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Abstract

We revisit Free-Electron Laser (FEL) equations in the presence of a planar undulator to investigate effects on longitudinal FEL dynamics upon seeding by a harmonic for both low-gain and high-gain cases. Then, we extend the FEL equations in 3D to include electron beam emittance and radiation diffraction effects.

I. INTRODUCTION

W. B. Colson derived the FEL pendulum equations by solving for longitudinal electron motion in the presence of radiation and undulator fields [1]. He extended this treatment to apply for the generation of harmonics [2]. Later, K.-J. Kim applied this formulation along with the microscopic description of electron beams by Klimontovich distribution function to formulate three-dimensional models for self-amplified spontaneous emission (SASE) in high-gain FELs [3] and brightness functions based gain formula for low-gain FELs [4]. Recently, L. H. Yu extended the low-gain formula with no focusing approximation to achieve harmonic lasing in transverse gradient undulator based FEL [5, 6]. The conventional treatment used by Colson, Kim and Yu for deriving Free-Electron Laser (FEL) pendulum equations and gain assumes that the pondermotive potential is provided by the fundamental radiation. While such treatment is valid for FELs during startup from noise, it is feasible to seed the FEL with a harmonic radiation ($h > 1$) via cavity configurations or outcoupling schemes. When the field amplitude of this harmonic is higher than that of the fundamental from spontaneous emission, FEL dynamics is governed by the harmonic seed.

In this report, we derive equations governing longitudinal dynamics for a FEL in the presence of harmonic seeding and a planar undulator. Then, we include transverse beam dynamics to arrive at three-dimensional FEL model. We note that calculation techniques explored here have been previously applied and most expressions throughly derived and investigated previously for fundamental radiation seeding (please see Refs. [7, 8] and references within). We tried our best to minimize repetitions of derivations and keep only the pertinent ones necessary for our study here.

The rest of this article is organized as follows: in section II, we obtain relevant electron parameters such as velocities and arrival times for an individual electron and an average electron. Section III applies Lorentz force equation and slowly changing approximation to obtain pendulum equations governing longitudinal dynamics for single electrons using expressions from section II. In section IV, we apply slowly varying envelope approximation to paraxial Maxwell equation in one-dimension to obtain evolution equation for emitted harmonic field amplitude. Then, we introduce Klimontovich distribution function for electron beam to formulate coupled equations governing longitudinal FEL dynamics in section V; we solve these equations to obtain expressions for power gain for low-gain and high-gain FELs in sections VI and VII using different techniques. In section VIII, we relax the 1D limit restriction to include radiation diffraction and electron beam emittance effects to formulate 3D version of FEL equations. We wrap up this report with our findings in section IX.

II. SINGLE ELECTRON MOTION

For convenience, we solve for electron motion along the $z-$axis (at $x = 0$ and $y = 0$). For the undulator period λ_u, the magnetic field of a planar undulator along $z-$axis is

$$B_y = B_0 \sin(k_u z), \tag{1}$$

where $k_u = 2\pi/\lambda_u$ is the undulator wavenumber, B_0 is the peak magnetic field and the field component is along vertical direction. For a relativistic electron with relativistic Lorentz factor γ, velocity $\boldsymbol{v}$, the Lorentz force equation becomes

$$\frac{d}{dt}(\gamma m \boldsymbol{v}) = -e\boldsymbol{E} - e\left[\hat{x}(v_y B_z - v_z B_y) - \hat{y}(v_x B_z - v_z B_x) + \hat{z}(v_x B_y - v_y B_x)\right], \tag{2}$$

where $\boldsymbol{v} = (v_x, v_y, v_z)$ and v_i is the electron's velocity along ith direction, m is electron mass and e is electron charge. The electric field $\boldsymbol{E}$ contribution comes from the seed radiation and by using Lorentz force equation we have effectively ignored the recoil effects of emitted radiation on electron motion. Since $|\boldsymbol{E}| \propto \sin(kz - \omega t + \phi)$ for a radiation with frequency ω and phase ϕ, the effect of this electric field amplitude on electron velocity is proportional to $\hbar/\gamma \lll 1$ for a relativistic electron ($\gamma \gg 1$), where $\hbar = h/2\pi$ with h being the Planck's constant. Thus, we ignore the electric field contribution in equation (2) and take the only contribution for the magnetic field from equation (1). This results in wiggle motion in $x-$direction as well as reduction in the longitudinal velocity. Assuming electron energy loss is negligible along the undulator (generally true for low-gain FELs), it is easy to show that

$$v_x = -\frac{Kc}{\gamma}\cos(k_u z), \tag{3a}$$

$$
\begin{aligned}
v_z &= \sqrt{v^2 - v_x^2 - v_y^2} \\
&= c\sqrt{\left(1 - \frac{1}{\gamma^2}\right) - \left(\frac{v_x}{c}\right)^2} \\
&\approx \bar{v}_z - \frac{K^2 c}{4\gamma^2}\cos(2k_u z).
\end{aligned} \tag{3b}
$$

Here c is the speed of light and we introduced $K = \frac{eB_0}{mck_u} = 0.9343\lambda_u[\text{cm}]B_0[\text{T}]$ as the undulator deflection parameter. The average longitudinal electron velocity is given by $\bar{v}_z = c\left[1 - \frac{1+K^2/2}{2\gamma^2}\right]$. Likewise, the time t it takes an electron to arrive at location z can be obtained from v_z as follows

$$
t(z) = t(0) + \int \frac{dz}{v_z} = \bar{t}(z) + \frac{K^2}{8k_u\gamma^2 c}\sin(2k_u z), \tag{4}
$$

where $\bar{t}(z) = t(0) + \frac{z}{c}\left(1 + \frac{1+K^2/2}{2\gamma^2}\right)$ represents average particle time.

III. FEL PENDULUM EQUATIONS

We proceed to derive the FEL pendulum equations for electrons. As we already discussed in section II, the electron is under the unfluence of both undulator and radiation fields. The magnetic field itself does no work to the electron so energy exchange occurs only between electrons and radiation fields. Assuming the seed radiation spectrum consists of all harmonics, we can model the radiation fields as a collection of discrete electromagnetic waves given by

$$
\boldsymbol{E}(z;t) = \hat{x}\sum_h E_0^h \cos(k_h z - \omega_h t + \phi_h), \tag{5}
$$

where the waves are polarized in $x-$direction and co-propagating with the electron beam. E_0^h is the field amplitude and ϕ_h is the phase of a harmonic h with wavenumber k_h and frequency ω_h. The rate of energy transfer from the radiation spectrum of equation (5) to an electron is given by the incremental work $W = \boldsymbol{F}.\boldsymbol{v} = -e\boldsymbol{E}.\boldsymbol{v}$. Substituting the electron velocity from equation (3a) and field amplitude from equation (5), we obtain

$$
W = -e\boldsymbol{E}.\boldsymbol{v} = \frac{eKc}{\gamma}\cos(k_u z)\sum_h E_0^h \cos(k_h z - \omega_h t + \phi_h). \tag{6}
$$

Since the rate of energy change for an electron is $d(\gamma mc^2)/dt$, we obtain

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{eK}{\gamma mc}\sum_h E_0^h \cos(k_u z)\cos(k_h z - \omega_h t + \phi_h) \\ &= \frac{eK}{2\gamma mc}\sum_h E_0^h \left(\cos\left[(k_u + k_h)z - \omega_h t + \phi_h\right] + \cos\left[(k_u - k_h)z + \omega_h t - \phi_h\right]\right). \qquad (7)\end{aligned}$$

The argument in the first cosine term give rise to particle's slowly varying pondermotive phase in the presence of combined radiation and undulator fields whereas the second cosine term give rise to fast oscillations which tend to average to zero. We note that there is a slowly varying pondermotive phase (θ^h) associated with each harmonic h and the overall pondermotive phase of an electron is the result of contribution from all harmonics i. e. $\theta = \sum_h \theta^h$. Since the particle arrival time t is fastly varying component as defined in (4), we assign the average particle time $\bar{t}$ to define a slowly varying phase ψ^h given by

$$\psi^h = (k_h + k_u)z - \omega_h \bar{t}. \qquad (8)$$

Then, we differentiate on both sides with respect to $\bar{t}$ to get

$$\frac{d}{d\bar{t}}\psi^h = \frac{d\psi^h}{dz}\bar{v}_z = (k_h + k_u)\bar{v}_z - \omega_h;$$

since $\omega_h = ck_h$ and $\bar{v}_z = c\left[1 - \frac{1+K^2/2}{2\gamma^2}\right]$, after rearranging and expanding for $\gamma \gg 1$, we get

$$\begin{aligned}\frac{d\psi^h}{dz} &= k_u + k_h - \frac{ck_h}{\bar{v}_z} \\ &= k_u + k_h - k_h\left[1 + \frac{1+K^2/2}{2\gamma^2}\right] \\ &= k_u\left[1 - \frac{k_h}{k_u}\frac{1+K^2/2}{2\gamma^2}\right]. \qquad (9)\end{aligned}$$

The resonant condition for a harmonic radiation emission in planar undulator is given by

$$\frac{k_h}{k_u} = \frac{2h\gamma_r^2}{1+K^2/2}, \qquad (10)$$

where γ_r is the resonant Lorentz factor associated with the resonant energy. Assuming the electron's energy differs from the reference (resonant) energy by $\eta = (\gamma - \gamma_r)/\gamma_r$, we can rewrite equation (9) as

$$\begin{aligned}\frac{d\psi^h}{dz} &= k_u\left[1 - h(1+\eta^h)^{-2}\right] \\ \frac{d\psi^h}{dz} &\approx k_u(1-h) + 2hk_u\eta^h \\ \frac{d\theta^h}{dz} &\approx 2hk_u\eta^h. \qquad (11)\end{aligned}$$

Here we have introduced the reduced phase notation $\theta^h = \psi^h + (h-1)k_u z$ to track slowly changing dependence on particle's energy offset from the FEL resonance condition. Equation (11) is one of the equations that describes pendulum like behavior of the longitudinal electron motion in FEL.

Now we derive second equation relevant to the pendulum behavior. Using the definition of energy deviation $\eta = (\gamma - \gamma_r)/\gamma_r$, we can rewrite equation (7) as

$$\begin{aligned}\frac{d\eta}{dt} &= \frac{eK}{\gamma_r \gamma mc}\sum_h E_0^h \cos(k_u z)\cos(k_h z - \omega_h t + \phi_h) \\ &= \frac{eK}{2\gamma_r \gamma mc}\sum_h E_0^h (e^{ik_u z} + e^{-ik_u z})\cos(k_h z - \omega_h t + \phi_h). \qquad (12)\end{aligned}$$

From the definition of pondermotive phase and average particle time expression of equation (4), we have

$$k_h z - \omega_h t = \theta^h - hk_u z - \frac{k_h}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z).$$

Upon substituting the above expression and expanding cosine in terms of complex exponential, equation (12) becomes

$$\frac{d\eta^h}{dt} = \frac{eKE_0^h}{4\gamma_r\gamma mc}(e^{ik_u z} + e^{-ik_u z})\left(e^{i\theta^h - ihk_u z + i\phi_h}\exp\left[-i\frac{k_h}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z)\right] + c.c.\right).$$

Using Jacobi-Anger identity $e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\varphi}$, we get

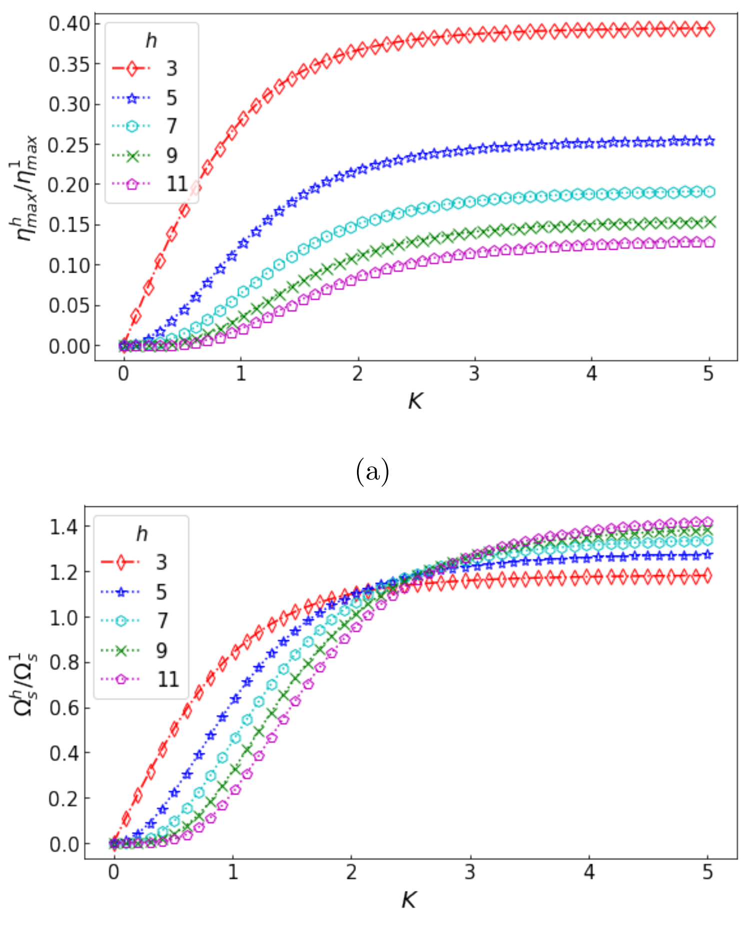
$$\begin{aligned}&= \frac{eKE_0^h}{4\gamma_r\gamma mc}e^{i(\theta^h+\phi_h)}\sum_{n=-\infty}^{\infty} J_n\left(\frac{k_h}{k_u}\frac{K^2}{8\gamma^2}\right)\left[e^{i(1-h-2n)k_u z} + e^{-i(1+h+2n)k_u z}\right] + c.c \\ &= \frac{eKE_0^h}{4\gamma_r\gamma mc}e^{i(\theta^h+\phi_h)}\left[J_{-\left(\frac{h-1}{2}\right)}\left(\frac{k_h}{k_u}\frac{K^2}{8\gamma^2}\right) + J_{-\left(\frac{h+1}{2}\right)}\left(\frac{k_h}{k_u}\frac{K^2}{8\gamma^2}\right)\right] + c.c. \\ &= \frac{eKE_0^h[JJ]_h}{2\gamma_r\gamma mc}\cos(\theta^h + \phi_h). \qquad (13)\end{aligned}$$

We used the fact that overall energy change is due to contribution from each harmonic such that $\eta = \sum_h \eta^h$ and kept only the terms associated with factors $n = (1-h)/2$ and $n = -(1+h)/2$ which give rise to slowly changing phase. We apply two more approximations to obtain a simpler expression for equation (13). The first approximation is $v_z = dz/dt \approx c$ and the second approximation that $\gamma = \gamma_r(1+\eta) \approx \gamma_r$. This results in simplified version of energy equation given by

$$\frac{d\eta^h}{dz} = \frac{eKE_0^h[JJ]_h}{2\gamma_r^2 mc^2}\cos(\theta^h + \phi_h) = \frac{\epsilon_h}{2k_u L_u^2}\cos(\theta^h + \phi_h), \qquad (14)$$

where $\epsilon_h = \frac{eE_0^h K[JJ]_h}{\gamma_r^2 mc^2} k_u L_u^2$ is the dimensionless field strength. While this treatment allows us to gather cumulative effect of all present harmonics on electron longitudinal phase space, this becomes indispensable for FEL analysis and deriving gain formula in the presence of a single harmonic h. Now we can also express the Bessel factor of (13) in a more convenient form by using the resonance condition (eq. (10)) and Bessel function identity $J_{-n}(x) = (-1)^n J_n(x)$ as

$$
\begin{aligned}
[JJ]_h &= J_{-\left(\frac{h-1}{2}\right)}\left(\frac{hK^2}{4+2K^2}\right) + J_{-\left(\frac{1+h}{2}\right)}\left(\frac{hK^2}{4+2K^2}\right) \\
&= (-1)^{(h-1)/2}\left[J_{\left(\frac{h-1}{2}\right)}\left(\frac{hK^2}{4+2K^2}\right) - J_{\left(\frac{1+h}{2}\right)}\left(\frac{hK^2}{4+2K^2}\right)\right]. \qquad (15)
\end{aligned}
$$



(a)

(b)

FIG. 1. Ratios of (a) maximum bucket height in the presence of harmonic h with respect to that with fundamental and (b) oscillation frequency near stable points in harmonic h potential with respect to that in fundamental potential.

In the presence of a constant electric field of a harmonic h (i. e. ϵ_h = constant), θ^h and η^h become conjugate variables of the constant of motion given by

$$H_h = hk_u\eta^2 + \frac{\epsilon_h}{2k_uL_u^2}\left[\sin(\theta^h + \phi_h) - \sin(\phi_h)\right]. \tag{16}$$

At the separatices ($\eta^h = 0$ and $\theta^h = \pm\pi$), $H_h = \frac{|\epsilon_h|}{k_uL_u^2}\sin\phi_h$ which means

$$\eta^h = \pm\frac{\sqrt{|\epsilon_h|}}{k_uL_u\sqrt{h}}\sqrt{\sin(\theta^h + \phi_h)\cos(\phi_h)} = \pm\eta^h_{\mathrm{max}}\sqrt{\sin(\theta^h + \phi_h)\cos(\phi_h)},$$

with $\eta^h_{\mathrm{max}} = \sqrt{|\epsilon_h|}/(k_uL_u\sqrt{h})$. The motion outside the separatices are unbounded and unidirectional whereas the particle exhibit periodic behavior about the stable orbit inside these separatices which corresponds to the vibrational motion of the pendulum (see Ref. [8] for more details). The latter region is often referred to as pondermotive bucket in which particles are trapped for a constant field. The maximum height of this bucket is η^h_{max} and is directly proportional to the square root of Bessel function factor and inversly proportional to square root of the harmonic number h. Figure 1a shows the ratio of maximum bucket height for different harmonic with respect to that of the fundamental radiation with $E_0^h = E_0^1$ for K ranging from 0 to 5. It is clear that bucket height decreases with increasing harmonic number. For each harmonic number, the maximum bucket height keeps on increasing until a threshold value for K is reached, beyond which the the bucket height remains fixed. $K_{\mathrm{thres}} \approx 3$ for $h < 10$ as shown in Fig. 1a.

The oscillatory motion in the pondermotive bucket occurs at frequency that is dependent on the energy H_h. However, the motion close to the stable fixed point are similar to that of a simple harmonic oscillator for small phase ($|\theta^h| \ll 1$). In this case, the oscillation wavenumber is given by

$$\Omega_s = \frac{\sqrt{h|\epsilon_h|}}{L_u}, \tag{17}$$

also known as the synchrotron wavenumber. The corresponding synchrotron period of the particle in the pondermotive bucket is

$$T_s \equiv \frac{2\pi}{\Omega_s} = \frac{2\pi L_u}{\sqrt{h|\epsilon_h|}}. \tag{18}$$

Figure 1b shows the ratio of synchrotron frequency of a particle in the bucket formed by harmonic h to that in the bucket formed by the fundamental radiaition with same electric field amplitude. The frequency increases with increasing undulator strength and become greater for potentials formed by harmonic $h > 1$ compared to frequencies of oscillation in a bucket formed by fundamental radiation. There exists a threshold undulator strength after which the frequency of oscillation remains more or less constant for a given harmonic h. These peculiar behaviors in figure 1 may be formulated in terms of scaling laws, however is beyond the scope of this article.

Following discussion on sections 3.3.2 and 7.1.1 of Ref. [8], we can roughly estimate the saturated radiation power in low-gain scenarios in single pass and oscillator based FELs. For a harmonic h, the maximum saturated power in a single pass FEL would be given by $P^{\rm s}_{\rm sat} \approx P_{\rm beam}/(\sqrt{h}N_u)$, where N_u is the total undulator periods and $P_{\rm beam}$ is the power of the electron beam. For cavity based oscillator, the saturated power becomes $P^{\rm osc}_{\rm sat} = P^{s}_{\rm sat}/(1-R) \approx P_{\rm beam}/[(1-R)\sqrt{h}N_u]$. Here R is the net power reflectivity of the optical cavity forming the oscillator. It is clear that the saturated power of a harmonic h gets reduced by the square root of the harmonic number itself when compared to the fundamental. This means that the saturated power of an FEL operating at harmonic h is much lower than that for fundamental radiation, which sets the upper limit on maximum achievable radiation power in an operating low-gain FEL.

IV. MAXWELL EQUATION

Now we switch our attention to the emitted radiation as a result of wiggle motion of electrons in the undulator. The electric field amplitude of the emitted radiation due to the motion of an electron beam current in the undulator can be obtained by solving Maxwell equation. The slowly varying envelope approximation to paraxial wave equation results in the angular field representation given by (equation (3.59) of Ref.[8])

$$\left[\frac{\partial}{\partial z} + \frac{ik}{2}\boldsymbol{\phi}^2\right]\tilde{\mathcal{E}}_\omega(\boldsymbol{\phi}; z) = \sum_{j=1}^{N_e} \frac{e(v_{xj}/c - \phi_x)}{4\pi\epsilon_0 c\lambda^2} e^{ik[ct_j(z)-z]} \int d\boldsymbol{x} e^{-ik\boldsymbol{\phi}.\boldsymbol{x}} \delta(\boldsymbol{x} - \boldsymbol{x}_j), \qquad (19)$$

where ϵ_0 is the free-space permittivity and λ is the radiation wavelength. In one-dimension, we can use the approximation $\delta(\boldsymbol{x}-\boldsymbol{x}_j) \to \mathcal{A}_{tr}^{-1}$ with $\mathcal{A}_{tr}$ being the transverse area. From now on, we will be using shorthand notation $\boldsymbol{\phi}_\perp = \boldsymbol{\phi}$ for all vectors since z is an independent variable; in other words $\boldsymbol{\phi} = (\phi_x, \phi_y)$. Also, $\int d\boldsymbol{x} e^{-ik\boldsymbol{\phi}.\boldsymbol{x}} \delta(\boldsymbol{x} - \boldsymbol{x}_j) \to \mathcal{A}_{tr}^{-1} \int d\boldsymbol{x} e^{-ik\boldsymbol{\phi}.\boldsymbol{x}} = \lambda^2 \delta(\boldsymbol{\phi})/\mathcal{A}_{tr}$. We complete the 1D limit by defining the one dimensional angular electric field

$\tilde{\mathcal{E}}_\omega(\boldsymbol{\phi}; z) = \tilde{E}_\omega(z)\delta(\boldsymbol{\phi}) = \tilde{E}_\nu(z)\delta(\boldsymbol{\phi})$. Upon substituting the velocity from equation (3a) and integrating over angles, equation (19) becomes

$$\frac{\partial}{\partial z}\tilde{E}_\nu(z) = -\frac{eK\cos(k_u z)}{4\pi\epsilon_0 c\mathcal{A}_{tr}}\sum_{j=1}^{N_e}\frac{e^{ik[ct_j(z)-z]}}{\gamma_j}. \tag{20}$$

The slowly varying radiation field envelope requires finding slowly varying current which we can do by substituting $t(z)$ with the average particle time from equation (4). For simplicity, we will assume that only harmonic h is the dominating field providing the pondermotive potential in conjuction with the undulator field so that $\theta \equiv \theta^h$. This means

$$\begin{aligned}
k[ct_j(z) - z] &= k\left[c\bar{t}_j(z) + \frac{K^2}{8k_u\gamma^2}\sin(2k_u z)\right] - kz \\
&= \frac{\nu}{h}(\omega_h\bar{t}_j(z) - k_h z) + \frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z) \\
&= -\frac{\nu}{h}(\theta_j^h - hk_u z) + \frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z) \\
&= -\frac{\nu}{h}\theta_j^h + pk_u z + \Delta\nu k_u z + \frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z), \qquad (21)
\end{aligned}$$

where we have substituted $\nu = k/k_1 = \omega/\omega_1$ and $\Delta\nu = \nu - p$ as a factor of deviation from harmonic p, where p is not necessarily equal to h. In fact, $p \neq h$ would allow us to explore the potential of achieving harmonic lasing via non-overlapping harmonic seeding. Now equation (20) reduces to

$$\frac{\partial}{\partial z}\tilde{E}_\nu(z) = -\frac{eK\cos(k_u z)e^{ipk_u z}\exp\left[i\frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z)\right]}{4\pi\epsilon_0 c\mathcal{A}_{tr}}\sum_{j=1}^{N_e}\frac{e^{-i\nu\theta_j^h/h}e^{i\Delta\nu k_u z}}{\gamma_j}. \tag{22}$$

Again applying the Jacobi-Anger identity and keeping only the terms that give rise to pondermotive phase, we can write

$$\begin{aligned}
\cos(k_u z)e^{ipk_u z}\exp\left[i\frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\sin(2k_u z)\right] &= \frac{1}{2}\sum_{n=-\infty}^{\infty}J_n\left(\frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\right)\left[e^{i(1+p+2n)k_u z} + e^{-i(1-p-2n)k_u z}\right] \\
&= \frac{1}{2}\left[J_{-\left(\frac{1+p}{2}\right)}\left(\frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\right) + J_{-\left(\frac{p-1}{2}\right)}\left(\frac{k_\nu}{k_u}\frac{K^2}{8\gamma^2}\right)\right] \\
&= \frac{1}{2}[JJ]_p.
\end{aligned}$$

In order to obtain the last expression, we used $\nu \approx p$ for $\Delta\nu \ll 1.$ and $\gamma \approx \gamma_r$ for $\eta \ll 1$. Since we want to connect physical field to the spectral field, we define $E_\nu(z) = \omega_1 e^{-i\Delta\nu k_u z}\tilde{E}_\nu(z)$.

Finally, equation (22) takes the form

$$\left[\frac{\partial}{\partial z} + i\Delta\nu k_u\right] E_\nu(z) = -\frac{ek_1K[JJ]_p}{8\pi\epsilon_0\gamma_r\mathcal{A}_{tr}}\sum_{j=1}^{N_e} e^{-i\frac{\nu}{h}\theta_j^h} = -\frac{\kappa_p n_e}{N_{\lambda 1}}\sum_{j=1}^{N_e} e^{-i\frac{\nu}{h}\theta_j^h}. \tag{23}$$

Here $\kappa_p = \frac{eK[JJ]_p}{4\epsilon_0\gamma_r}$, $N_{\lambda 1} = \lambda_1 dN_e/dz = \lambda_1 I/ec$, and $n_e = \frac{dN_e/dz}{\mathcal{A}_{tr}}$. We can obtain the time domain wave equation from equation (23) by applying Fourier transform i. e. $E(\theta; z) = \int d\nu e^{i\Delta\nu\theta/h} E_\nu(z)$ to obtain

$$\left[\frac{\partial}{\partial z} + hk_u\frac{\partial}{\partial\theta}\right] E(\theta; z) = -\frac{2\pi\kappa_p n_e}{N_{\lambda h}}\sum_{j=1}^{N_e} e^{-ip\theta/h}\delta(\theta - \theta_j^h), \tag{24}$$

where $N_{\lambda h} = N_{\lambda 1}/h = \lambda_h dN_e/dz$ is the number of electrons contained within harmonic wavelength λ_h.

V. COUPLED MAXWELL-KLIMONTOVICH EQUATIONS

Various aspects of the SASE process as well as low-gain FELs can be understood by adopting the microscopic description of electron beam given by Klimontovich distribution function [3, 4]. In one-dimension, this discrete distribution function is written as

$$F(\theta, \eta; z) = \frac{k_1}{dN_e/dz}\sum_{j=1}^{N_e}\delta[\theta - \theta_j(z)]\delta[\eta - \eta_j(z)]. \tag{25}$$

Here $dN_e/dz = I/ec$ is electron line density for beam current I and N_e is the total number of electrons in the beam. Under the assumption that FEL interaction is a perturbative process, the Klimontovich distribution function can be expanded using a coasting beam approximation. Similar technique of perturbative expansions are also applied in plasma physics (see Ref. [9] for instance). In this expansion, the distribution is separated into the smooth background part and a perturbative part, where the perturbative part contains the shot noise and bunching like features as follows

$$F(\theta, \eta; z) = \bar{F}(\eta; z) + \delta F(\theta, \eta; z). \tag{26}$$

Here the smooth background function representated by $\bar{F}(\eta; z)$ is independent of phase and satisfies $\int d\eta\bar{F}(\eta; z) = 1$. The continuity equation for the Klimontovich distribution function $dF(\theta, \eta; z)/dz = 0$ can be broken into two parts as follows

$$\left[\frac{\partial}{\partial z}\bar{F} + \frac{d\eta}{dz}\frac{\partial}{\partial z}\delta F\right] + \left[\frac{\partial}{\partial z}\delta F + \frac{d\theta}{dz}\frac{\partial}{\partial\theta}\delta F + \frac{d\eta}{dz}\frac{\partial}{\partial\eta}\bar{F}\right] = 0. \tag{27}$$

The first bracket is for the terms that vary slowly along the bunch and also attributes nonlinear harmonic contributions whereas the second bracket groups terms with fluctuations from FEL interaction dominated by harmonics. Since these brackets indicate processes that occur at different time scales, each bracket should separately vanish to satisfy continuity condition [8]. Our interest lies in obtaining an equivalent expression for the second bracket in frequency space. In order to do so, we introduce the frequency representation of the distribution function as

$$F_\nu(\eta;z) = \frac{1}{2\pi}\int d\theta e^{-i\nu\theta/h}F(\theta,\eta;z) = \frac{1}{N_{\lambda 1}}\sum_{j=1}^{N_e} e^{-i\nu\theta_j/h}\delta(\eta-\eta_j). \tag{28}$$

This implies $\delta F(\theta,\eta;z) = \frac{1}{h}\int d\nu e^{i\nu\theta/h}F_\nu(\eta;z)$. Using the alternative expression for energy change equation of equation (11) given by

$$\frac{d\eta^h}{dz} = \chi_h \int d\nu E_\nu(z) e^{i\nu\theta^h/h} + c.c., \tag{29}$$

where $\chi_h = \frac{eK[JJ]_h}{2mc^2\gamma_r^2}$, we obtain the fourier transform version of second bracket in equation (27) to be

$$\left[\frac{\partial}{\partial z} + 2i\nu k_u \eta\right] F_v(\eta;z) = -h\chi_h E_v(z)\frac{\partial \bar{F}(\eta;z)}{\partial \eta}. \tag{30}$$

Here we are assuming that the fluctuations are induced by single harmonic h and $\eta \equiv \eta^h$. Using the definition of F_ν from equation (28) and since $\nu \approx p$ for $\Delta\nu \ll 1$ (i. e. $\kappa_\nu \equiv \kappa_p$), equation (23) for field evolution of harmonic p when seeded by harmonic h becomes

$$\left[\frac{\partial}{\partial z} + i\Delta\nu k_u\right] E_\nu(z) = -\kappa_p n_e \int d\eta F_\nu(\eta;z). \tag{31}$$

The evolution of radiation field and and electron beam fluctuations in the FEL can be now solved using equations (30) and (31).

VI. PERTURBATIVE SOLUTION FOR LOW-GAIN

The coupled equations (30) and (31) can be written in integral form as follows

$$E_\nu(z) = e^{-i\Delta\nu k_u z}\left[E_\nu(0) - \kappa_p n_e \int_0^z ds\ e^{i\Delta\nu k_u s}\int d\eta F_\nu(\eta;s)\right], \tag{32}$$

$$F_\nu(\eta;z) = e^{-i2\nu k_u \eta z}\left[F_\nu(\eta;0) - h\chi_h \int_0^z ds\ e^{i2\nu k_u \eta s} E_\nu(s)\frac{\partial \bar{F}(\eta;s)}{\partial \eta}\right]. \tag{33}$$

The solution for electric field given by (32) can be alternatively expressed in terms of the electric field at the undulator center as follows

$$E_\nu(z) = \mathcal{G}\left(z - \frac{L_u}{2}\right) E_\nu\left(\frac{L_u}{2}\right) - \kappa_p n_e \int_0^z ds\ \mathcal{G}(z-s) \int d\eta F_\nu(\eta; s)$$

where $\mathcal{G}(z) = e^{-i\Delta\nu k_u z}$ is the homogenous solution. Substituting F_ν from equation (33), we get

$$\begin{aligned} E_\nu(L_u) &= \mathcal{G}\left(\frac{L_u}{2}\right) E_\nu\left(\frac{L_u}{2}\right) - \kappa_p n_e \int_0^{L_u} dz\ \mathcal{G}(L_u - z) \int d\eta e^{-i2\nu k_u \eta z} F_\nu(\eta; 0) \\ &+ h\chi_h \kappa_p n_e \int_0^{L_u} dz\ \mathcal{G}(L_u - z) \int d\eta e^{-i2\nu k_u \eta z} \int_0^z ds e^{i2\nu k_u \eta s} E_\nu(s) \frac{\partial \bar{F}(\eta; s)}{\partial \eta}, \end{aligned}$$

where the first term appears from the input coherent radiation, the second term corresponds to the spontaneous undulator radiation and the third term is the result of FEL interaction between the electron beam and radiation field. Ideally we would solve iteratively to find the evolving radiation field in the undulator. However, appropriate approximation for $E_\nu(s) = \mathcal{G}\left(s - \frac{L_u}{2}\right) E_\nu\left(\frac{L_u}{2}\right)$ can be used for weak interaction between electron beam and radiation field. In this case, the radiation electric field can be conveniently expressed as

$$\begin{aligned} E_\nu(L_u) &= \mathcal{G}\left(\frac{L_u}{2}\right) E_\nu\left(\frac{L_u}{2}\right) - \kappa_p n_e \mathcal{G}\left(\frac{L_u}{2}\right) \int_0^{L_u} dz \int d\eta\ U_\nu(\eta; z) F_\nu(\eta; 0) \\ &+ h\chi_h \kappa_p n_e \mathcal{G}\left(\frac{L_u}{2}\right) E_\nu\left(\frac{L_u}{2}\right) \int d\eta \int_0^{L_u} dz\ U_\nu(\eta; z) \int_0^z ds\ U_\nu^*(\eta; s) \frac{\partial \bar{F}(\eta; L_u/2)}{\partial \eta}. \end{aligned} \tag{34}$$

Here we introduced 1D undulator field $U_\nu(\eta; z) = \exp\left[-i\Delta\nu k_u \left(\frac{L_u}{2} - z\right) - i2\nu\eta k_u z\right]$ and the transformation of $\bar{F}(\eta; s) \rightarrow \bar{F}(\eta; L_u/2)$ follows naturally with the field transformation. Since the field is complex in nature, the gain of the field amplitude is complex. Therefore, it is more convenient to obtain power gain by computing absolute square of field amplitude in the above expression. Absolute square of the first term gives input power whereas square of the second term gives spontaneous undulator radiation power. The cross terms involving the second term (spontaneous radiation) sums over all particles phases leading to zero. The lowest order power amplification appears from the cross terms involving the first and third term. Hence, 1D FEL power gain is given by

$$\begin{aligned} G &= \frac{P_{out} - P_{in}}{P_{in}} \\ &= h\chi_h \kappa_p n_e \int d\eta \int_0^{L_u} dz \int_0^{L_u} ds U_\nu(\eta; z) U_\nu^*(\eta; s) \frac{\partial \bar{F}(\eta)}{\partial \eta}. \end{aligned} \tag{35}$$

Further simplification comes from substituting $\bar{z} = z - L_u/2$ and $\bar{s} = s - L_u/2$. This changes the limits of z and s integrals to $-L_u/2$ and $L_u/2$ from 0 and L_u respectively. Moreover, the product of the undulator field takes the following form

$$U_\nu(\eta; z)U_\nu^*(\eta; s) = U_\nu(\eta; \bar{z})U_\nu^*(\eta; \bar{s}) = \exp\left[-i(2\nu\eta - \Delta\nu)k_u(\bar{z} - \bar{s})\right].$$

For a gaussian electron beam with rms energy spread of σ_η and centered at energy η_0, $\bar{F}(\eta) = \frac{e^{-(\eta-\eta_0)^2/2\sigma_\eta^2}}{\sqrt{2\pi}\sigma_\eta}$ and $\frac{\partial \bar{F}(\eta)}{\partial \eta} = -\frac{\eta-\eta_0}{\sigma_\eta^2}\bar{F}(\eta)$, the gain formula reduces to

$$G = h\chi_h\kappa_p n_e \int_{-L_u/2}^{L_u/2} dz \int_{-L_u/2}^{L_u/2} ds\ \mathcal{I}_\eta(z; s), \tag{36}$$

where

$$\begin{aligned}
I_\eta(z; s) &= \int d\eta \frac{\partial \bar{F}}{\partial \eta} U_\nu(\eta; z)U_\nu^*(\eta; s) \\
&= -\frac{1}{\sqrt{2\pi\sigma_\eta^3}} \int d\eta(\eta - \eta_0)e^{-(\eta-\eta_0)^2/2\sigma_\eta^2} e^{-i2(\nu\eta-\Delta\nu/2)k_u(z-s)} \\
&= 2i\nu k_u(z - s)e^{-i2(\nu\eta_0-\Delta\nu/2)k_u(z-s)} \exp\left[-2(k_u\nu\sigma_\eta(z - s))^2\right].
\end{aligned}$$

We got rid of bars over z and s for convenience. Finally, the gain formula reduces to

$$G = 2\nu hk_uL_u^3\chi_h\kappa_p n_e \int_{-1/2}^{1/2} dz \int_{-1/2}^{1/2} ds(z - s)\sin\left[2x_0(z - s)\right] e^{-2[2\pi N_u\nu\sigma_\eta(z-s)]^2}, \tag{37}$$

where $x_0 = 2\pi N_u(\nu\eta_0 - \Delta\nu/2)$. We now replace ν with p without breaking any assumptions of low-gain analysis. Following Colson, we define $j_{C,p,h} = 4phk_uL_u^3\chi_h\kappa_p n_e = hj_{C,p}[JJ]_p/[JJ]_h$ [2] (see page 108 of Ref. [8]). Hence, 1D gain formula for low-gain FEL for harmonic p upon seeding by harmonic h is given by

$$\begin{aligned}
G &= \frac{j_{C,p,h}}{2} \int_{-1/2}^{1/2} dz \int_{-1/2}^{1/2} ds(z - s)\sin\left[2x_0(z - s)\right] e^{-2[2\pi N_u p\sigma_\eta(z-s)]^2} \\
&= h\frac{[JJ]_h}{[JJ]_1}G_1.
\end{aligned} \tag{38}$$

The above formula suggests that the FEL gain at harmonic p increases by a factor $h[JJ]_h/[JJ]_1$ when the seeding laser is switched from fundamental to harmonic h. G_1 represents FEL gain for harmonic p upon seeding by fundamental radiation. We note that this extra factor of contribution comes from the phase rate equation (11) where the harmonic h increases phase change by a factor of h compared to that of fundamental.

Since the ratio of gain for harmonic seeding to fundamental seeding depends simply on the ratio of corresponding Bessel function factors, understanding the behaviors of $[JJ]_h$ becomes crucial for gain optimization. Figure 2a shows the Bessel function factors for odd harmonics plotted against K for h from 3 to 13. It is clear that alternate harmonics change signs and become negative. For instance, $h = 1, 5, 9, 13$ have values greater than zero whereas $h = 3, 7, 11$ have values less than zero for the range of K as shown in Fig. 2a. Since the gain formula of equation (38) contains a corresponding Bessel function factor $[JJ]_p$ associated with the desired harmonic p, gain amplification of harmonic p requires seeding with harmonic h that follows $h = ..., p-4, p, p+4, ...$. Likewise, negative gain is achieved when $h = ..., p-2, p+2, ...$; this would suppress the emission of harmonic p. Figure 2b shows the ratio of gain upon harmonic seeding to fundamental seeding versus K from 0 to 5 for harmonic h between 3 and 13. For $K < 2$, harmonic seeding has amplification less than 1. However, gain amplification of greater than 1.0 is easily achieved for all h when $K > 3$. The higher harmonic seems to provide the maximum amplification at higher K values. For instance, $h = 13$ amplifies gain by a factor of 2 for $K \approx 5$. Harmonic $h = 5$ amplifies gain by factor by 1.5 starting at around $K \approx 4$ and this amplification remains almost fixed with increasing K afterwards. This indicates that an optimal value of K exists for optimal gain amplification with harmonic seeding h. On the other hand, gain attenuation can easily be achieved by working with $K < 2$ as pointed out earlier (see Fig. 2).

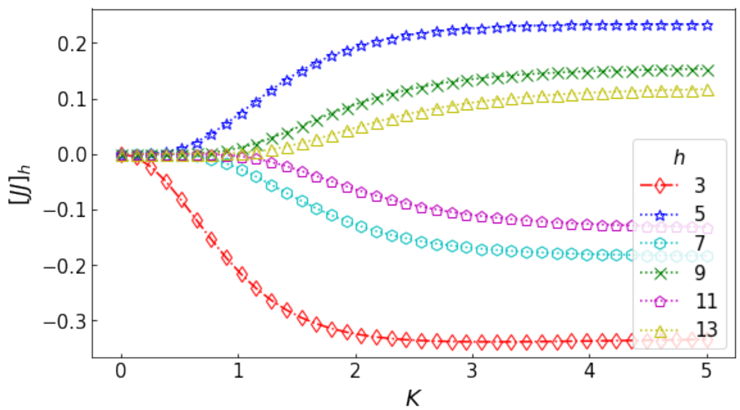
VII. LAPLACE TRANSFORM FOR ARBITRARY GAIN

A complete solution of the coupled Maxwell-Klimontovich equations (30) and (31) can be obtained by using Laplace transform [10, 11] given by

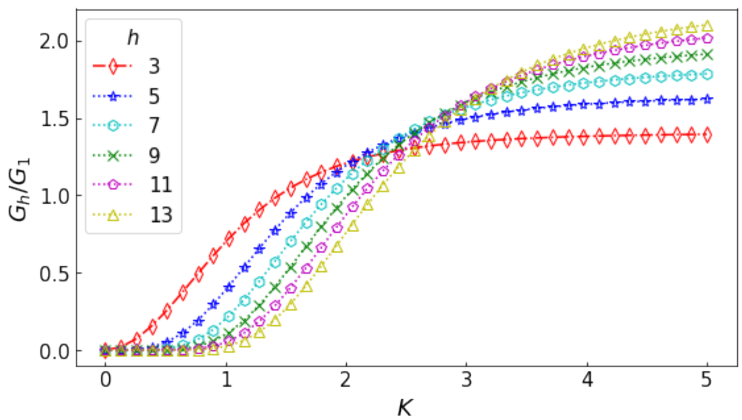
$$S_{\nu,\mu} = \int_0^\infty dz\; e^{i2\mu\rho k_u z} S_\nu(z). \tag{39}$$

Here S is dummy representation for E and F. This representation allows to obtain solution for $S_\nu(z)$ using inverse Laplace transform given by

$$S_\nu(z) = -\frac{\rho k_u}{\pi} \oint d\mu\; e^{-i2\mu\rho k_u z} S_{\nu,\mu}. \tag{40}$$



(a)



(b)

FIG. 2. Plot of (a) Bessel function factor $[JJ]_h$ and (b) gain ratio from equation (38) for harmonic from 3 to 13 versus undulator deflection parameter K.

ρ is the FEL scaling parameter that gets introduced when we attempt to make FEL equations unitless and is given by

$$\rho = \left[\frac{n_e \kappa_p \chi_h}{4k_u^2}\right]^{1/3} = \left(\frac{e^2 K^2 [JJ]_p [JJ]_h n_e}{32\epsilon_0 \gamma_r^2 m c^2 k_u^2}\right)^{1/3} \tag{41}$$

After careful calculation steps and substituting for $F_\nu(\eta, 0)$ from equation (28), the electric field amplitude takes the form

$$E_\nu(z) = \oint \frac{d\mu}{2\pi i} \frac{e^{-i2\mu\rho k_u z}}{D(\mu)} \left[E_\nu(0) + \frac{i\kappa_p n_e}{2k_u \rho N_{\lambda 1}} \sum_{j=1}^{N_e} \frac{e^{-i\nu\theta_j^h/h}}{(\frac{\mu\eta_j}{\rho} - \mu)}\right], \tag{42}$$

where the dispersion function $D(\mu)$ is given by

$$D(\mu) \equiv \mu - \frac{\Delta\nu}{2\rho} - h\nu \int d\eta \, \frac{\bar{F}(\eta)}{\left(\frac{\nu\eta}{\rho} - \mu\right)^2} \tag{43}$$

The radiation evolution in a FEL is mainly dictated by the poles of $1/D$, which can be obtained from the roots of $D(\mu) = 0$. In the limits of vanishing energy spread i. e. $\bar{F}(\eta) \rightarrow \delta(\eta)$, $D(\mu) = 0$ becomes

$$\mu - \frac{\Delta\nu}{2\rho} - \frac{h\nu}{\mu^2} = 0$$
$$\mu^2 \left(\mu - \frac{\Delta\nu}{2\rho}\right) = h\nu. \tag{44}$$

For the FEL resonance condition $\Delta\nu = 0$ and $\mu^3 = hp$ which has three solutions $\mu = (hp)^{1/3}\left[1, \frac{i\sqrt{3}-1}{2}, \frac{1-i\sqrt{3}}{2}\right]$. The first solution being the real solution gives rise to oscillatory behavior of input field. The imaginary parts of the second and third solutions result in exponentially decaying and growing modes respectively as per equation (42). In the exponentially growth regime, the radiation power grows as $P \propto e^{4\Im(\mu_3)\rho k_u z}$ along the undulator [8, 10], where $\Im(\mu_3) = (hp)^{1/3}\sqrt{3}/2$. The gain length in this exponential growth region is effectively given by

$$L_g = \frac{1}{(hp)^{1/3} 2\sqrt{3}\rho k_u} = \frac{\lambda_u}{(hp)^{1/3} 4\pi\sqrt{3}\rho} = \left(\frac{k_u}{6\sqrt{3} hp n_e \kappa_p \chi_h}\right)^{1/3}. \tag{45}$$

Hence, we have extended the gain length formula for SASE like processes that involves seeding with harmonic h for emitting harmonic p such that $h = ..., p-4, p, p+4, ...$. The ratio of gain length for harmonic p to fundamental when seeded by harmonic h is given by

$$\frac{L_{g,p}}{L_{g,1}} = \left(\frac{[JJ]}{p[JJ]_p}\right)^{1/3}. \tag{46}$$

Figure 3 shows the ratio of gain length for various harmonics from 5 to 25 plotted against undulator deflection parameter K upon seeding the FEL by harmonic h. For $K < 3$, the gain length for harmonic p is much higher than that for the fundamental, but for $K > 3$ the gain lengths for harmonic p approaches to that of the fundamental. This shows that it is feasible to achieve similar gain lengths as fundamental for higher harmonic in SASE based FELs with $K > 3$. Our model allows SASE analysis such as effects in gain due to detuning, energy spread, temporal coherence, saturation and so on for harmonic seeding as well as harmonic lasing. Since extensive analysis for fundamental radiation has been conducted already in Ref. [8], extension to harmonic is pretty straightforward. We welcome interested readers to carry out such studies and refer them to Chapter 4 of Ref. [8] and references within.

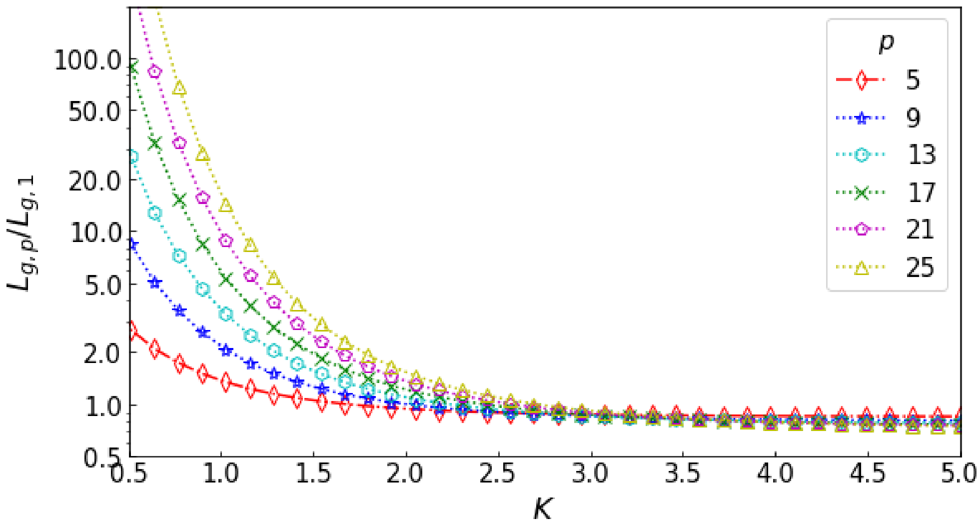


FIG. 3. Ratio of gain length for harmonic p to fundamental upon seeding by harmonic h such that $h = ..., p-4, p, p+4, ...$

VIII. 3D EXTENSION

Now we extend the FEL equations from sections II to V in 3D by adding transverse effects such as electron beam emittance and radiation diffraction.

A. FEL equations of motion

An average electron's longitudinal velocity is slightly reduced due to transverse focusing effect introduced by the average of the wiggle motion in the undulator field and betatron dynamics due to external lattice [7, 8]. Then, the correction to the average particle velocity can be written as

$$\begin{aligned}\bar{v}_z &= c\left[1 - \frac{1 + K^2/2}{2\gamma^2}\right] - \frac{1}{2}\overline{(v_x^2 + v_y^2)} \\ &= c\left[1 - \frac{1 + K^2/2}{2\gamma^2}\right] - cH_\perp,\end{aligned} \tag{47}$$

where we introduced the transverse constant of motion $H_\perp$ to represent the transverse betatron motion such that $H_\perp = \left[\boldsymbol{p}^2 + (\boldsymbol{k}_\beta.\boldsymbol{x})^2\right]/2$. Here $\boldsymbol{x} = (x, y)$ is the transverse position vector, $\boldsymbol{p} = (p_x, p_y)$ is the transverse angle vector and $\boldsymbol{k}_\beta$ represents the betatron focusing vector with each component having magnitude equal to the inverse of the average betatron function in that direction (i. e. $k_{\beta x,y} = 1/\bar{\beta}_{x,y}$). Also, the transverse degrees of freedom from $H_\perp =$ constant follow

$$\frac{dx}{dz} = p_x, \quad \text{and} \quad \frac{dp_x}{dz} = -k_{\beta x}^2 x; \tag{48a}$$

$$\frac{dy}{dz} = p_y, \quad \text{and} \quad \frac{dp_y}{dz} = -k_{\beta y}^2 x. \tag{48b}$$

Following equation (9) and the definition of pondermotive phase, the additional term in equation (47) appears in the rate of pondermotive phase change as follows

$$\begin{aligned}\frac{d\theta^h}{dz} &= k_h + k_u - \omega_h/\bar{v}_z \\ &= 2hk_u\eta^h - k_h H_\perp \\ &= 2hk_u\eta^h - \frac{k_h}{2}\left[p_x^2 + p_y^2 + k_{\beta x}^2 x^2 + k_{\beta y}^2 y^2\right]\end{aligned} \tag{49}$$

Equations (29), (48), and (49) collectively govern particle's motion in a FEL.

B. Maxwell equation

The radiation evolution in a FEL is governed by equation (19). In the 1D limit, we ignored ϕ_x contribution which appears from transverse derivative of charge. The relative magnitude of ϕ_x is $\phi_x \sim \sqrt{\Delta\lambda_1/\lambda_u}$, where Δ corresponds to bandwidth and $\Delta \sim 1/N_u$ for low-gain and $\Delta \sim \rho$ for high-gain [8]. Since $v_x/c \sim K/\gamma$, the contribution of ϕ_x can be effectively ignored for on-axis radiation. Now, for a 3D spatial field given by

$$E_\nu(\boldsymbol{x}; z) = \omega_1 e^{-i\Delta\nu k_u z}\int d\boldsymbol{\phi}\ \tilde{\mathcal{E}}(\phi; z)e^{ik\boldsymbol{\phi}.\boldsymbol{x}}, \tag{50}$$

equation (19) reduces to

$$\left[\frac{\partial}{\partial z} + i\Delta\nu k_u - \frac{i}{k_\nu}\nabla^2\right]E_\nu(\boldsymbol{x}; z) = -\frac{\kappa_p}{\lambda_1}\sum_{j=1}^{N_e} e^{-i\nu\theta_j^h/h}\delta[\boldsymbol{x} - \boldsymbol{x}_j], \tag{51}$$

which is the required expression for radiation field evolution of harmonic p close to $\nu = p+\Delta\nu$ for $\Delta\nu \ll 1$ upon seeding by harmonic h. The additional ∇^2 term compared to the 1D equivalent (eq. (23)) introduces diffraction effect as anticipated.

C. Coupled Maxwell-Klimontovich equations

The 3D version of Klimontovich distribution function is given by

$$F(\theta,\eta,\boldsymbol{x},\boldsymbol{p};z)=\frac{k_1}{n_e}\sum_{j=1}^{N_e}\delta[\theta-\theta_j(z)]\delta[\eta-\eta_j(z)]\delta[\boldsymbol{x}-\boldsymbol{x}_j]\delta[\boldsymbol{p}-\boldsymbol{p}_j], \tag{52}$$

where $n_e=dN_e/dV$ is the electron volume density as we introduced earlier in section IV. Then, the continuity equation takes the form

$$\left[\frac{\partial}{\partial z}+\frac{d\theta}{dz}\frac{\partial}{\partial\theta}+\frac{d\eta}{dz}\frac{\partial}{\partial\eta}+\frac{d\boldsymbol{x}}{dz}.\frac{\partial}{\partial\boldsymbol{x}}+\frac{d\boldsymbol{p}}{dz}.\frac{\partial}{\partial\boldsymbol{p}}\right]F=0. \tag{53}$$

Applying coasting beam approximation (as we did in section V), we can write the distribution function in terms of a smooth background and a perturbative term such that $F=\bar{F}+\delta F$. This time, the frequency representation takes the form

$$\begin{aligned}F_\nu(\eta,\boldsymbol{x},\boldsymbol{p};z)&=\frac{1}{2\pi}\int d\theta\ e^{-i\nu\theta/h}F(\theta,\eta,\boldsymbol{x},\boldsymbol{p};z)\\&=\frac{1}{n_e\lambda_1}\sum_{j=1}^{N_e}e^{-i\nu\theta_j/h}\delta[\eta-\eta_j]\delta[\boldsymbol{x}-\boldsymbol{x}_j]\delta[\boldsymbol{p}-\boldsymbol{p}_j].\end{aligned} \tag{54}$$

This allows us to write the 3D Maxwell equation for the radiation field evolution in terms of F_ν. Equation (51) becomes

$$\left[\frac{\partial}{\partial z}+i\Delta\nu k_u-\frac{i}{k_\nu}\nabla^2\right]E_\nu(\boldsymbol{x};z)=-\kappa_p n_e\int d\eta\int d\boldsymbol{p}\ F_\nu(\eta,\boldsymbol{x},\boldsymbol{p};z), \tag{55}$$

where we have assumed that the FEL is seeded with only one harmonic h. One could solve for the radiation field evolution near harmonic p by using the above expression with the information regarding F_ν. Likewise, the lowest order contribution to perturbation evolution in the FEL as a result of interaction with radiation is given by

$$\left[\frac{\partial}{\partial z}+\frac{d\theta}{dz}\frac{\partial}{\partial\theta}++\frac{d\boldsymbol{x}}{dz}.\frac{\partial}{\partial\boldsymbol{x}}+\frac{d\boldsymbol{p}}{dz}.\frac{\partial}{\partial\boldsymbol{p}}\right]\delta F=-\frac{d\eta}{dz}\frac{\partial\bar{F}}{\partial\eta}. \tag{56}$$

Upon taking the Fourier transform and making appropriate substitutions from FEL equations of motion, we obtain

$$\begin{aligned}&\left[\frac{\partial}{\partial z}+i\left(2\nu k_u\eta-k_\nu H_\perp\right)+\boldsymbol{p}.\frac{\partial}{\partial\boldsymbol{x}}-k_{\beta x}^2x\frac{\partial}{\partial p_x}-k_{\beta y}^2y\frac{\partial}{\partial p_y}\right]F_v(\eta,\boldsymbol{x},\boldsymbol{p};z)\\&=-h\chi_h E_v(z)\frac{\partial\bar{F}(\eta,\boldsymbol{x},\boldsymbol{p};z)}{\partial\eta},\end{aligned} \tag{57}$$

which is the desired evolution expression for F_ν. Equations (55) and (57) are coupled equations which contain the 3D solutions for weakly interacting radiation and electron distribution in a FEL.

IX. CONCLUSION

To sum up, we have derived governing equations for longitudinal FEL dynamics for harmonic lasing by harmonic seeding in the absence of transverse effects; then, we explored scenarios of harmonic lasing with harmonic seeding for low-gain and high-gain cases. We relaxed the 1D restriction to include transverse effects from electron motion and radiation diffraction to formulate 3D FEL equations. This model allows us to generalize FEL solutions and gain expressions for emission of any harmonic radiation. This model could find potential applications in design and analysis of FEL devices operating at various range of parameters.

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