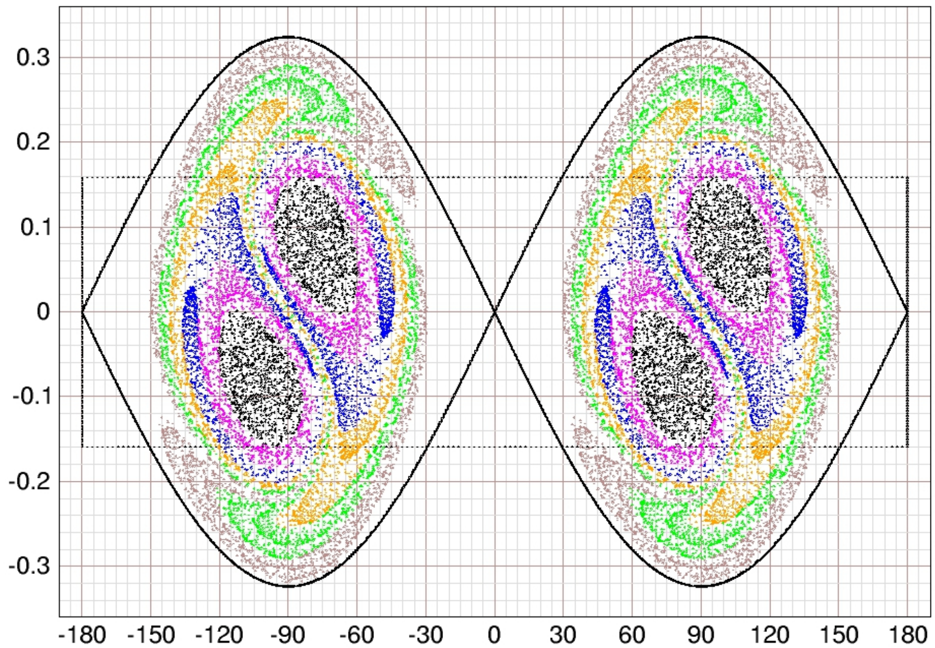
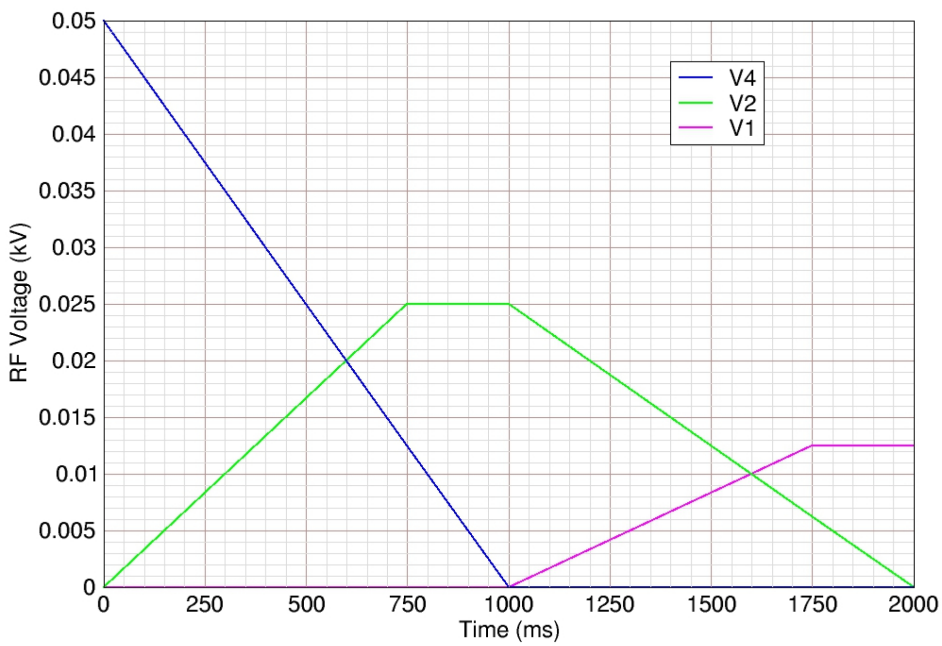


Figure 98: 4 to 2 to 1 merge at time $\underline{t = 1000 \text{ ms}}$.



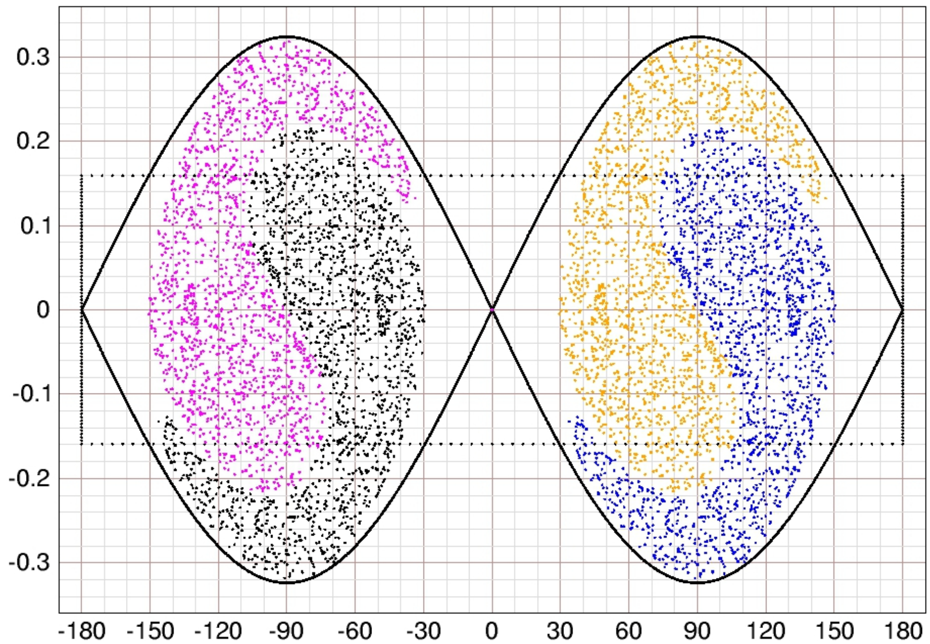
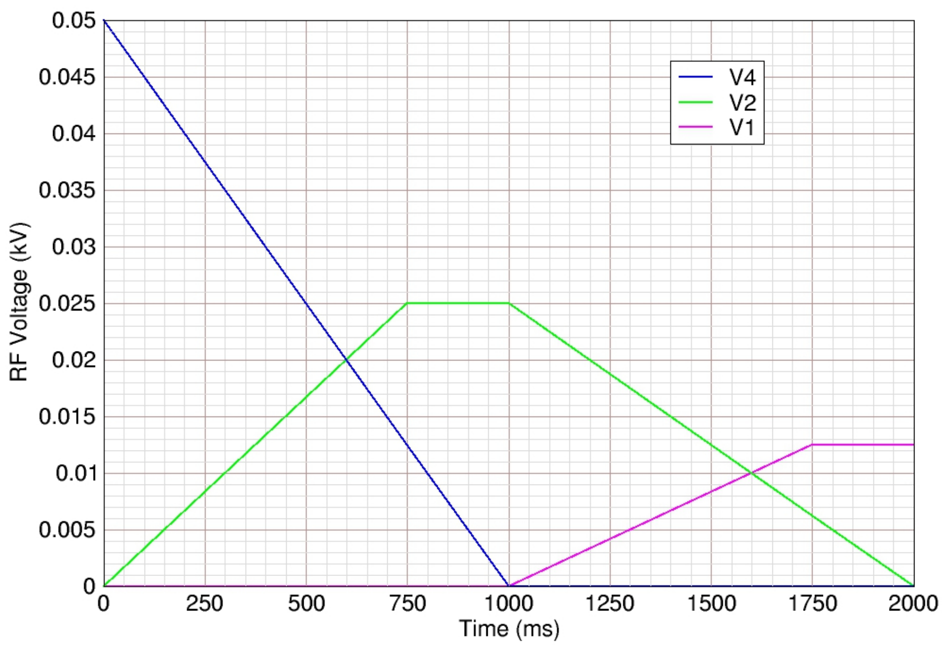


Figure 99: 4 to 2 to 1 merge at time $\underline{t = 1000 \text{ ms}}$.



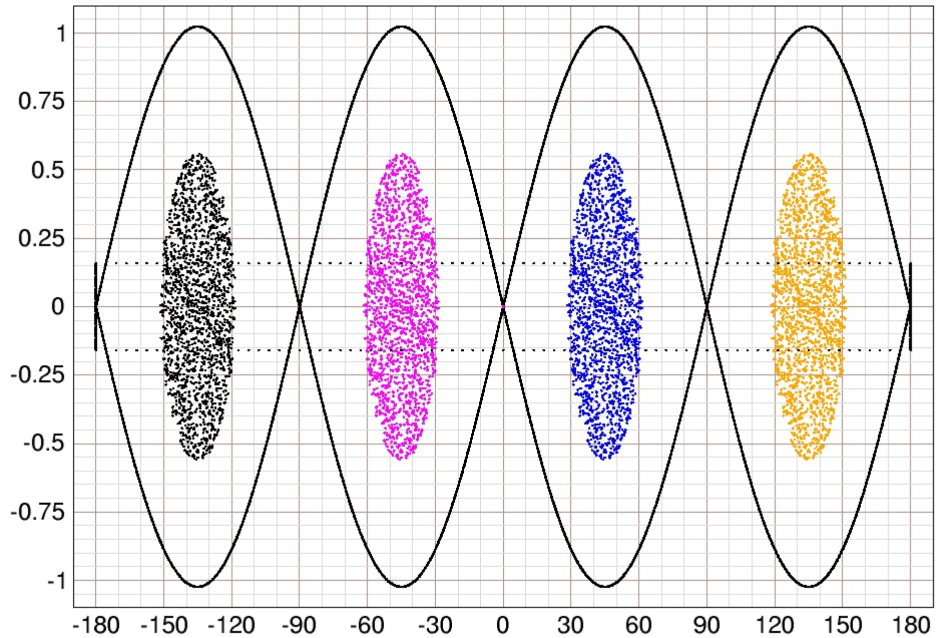
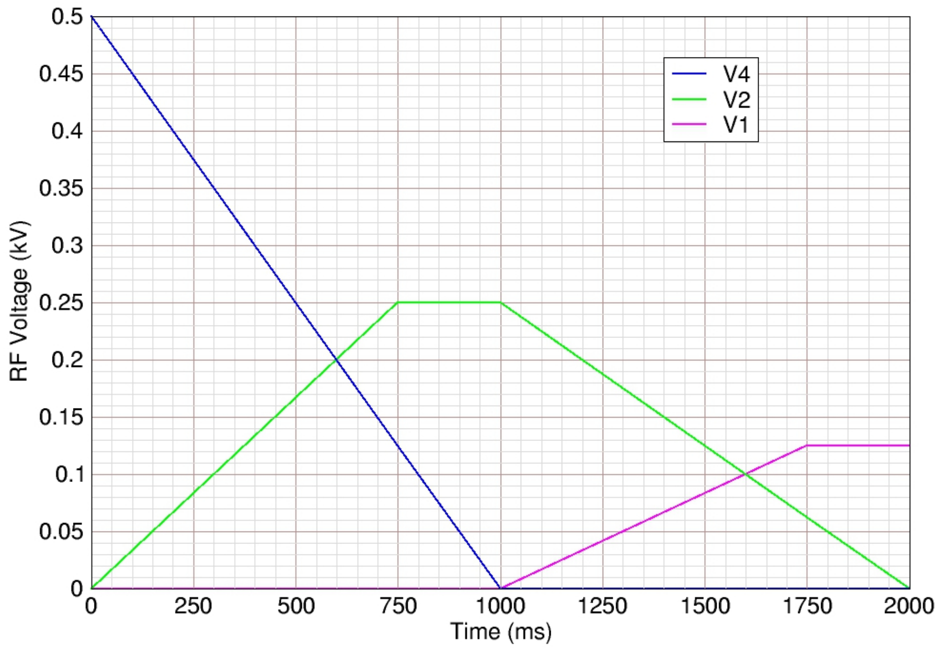


Figure 100: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



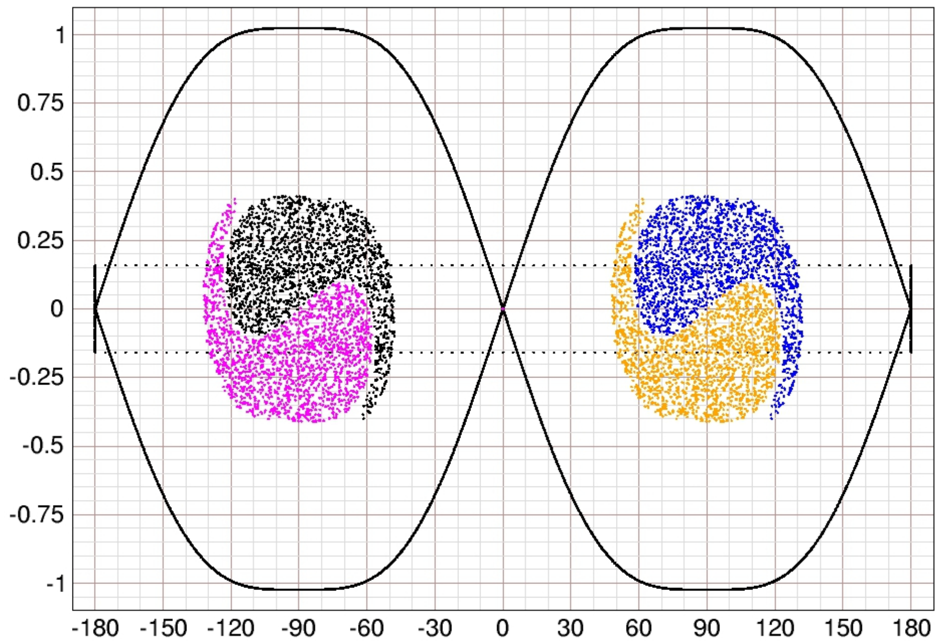
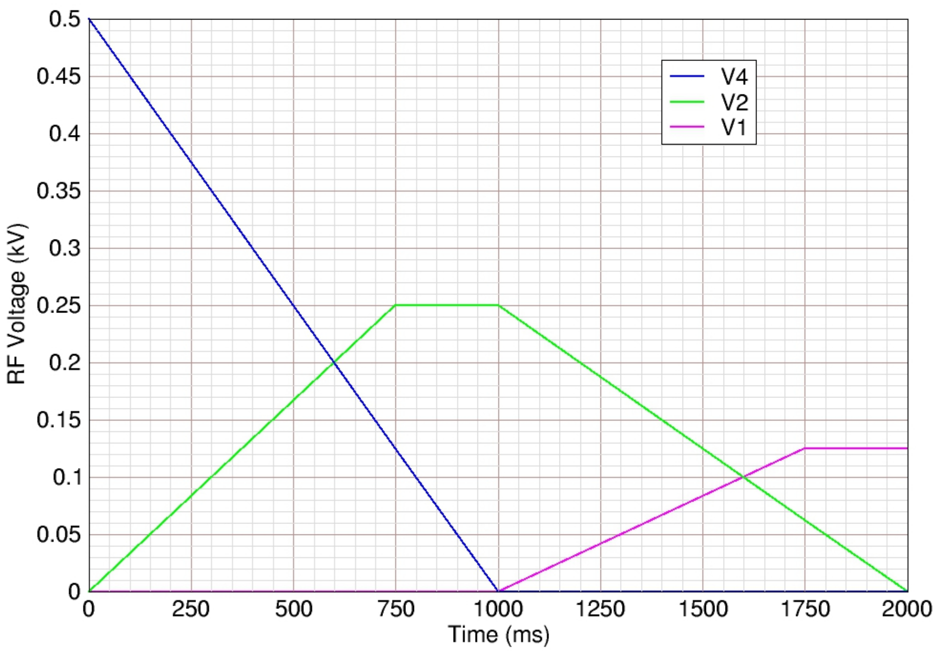


Figure 101: 4 to 2 to 1 merge at time $\underline{t = 750 \text{ ms}}$.



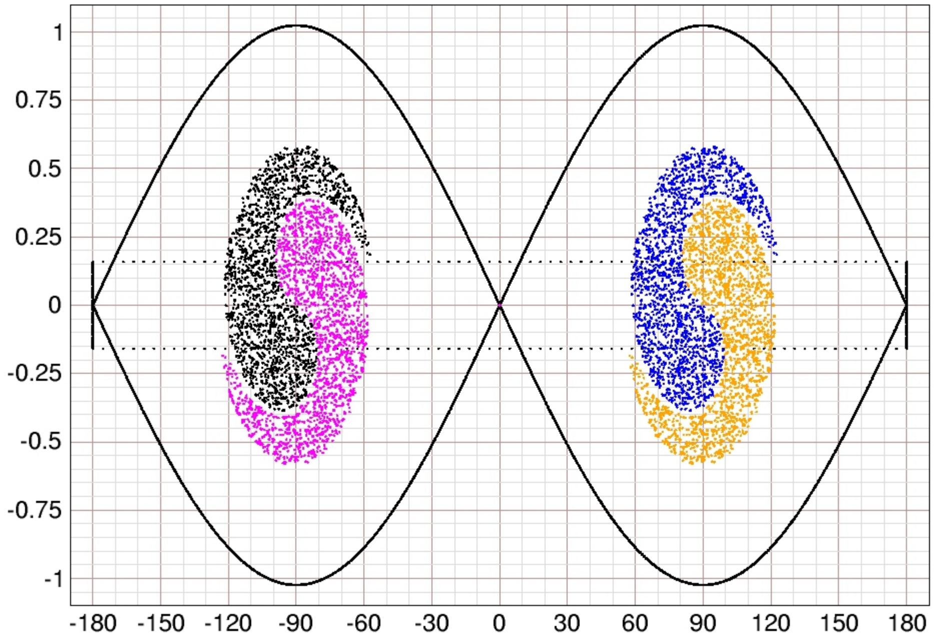
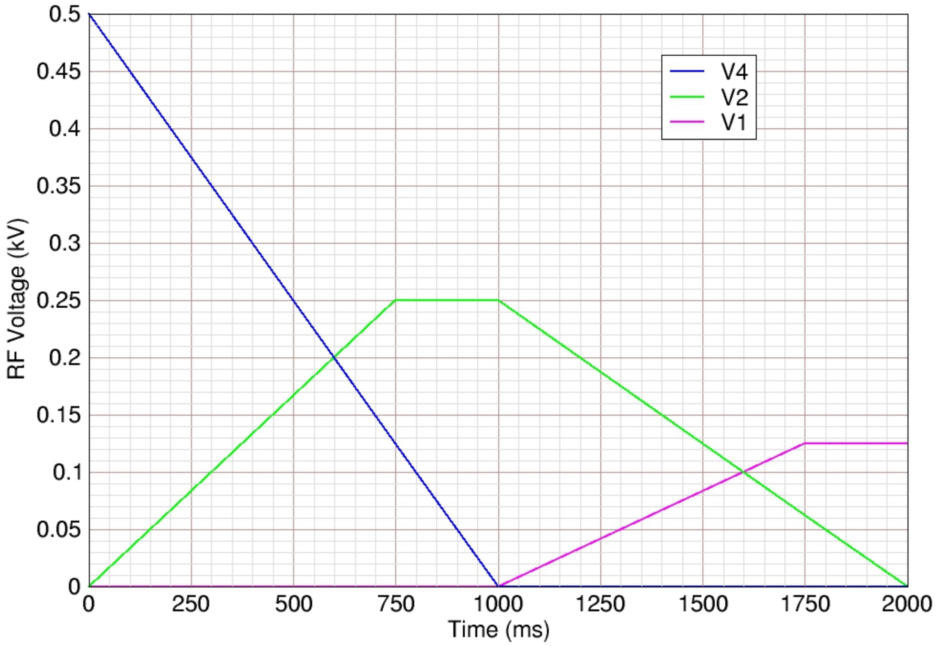


Figure 102: 4 to 2 to 1 merge at time <u>$t = 1000$ ms</u>.



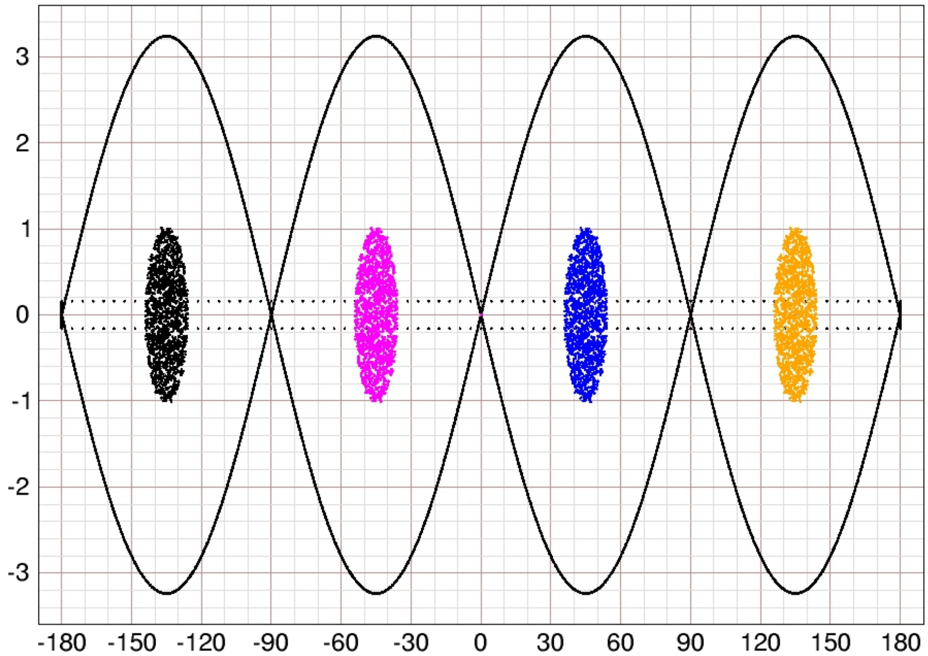
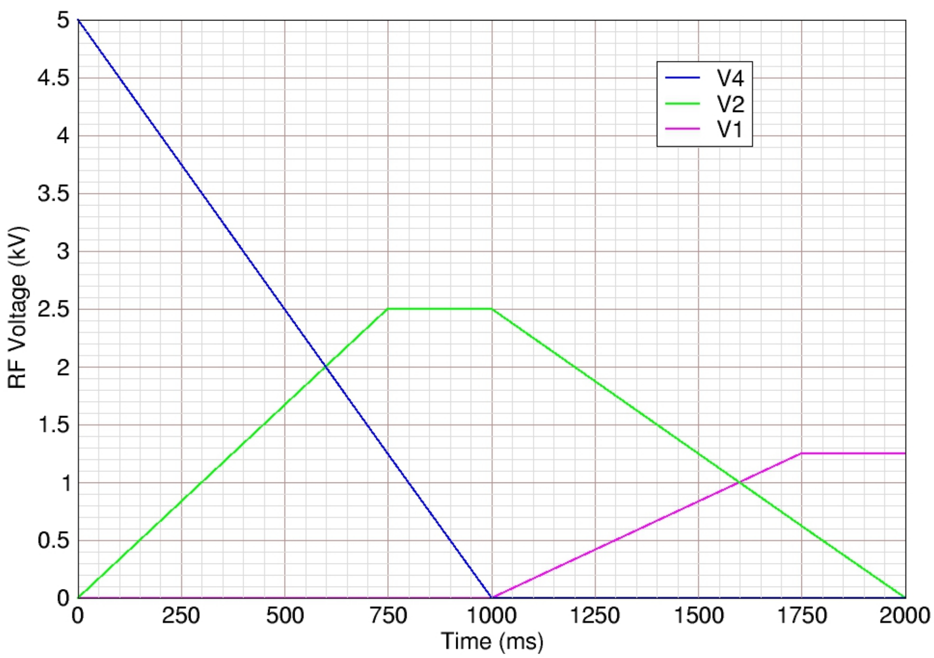


Figure 103: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



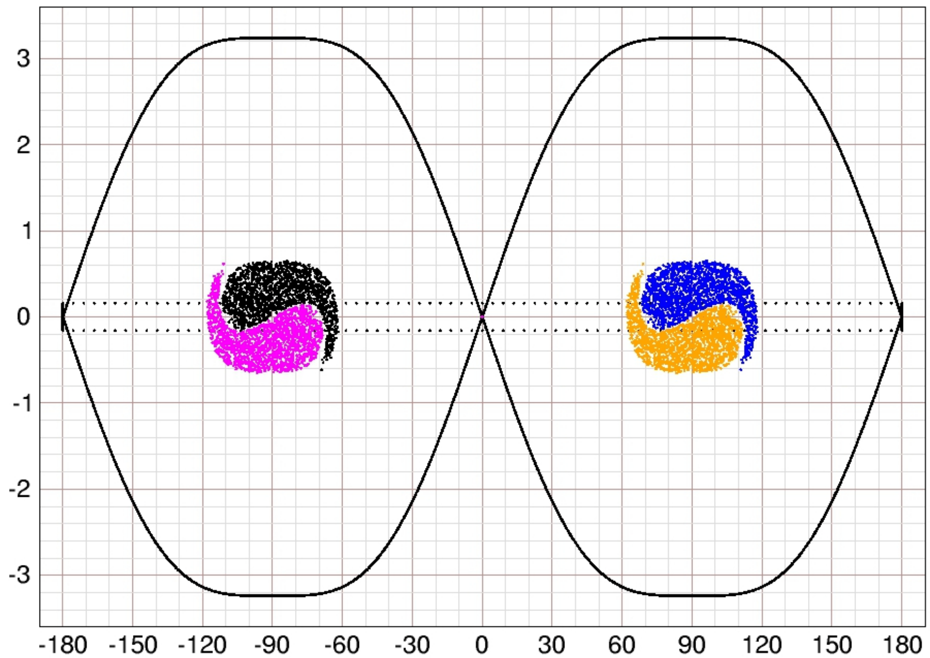
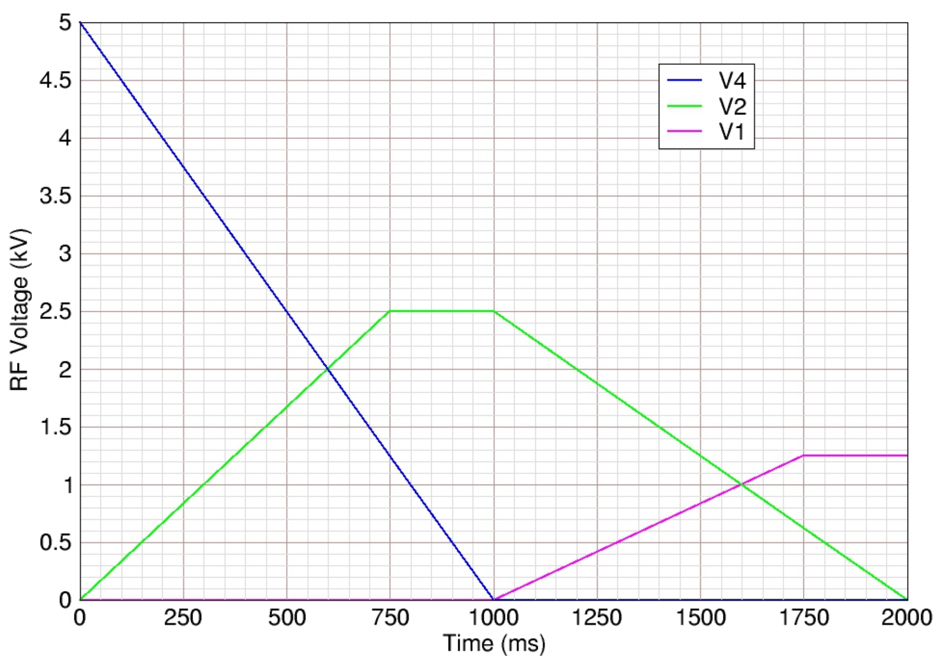


Figure 104: 4 to 2 to 1 merge at time $\underline{t = 750 \text{ ms}}$.



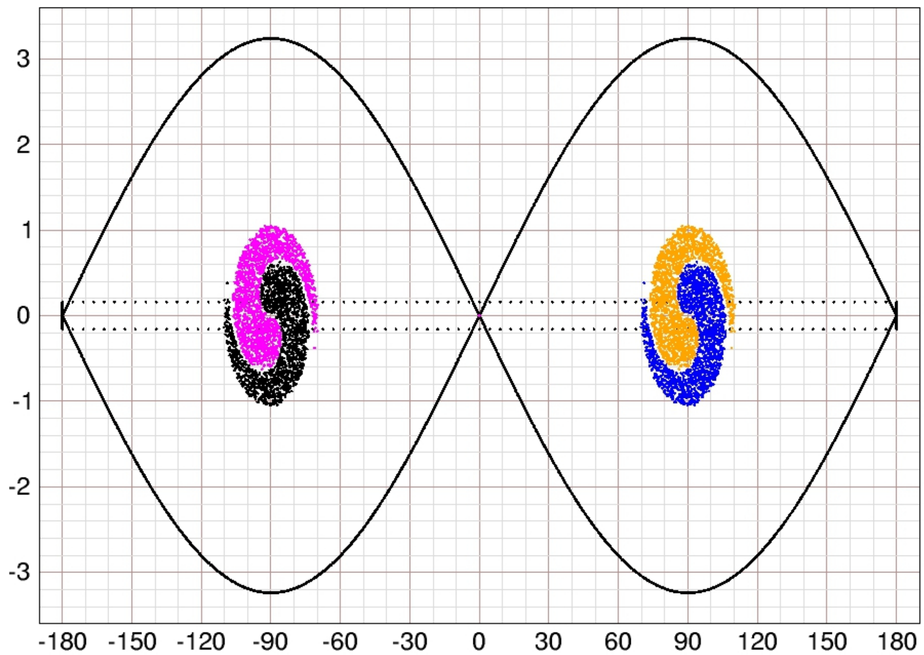
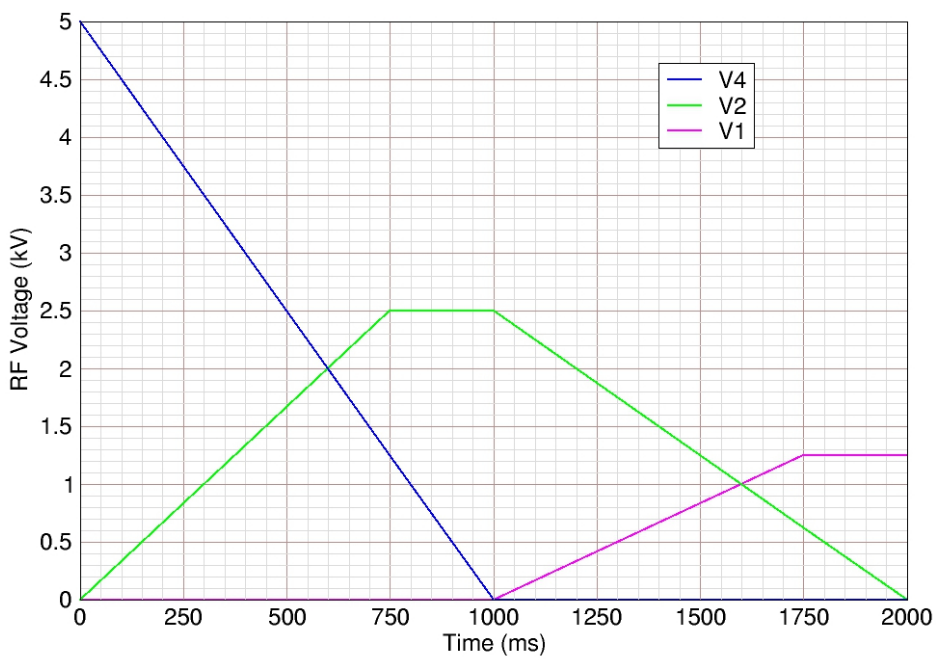


Figure 105: 4 to 2 to 1 merge at time $\underline{t = 1000 \text{ ms}}$.



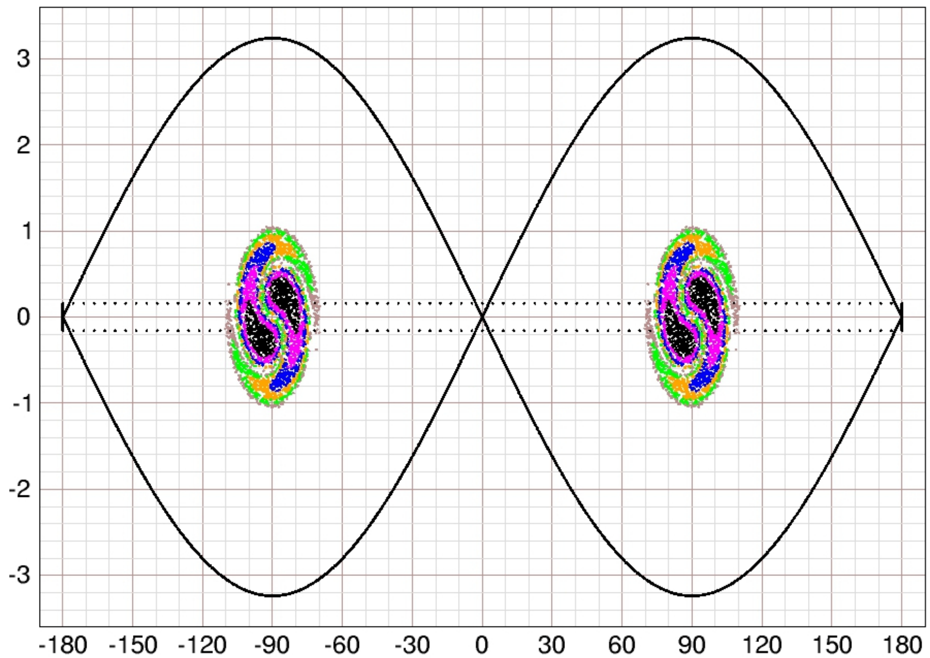
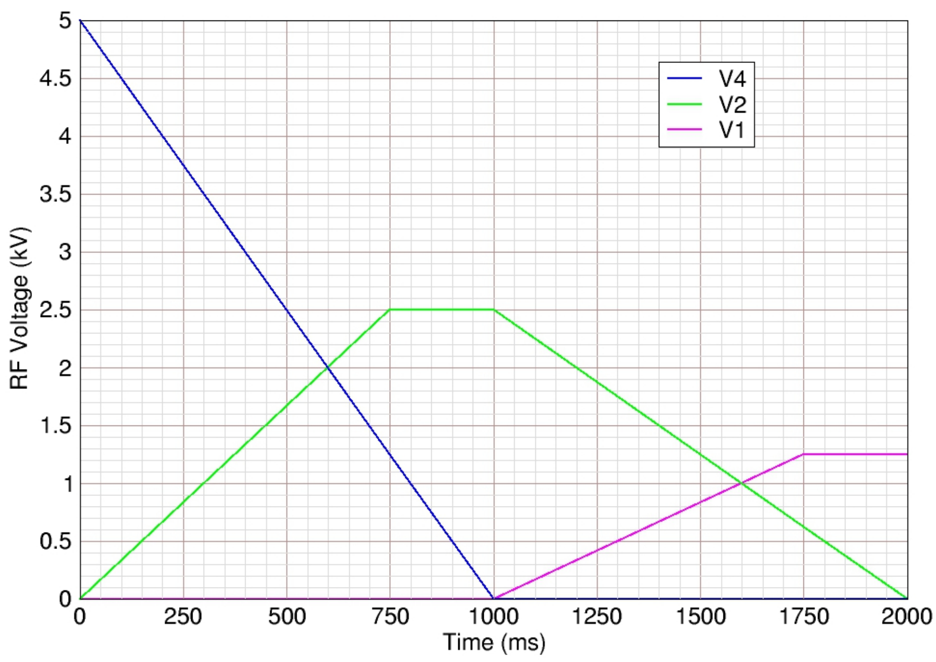


Figure 106: 4 to 2 to 1 merge at time $\underline{t = 1000 \text{ ms}}$.



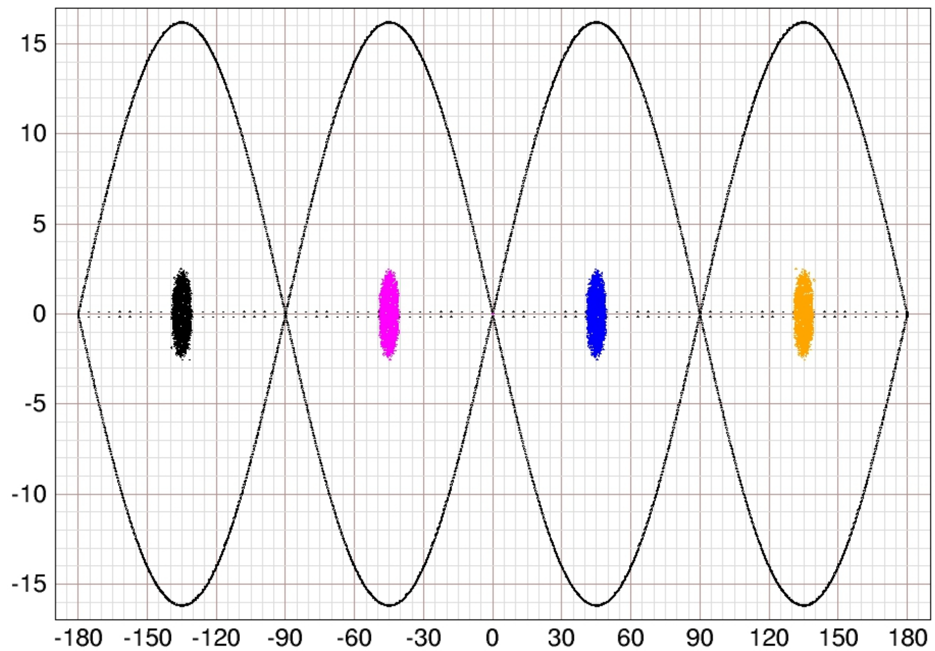
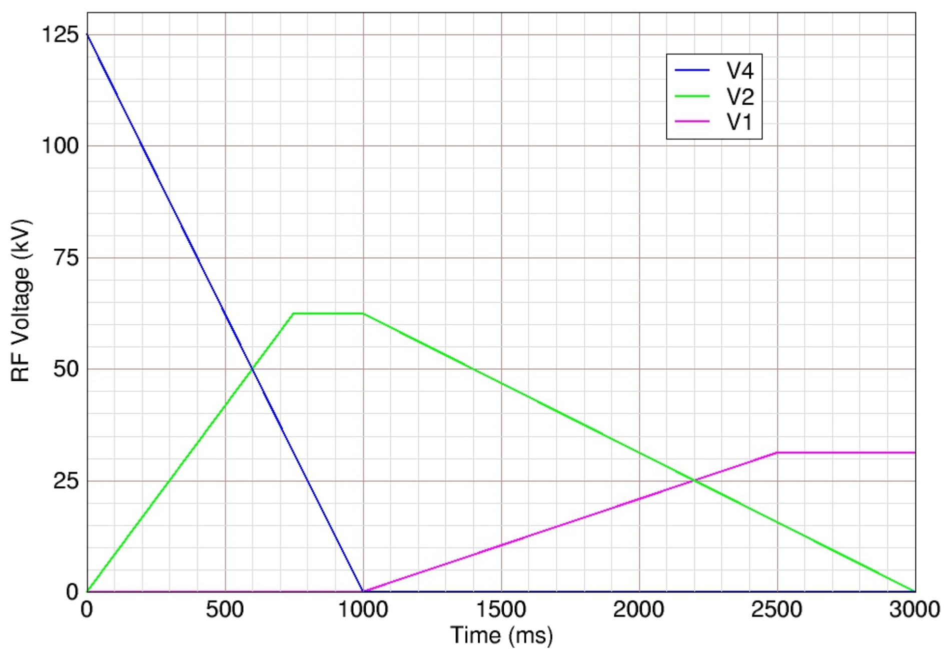


Figure 107: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



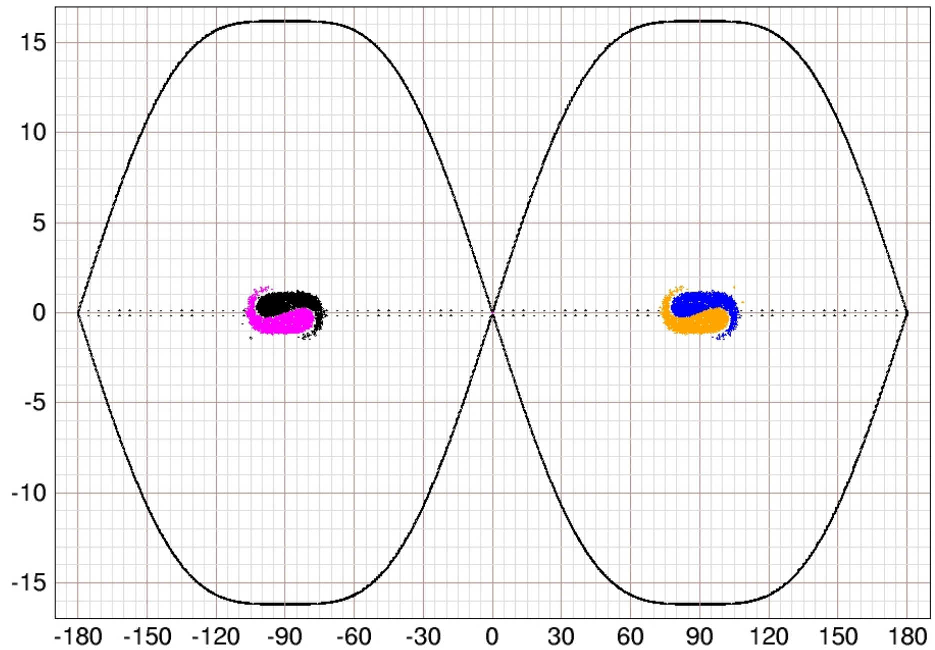
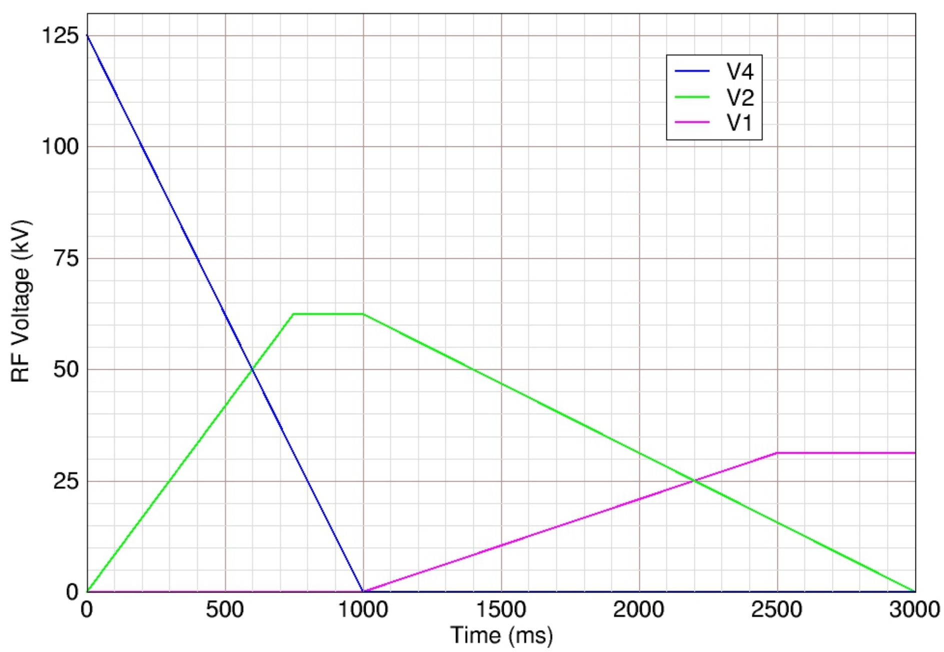


Figure 108: 4 to 2 to 1 merge at time $\underline{t = 750\text{ ms}}$.



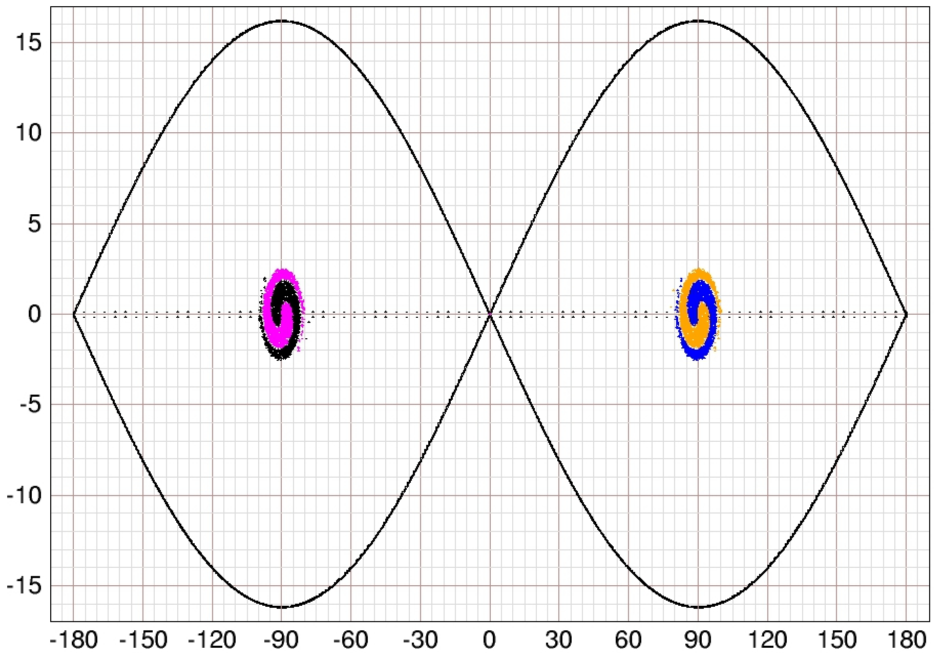
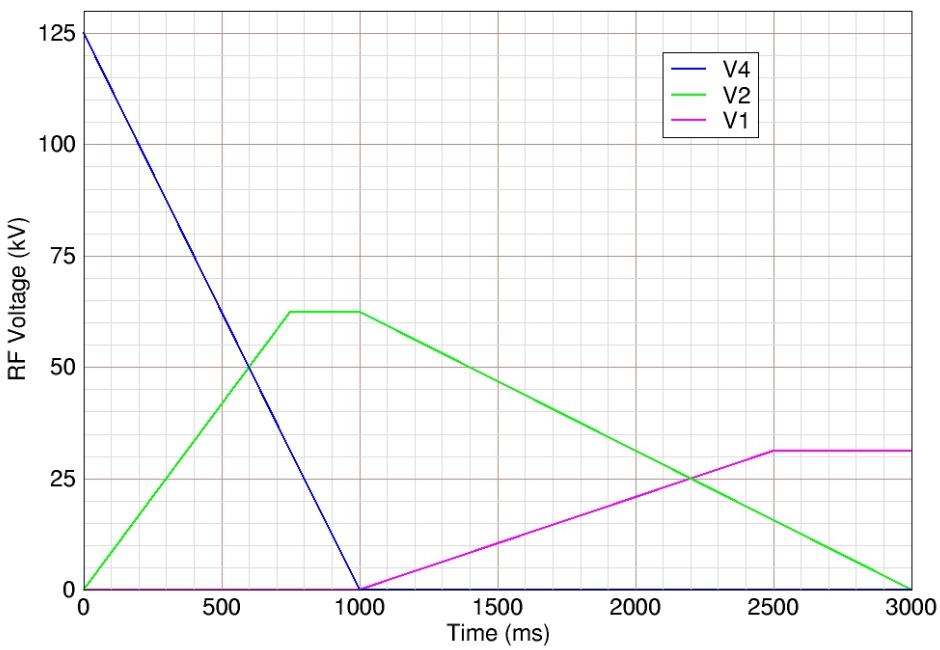


Figure 109: 4 to 2 to 1 merge at time $\underline{t = 1000 \text{ ms}}$.



28 500 ms merge in AGS at $G\gamma = 45.5$

As in the previous sections, the upper half of **Figure 110** shows the start of a simulation in which polarized proton bunches undergo two 2 to 1 merges on the AGS flattop at $G\gamma = 45.5$. The bunches initially are sitting in four adjacent harmonic 12 buckets as shown in the figure. The area of each bucket is 0.648 eV-s. The longitudinal emittance of each bunch is 0.5 eV-s. The bunches are formed by adiabatic capture of a uniform distribution of unbunched particles as explained in [6]. The particles in the bunches are color coded according to their initial longitudinal oscillation amplitudes. In order of increasing amplitude, the colors are black, pink, blue, orange, green, and brown.

The first 2 to 1 merge is done over a period of 500 ms. **Figures 110** through **112** show the evolution of the merge. In **Figure 112** one sees that the oscillation amplitude layers of each bunch are essentially intact with no mixing of bunch particles. One also sees that some particles are outside the buckets, indicating increased emittance growth with the shorter merge time.

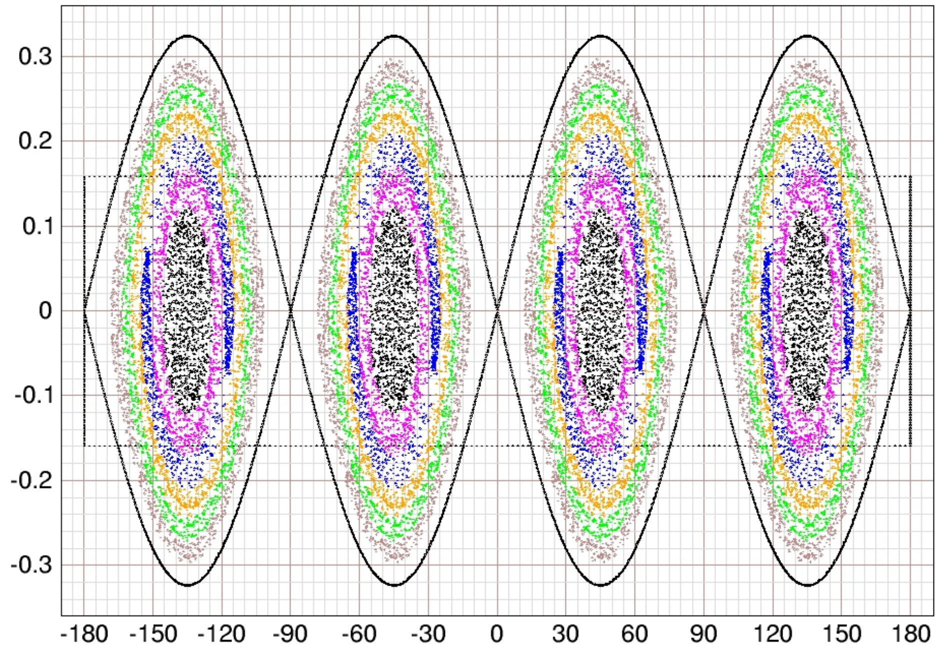
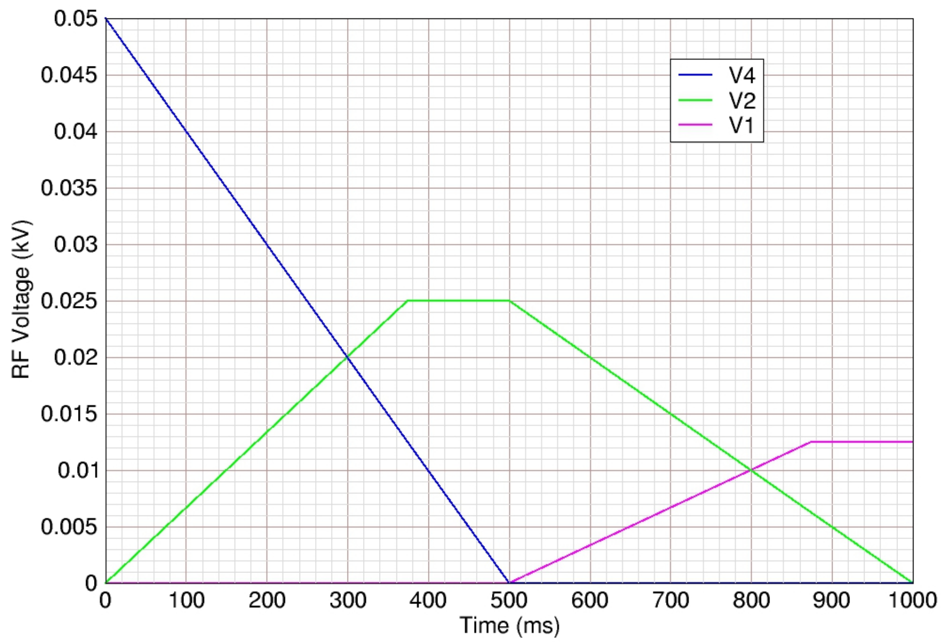


Figure 110: 4 to 2 to 1 merge at time $\underline{t = 0 \text{ ms}}$.



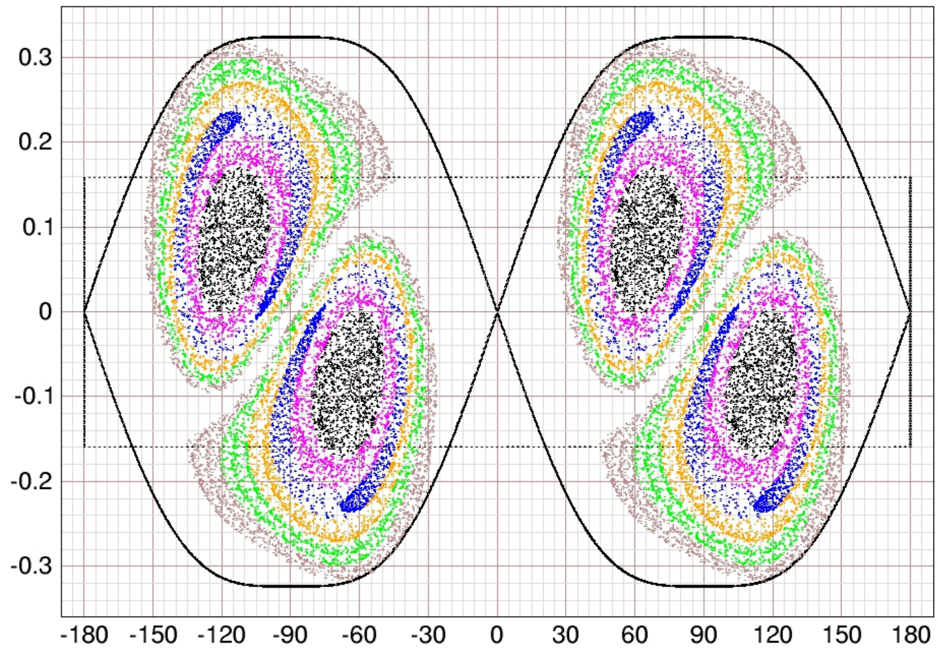
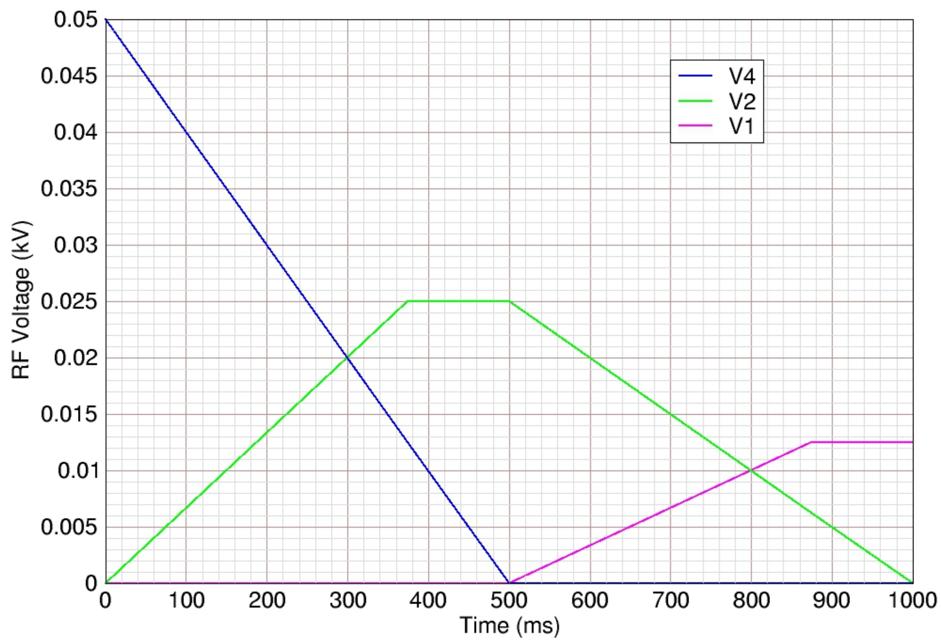


Figure 111: 4 to 2 to 1 merge at time <u>$t = 375$ ms</u>.



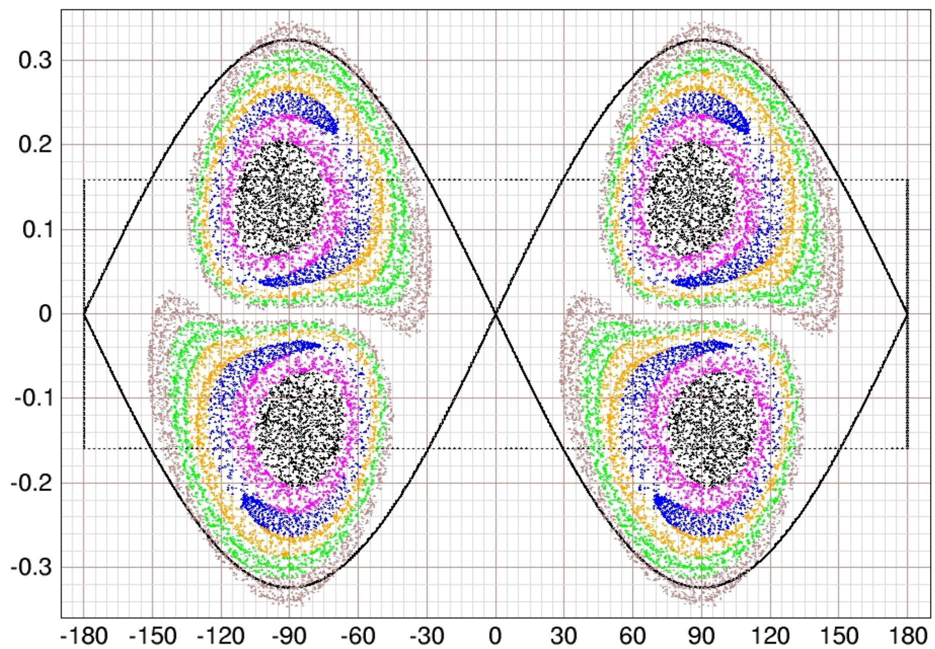
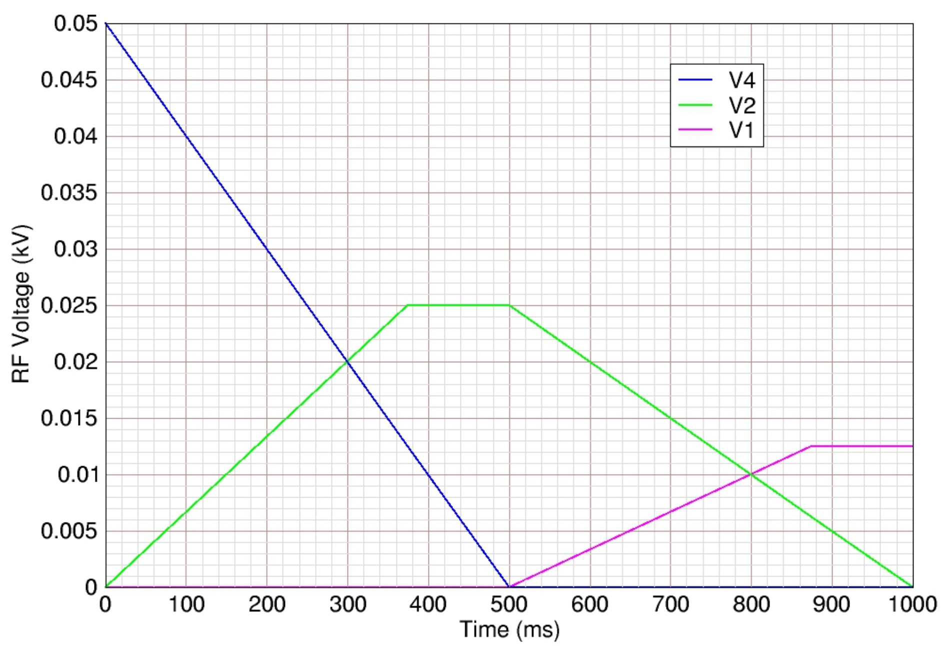


Figure 112: 4 to 2 to 1 merge at time $\underline{t = 500 \text{ ms}}$.



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[1] K.L. Zeno, Booster-AGS-pp elog, December 20–23, 2021

[2] C.J. Gardner, "Simulations of Merging Helion Bunches on the AGS Injection Porch," C-A/AP/Note 527, August 2014

[3] C.J. Gardner, "FY2016 Parameters for gold ions in Booster, AGS, and RHIC," C-A/AP/Note 574, October 2016

[4] C.J. Gardner, "Simulations of Merging Helion Bunches on the AGS Injection Porch," C-A/AP/Note 527, Section 16, August 2014

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[6] C.J. Gardner, "Simulations of Merging Helion Bunches on the AGS Injection Porch," C-A/AP/Note 527, Section 9, August 2014