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Redistribution of cooling rates for electron bunch with nonuniform density

S. Seletskiy*

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Abstract

In this paper we derive explicit formulas for the friction force and the cooling rates in non-magnetized electron coolers in the presence of redistribution of cooling decrements. In the derived formulas we take into account that the density of the electron bunch can be non-uniform.

1 Friction force

1.1 General remarks

Electron Cooling (EC) [1, 2] is a technique that allows increasing a 6-D phase space density of stored hadron beams.

In EC a beam of “cold” electrons co-propagate with a hadron beam with the same average velocity in a straight section of the storage ring, called a cooling section (CS). A hadron interacts with electrons in a CS via Coulomb force, which introduces dynamical friction [3] acting on each hadron. Over many revolutions in the accelerator the average friction reduces both the transverse and the longitudinal momentum spread of the ion bunch.

The friction force (in the beam frame) acting on an ion co-traveling with an electron bunch is given by [4, 5]:

$$\vec{F} = -\frac{4\pi e^4 Z_i^2}{m_e} \int \Lambda_C \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f(r_e, v_e) d^3 v_e \quad (1)$$

Here, e is the electron charge, $Z_i \cdot e$ is the ion charge, m_e is the mass of electron, \vec{v}_i and \vec{v}_e are ion and electron velocities in the beam frame, Λ_C

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is the Coulomb logarithm, which has a weak dependence on v_e and can be moved from under the integral, and $f(r_e, v_e)$ is a six-dimensional distribution function of the electrons.

The cooling rate ($\lambda_{x,y,z}$) in the laboratory frame can be obtained from:

$$\lambda_{x,y,z} = \frac{F_{x,y,z}\eta}{\gamma m_i v_{xi,yi,zi}} \quad (2)$$

where the duty factor $\eta = L_{CS}/(2\pi R)$, R is the storage ring radius, L_{CS} is the length of the cooling section, $m_i = A_i m_p$, m_p is the proton mass, A_i is the ion mass number, and indexes x, y, z correspond to the horizontal, vertical and longitudinal components of F , v and λ .

Similarly to Eq. (2), the change in the ion's velocity on a single pass through the CS is given by:

$$\Delta v_{i(x,y,z)} = \frac{F_{x,y,z} L_{CS}}{m_i \gamma \beta c} \quad (3)$$

It is well known [5] that the cooling rates can be "redistributed" between the longitudinal and transverse directions.

The redistribution requires two conditions. The first one is a coupling between the longitudinal and the transverse (we will consider the horizontal one) motion of an ion. This is created by the ions' dispersion in the CS. The second condition is a nonzero longitudinal drag force dependent on the horizontal position of an ion in the cooling section (a so called transverse gradient of the longitudinal friction force). A robust way to create the required gradient is to introduce the electron beam dispersion in the CS.

The cooling force can be well approximated by a linear function of the ion velocity (v_i) if v_i is less than the rms velocity spread of the electrons.

For the following considerations we will assume:

$$\begin{aligned} F_x &= -C_x n_e v_{xi} \\ F_z &= -C_z n_e (v_{zi} - K \cdot x_i) \end{aligned} \quad (4)$$

where n_e is the density distribution of the e-bunch and K is the horizontal gradient of the longitudinal cooling force.

Substituting Eq. (4) into Eq. (3) and noticing that an ion's relative momentum $\delta_i = v_{zi}/(\beta c)$ and an ion's angle $x'_i = v_{xi}/(\gamma \beta c)$, we get the following changes in an ion's angle ($\Delta x'_i$) and its relative momentum ($\Delta \delta_i$) for a single pass through the CS:

$$\begin{aligned} \Delta x'_i &= -c_x n_e x'_{iCS} \\ \Delta \delta_i &= -c_z n_e (\delta_{iCS} - k \cdot x_{iCS}) \end{aligned} \quad (5)$$

where $c_x = \frac{C_x L_{CS}}{\gamma \beta c m_i}$, $c_z = \frac{C_z L_{CS}}{\gamma \beta c m_i}$, $k = \frac{K}{\beta c}$, and x_{iCS} , x'_{iCS} and δ_{iCS} are respectively an ion's positions, angle and relative momentum in the cooling section.

1.2 Explicit expression for linear part of friction force

We consider an electron bunch with Gaussian 6-D distribution in the presence of e-beam dispersion (D_e) in the cooling section. To simplify the resulting formulas we will assume electron Twiss $\alpha_e = 0$ in the cooling section, which is a reasonable approximation for an electron cooler (a treatment of $\alpha_e \neq 0$ case can be found in [6]). Then the electron bunch distribution in the beam frame is:

$$\begin{aligned} f(r_e, v_e) &= \frac{1}{\gamma(2\pi)^3 \Delta_x \Delta_y \Delta_z \sigma_{xe} \sigma_{ye} \sigma_{ze}} f_x f_y f_z \\ f_x &= \exp \left[-\frac{(x - D_e \delta_e)^2}{2\sigma_{xe}^2} - \frac{v_{xe}^2}{2\Delta_x^2} \right] \\ f_y &= \exp \left(-\frac{y^2}{2\sigma_{ye}^2} - \frac{v_{ye}^2}{2\Delta_y^2} \right) \\ f_z &= \exp \left(-\frac{z^2}{2\sigma_{ze}^2} - \frac{v_{ze}^2}{2\Delta_z^2} \right) = \exp \left(-\frac{z^2}{2\sigma_{ze}^2} - \frac{\delta_e^2}{2\sigma_{\delta_e}^2} \right) \end{aligned} \quad (6)$$

Here $\Delta_{x,y,z}$ are the electrons' rms velocity spreads in the beam frame and $\sigma_{xe,ye,ze}$ are the e-bunch rms sizes along horizontal, vertical and longitudinal directions in the laboratory frame respectively.

With simple algebraic manipulations the distribution can be split in a density and a velocity distribution parts:

$$\begin{aligned} f(r_e, v_e) &= n_e f_{ve} \\ n_e &= \frac{1}{\gamma(2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp \left[-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{2\sigma_{ye}^2} - \frac{z^2}{2\sigma_{ze}^2} \right] \\ f_{ve} &= \frac{1}{(2\pi)^{3/2} \Delta_x \Delta_y \Delta_{1z}} \exp \left[-\frac{v_{xe}^2}{2\Delta_x^2} - \frac{v_{ye}^2}{2\Delta_y^2} - \frac{(v_{ze} - \mu_z)^2}{2\Delta_{1z}^2} \right] \\ \sigma_{1xe} &= \sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta_e}^2} \\ \Delta_{1z} &= \Delta_z \frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta_e}^2}} \\ \mu_z &= x \Delta_z \frac{D_e \sigma_{\delta_e}}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta_e}^2} \end{aligned} \quad (7)$$

Substituting Eq. (7) into Eq. (1) we get the expression for the friction force of the form:

$$\vec{F} = -C n_e \int \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f_{ve} d^3 v_e \quad (8)$$

where $C = \frac{4\pi N_e e^4 Z_i^2 \Lambda_C}{m_e}$ and N_e is the number of electrons in the e-bunch.

Equation (8) with distribution f_{ve} of the form (7) can be reduced to 1-D integrals for each component of the friction force (so called Binney's formulas

[7]) by noticing that the friction force in v-space looks like a point-charge Coulomb force in the physical space. Then we can introduce an effective potential in a velocity-space:

$$U = C \int \frac{f_{ve}}{|\vec{v}_i - \vec{v}_e|} d^3 v_e \quad (9)$$

such that

$$F_{x,y,z} = \partial U / \partial v_{xi,yi,zi} \quad (10)$$

The detailed step-by-step derivation of the Binney's formulas for the considered case is given in [8]. Here we simply give the final result for $\Delta_x = \Delta_y$. Considerations of unequal v-distributions in all three directions can be found in [9].

For the case of $\Delta_x = \Delta_y \equiv \Delta_\perp$ the friction force is:

$$\left\{ \begin{array}{l} F_{x,y} = -\tilde{C} n_e v_{xi,yi} \int_0^\infty g_\perp(q) dq \\ F_z = -\tilde{C} n_e (v_{zi} - \mu_z) \int_0^\infty g_z(q) dq \\ g_\perp(q) = \frac{1}{\Delta_\perp^2 (1+q)^2 \sqrt{\Delta_\perp^2 q + \Delta_{1z}^2}} \exp \left[-\frac{v_{xi}^2 + v_{yi}^2}{2\Delta_\perp^2 (1+q)} - \frac{(v_{zi} - \mu_z)^2}{2(\Delta_\perp^2 q + \Delta_{1z}^2)} \right] \\ g_z(q) = \frac{1}{(1+q)(\Delta_\perp^2 q + \Delta_{1z}^2)^{3/2}} \exp \left[-\frac{v_{xi}^2 + v_{yi}^2}{2\Delta_\perp^2 (1+q)} - \frac{(v_{zi} - \mu_z)^2}{2(\Delta_\perp^2 q + \Delta_{1z}^2)} \right] \end{array} \right. \quad (11)$$

where $\tilde{C} = 2\sqrt{2\pi} N_e r_e^2 m_e c^4 Z_i^2 \Lambda_C$.

The integrals (11) can be taken analytically in the approximation of "small amplitudes":

$$\begin{aligned} F_{x,y} &= -v_{xi,yi} \frac{\tilde{C} n_e}{\Delta_\perp^2 \Delta_{1z}} \Phi(\Delta_{1z} / \Delta_\perp) \\ F_z &= -(v_{zi} - \mu_z) \frac{2\tilde{C} n_e}{\Delta_\perp^2 \Delta_{1z}} (1 - \Phi(\Delta_{1z} / \Delta_\perp)) \\ \Phi(d) &= \begin{cases} \frac{d}{1-d^2} \left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d \right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1} \left(\frac{\log(d-\sqrt{d^2-1})}{\sqrt{d^2-1}} + d \right), & d > 1 \end{cases} \end{aligned} \quad (12)$$

where $d = \Delta_{1z} / \Delta_\perp$.

Equations (12) can be rewritten in the form of Eq. (4):

$$\begin{aligned}
F_x &= -C_x \cdot n_e \cdot v_{xi} \\
F_z &= -C_z \cdot n_e \cdot (v_{zi} - K \cdot x_i) \\
C_x &= C_0 \cdot h \cdot \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right) \\
C_z &= 2C_0 \cdot h \cdot \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right)\right] \\
C_0 &= \frac{2\sqrt{2\pi} N_e r_e^2 m_e c Z_i^2 \Lambda_C}{\gamma^2 \beta^3 \sigma_{\theta e}^2 \sigma_{\delta e}} \\
h &= \sqrt{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta e}^2} \\
K &= \beta c \frac{D_e \sigma_{\delta e}}{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned} \tag{13}$$

Here, we used for the e-bunch angular spread we used $\sigma_{\theta e} = \Delta_{\perp}/(\gamma\beta c)$ and for the e-bunch relative momentum spread we used $\sigma_{\delta e} = \Delta_z/(\beta c)$.

Finally, from Eqs. (4) (5) and (13) we get:

$$\begin{aligned}
\Delta x'_i &= -c_x \cdot n_{e1} \cdot x'_{iCS} \\
\Delta \delta_i &= -c_z \cdot n_{e1} \cdot (\delta_{iCS} - k \cdot x_{iCS}) \\
c_x &= \tilde{c}_0 \cdot h \cdot \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right) \\
c_z &= 2\tilde{c}_0 \cdot h \cdot \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right)\right] \\
k &= \frac{D_e \sigma_{\delta e}^2}{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta e}^2} \\
n_{e1} &= \frac{1}{(2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp\left[-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{2\sigma_{ye}^2} - \frac{z^2}{2\sigma_{ze}^2}\right] \\
\tilde{c}_0 &= \frac{2\sqrt{2\pi} N_e r_e^2 Z_i^2 \Lambda_C L_{CS}}{\gamma^4 \beta^4} \frac{m_e}{A_i m_p} \frac{1}{\sigma_{\theta e}^2 \sigma_{\delta e}} \\
h &= \sqrt{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta e}^2} \\
\sigma_{1xe} &= \sqrt{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned} \tag{14}$$

2 Cooling rate redistribution

2.1 General considerations

We assume that there is a non-zero horizontal ion dispersion in the cooling section D_i and that in the CS $D'_i = 0$. We will further assume that D_i is localized to the CS.

Let us denote all the coordinates of the ion upstream of the cooling section with an index “0”, at the entrance of the cooling section with an index “CS”, and downstream of the cooling section with an index “1”.

We are interested in redistribution of cooling rate between the longitudinal and the horizontal directions. Derivation of the cooling rate for vertical direction (cooling without redistribution) can be found in [9].

We assume that the cooling is weak enough to produce the “velocity kick” only and not to change an ion’s position through the CS. Then on a single pass through the CS the ion’s coordinate x changes as follows:

$$\begin{aligned} x_{iCS} &= x_{i0} + D_i \cdot \delta_{i0} \\ x_{i1} &= x_{iCS} - D_i \cdot (\delta_{iCS} + \Delta\delta_i) = x_{iCS} - D_i \cdot (\delta_{i0} + \Delta\delta_i) \end{aligned} \quad (15)$$

Combining Eqs. (15) and (14) we get:

$$\begin{aligned} \Delta x_i &= c_z \cdot n_{e1} \cdot D_i \cdot [\delta_{i0} - k \cdot (x_{i0} + D_i \cdot \delta_{i0})] \\ \Delta x'_i &= -c_x \cdot n_{e1} \cdot x'_{i0} \\ \Delta \delta_i &= -c_z \cdot n_{e1} \cdot [\delta_{i0} - k \cdot (x_{i0} + D_i \cdot \delta_{i0})] \end{aligned} \quad (16)$$

Notice, that the density of the e-bunch “probed” by the considered ion in the CS is:

$$n_{e1} = \frac{1}{(2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp \left[-\frac{(x_i + D_i \delta_i)^2}{2\sigma_{1xe}^2} - \frac{y_i^2}{2\sigma_{ye}^2} - \frac{z_i^2}{2\sigma_{ze}^2} \right] \quad (17)$$

Next, an action J_x characterizing horizontal motion of an ion is given by:

$$J_x = \frac{1}{2} (\gamma_x x_i^2 + \beta_x x_i'^2) \quad (18)$$

where β_x is a Twiss β -function, $\gamma_x = 1/\beta_x$ is the Twiss γ -function and we assumed that the ions’ Twiss α -function in the CS is equal to zero.

On a single pass through the cooling section, taking into account only the first order of small changes Δx_i and $\Delta x'_i$, we get:

$$\begin{aligned} \Delta J_x &\approx \frac{x_{i0} \Delta x_i}{\beta_x} + \beta_x x'_{i0} \Delta x'_i = \\ &= \frac{c_z D_i}{\beta_x} n_{e1} (x_{i0} \delta_{i0} (1 - k D_i) - k x_{i0}^2) - c_x n_{e1} \beta_x x_{i0}'^2 \end{aligned} \quad (19)$$

We will further assume that the ions have a Gaussian distribution with the distribution function:

$$f_i = \frac{\exp \left(-\frac{x^2}{2\sigma_{xi}^2} - \frac{y^2}{2\sigma_{yi}^2} - \frac{z^2}{2\sigma_{zi}^2} - \frac{x'^2}{2\sigma_{\theta xi}^2} - \frac{y'^2}{2\sigma_{\theta yi}^2} - \frac{\delta^2}{2\sigma_{\delta i}^2} \right)}{(2\pi)^3 \sigma_{xi} \sigma_{yi} \sigma_{zi} \sigma_{\theta xi} \sigma_{\theta yi} \sigma_{\delta i}} \quad (20)$$

where $\sigma_{xi} \sigma_{yi} \sigma_{zi}$ are the rms sizes of the ion bunch, $\sigma_{\theta xi}, \sigma_{\theta yi}$ are the ions’ rms angular spreads, and $\sigma_{\delta i}$ is the ion bunch relative momentum spread in the CS.

Finally, the change in horizontal emittance on a single pass through the CS is given by:

$$\begin{aligned}
\Delta\varepsilon_x &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i \Delta J_x dx_i dy_i dz_i dx'_i dy'_i d\delta_i = \\
&= \frac{1}{(2\pi)^{7/2} \sigma_{x_i} \sigma_{y_i} \sigma_{z_i} \sigma_{\delta_i} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \left(\frac{c_z D_i}{\beta_x} I_1 I_y I_z - c_x \beta_x \sigma_{\theta x_i}^2 I_2 I_y I_z \right)
\end{aligned} \tag{21}$$

Where integrals I_1, I_2, I_y, I_z are given by the following expressions:

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_i \delta_i (1 - k D_i) - k x_i^2) \exp \left[-\frac{(x_i + D_i \delta_i)^2}{2\sigma_{1xe}^2} - \frac{x_i^2}{2\sigma_{x_i}^2} - \frac{\delta_i^2}{2\sigma_{\delta_i}^2} \right] dx_i d\delta_i = \\
&= -\frac{2\pi \sigma_{1xe} \sigma_{\delta_i} \sigma_{x_i}^3 (D_i \sigma_{\delta_i}^2 + k \sigma_{1xe}^2)}{(D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1xe}^2)^{3/2}}
\end{aligned} \tag{22}$$

$$I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[-\frac{(x_i + D_i \delta_i)^2}{2\sigma_{1xe}^2} - \frac{x_i^2}{2\sigma_{x_i}^2} - \frac{\delta_i^2}{2\sigma_{\delta_i}^2} \right] dx_i d\delta_i = \frac{2\pi \sigma_{1xe} \sigma_{\delta_i} \sigma_{x_i}}{\sqrt{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1xe}^2}} \tag{23}$$

$$I_y = \int_{-\infty}^{\infty} \exp \left(-\frac{y_i^2}{2\sigma_{ye}^2} - \frac{y_i^2}{2\sigma_{y_i}^2} \right) dy_i = \frac{\sqrt{2\pi} \sigma_{ye} \sigma_{y_i}}{\sqrt{\sigma_{y_i}^2 + \sigma_{ye}^2}} \tag{24}$$

$$I_z = \int_{-\infty}^{\infty} \exp \left(-\frac{z_i^2}{2\sigma_{ze}^2} - \frac{z_i^2}{2\sigma_{z_i}^2} \right) dz_i = \frac{\sqrt{2\pi} \sigma_{ze} \sigma_{z_i}}{\sqrt{\sigma_{z_i}^2 + \sigma_{ze}^2}} \tag{25}$$

Substituting Eqs. (22)-(25) into Eq. (21) we get:

$$\begin{aligned}
\frac{\Delta\varepsilon_x}{\varepsilon_x} &= -\frac{1}{(2\pi)^{3/2} \sqrt{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1xe}^2} \sqrt{\sigma_{y_i}^2 + \sigma_{ye}^2} \sqrt{\sigma_{z_i}^2 + \sigma_{ze}^2}} \cdot \\
&\cdot \left(c_x + c_z \frac{D_i \sigma_{\delta_i}^2 + k D_i \sigma_{1xe}^2}{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1xe}^2} \right)
\end{aligned} \tag{26}$$

Similar to above considerations, we introduce a longitudinal action J_z :

$$J_z = \frac{1}{2} \left(\frac{z_i^2}{\beta_z} + \beta_z \delta_i^2 \right) \tag{27}$$

where $\beta_z = \sigma_{z_i} / \sigma_{\delta_i}$.

Then, for a single pass through the CS we get:

$$\Delta J_z \approx \beta_z \delta_{i0} \Delta \delta_i = -c_z n_{e1} \beta_z (\delta_{i0}^2 (1 - k D_i) - k x_{i0} \delta_{i0}) \tag{28}$$

The change in the longitudinal emittance ($\varepsilon_z = \sigma_{z_i} \sigma_{\delta_i}$) on a single pass through the CS is given by:

$$\begin{aligned}
\Delta\varepsilon_z &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i \Delta J_z dx_i dy_i dz_i dx'_i dy'_i d\delta_i = \\
&= \frac{-c_z \beta_z}{(2\pi)^{7/2} \sigma_{x_i} \sigma_{y_i} \sigma_{z_i} \sigma_{\delta_i} \sigma_{1x_e} \sigma_{y_e} \sigma_{z_e}} I_3 I_y I_z
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
I_3 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\delta_i^2 (1 - kD_i) - kx_i \delta_i) \exp \left[-\frac{(x_i + D_i \delta_i)^2}{2\sigma_{1x_e}^2} - \frac{x_i^2}{2\sigma_{x_i}^2} - \frac{\delta_i^2}{2\sigma_{\delta_i}^2} \right] dx_i d\delta_i = \\
&= -\frac{2\pi \sigma_{1x_e} \sigma_{\delta_i}^3 \sigma_{x_i} (\sigma_{x_i}^2 + \sigma_{1x_e}^2) (1 - Dik)}{(D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1x_e}^2)^{3/2}}
\end{aligned} \tag{30}$$

Substituting Eqs. (24), (25) and (30) into Eq. (29) we get:

$$\begin{aligned}
\frac{\Delta\varepsilon_z}{\varepsilon_z} &= -\frac{1}{(2\pi)^{3/2} \sqrt{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1x_e}^2} \sqrt{\sigma_{y_i}^2 + \sigma_{y_e}^2} \sqrt{\sigma_{z_i}^2 + \sigma_{z_e}^2}} \cdot \\
&\cdot \left(c_z - c_z \frac{D_i^2 \sigma_{\delta_i}^2 + kD_i \sigma_{1x_e}^2}{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + \sigma_{1x_e}^2} \right)
\end{aligned} \tag{31}$$

2.2 Explicit formulas for redistributed cooling rates

We can get the resulting horizontal and vertical cooling rates by dividing Eqs. (26) and (31) by a revolution period $T_{rev} = \frac{2\pi R}{\beta c}$:

$$\begin{aligned}
\lambda_x &= -P \left(c_{x0} + c_{z0} \frac{D_i^2 \sigma_{\delta_i}^2 + D_e D_i \sigma_{\delta_e}^2}{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + D_e^2 \sigma_{\delta_e}^2 + \sigma_{x_e}^2} \right) \\
\lambda_z &= -P \left(c_{z0} - c_{x0} \frac{D_i^2 \sigma_{\delta_i}^2 + D_e D_i \sigma_{\delta_e}^2}{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + D_e^2 \sigma_{\delta_e}^2 + \sigma_{x_e}^2} \right) \\
P &= \frac{\sqrt{D_e^2 \sigma_{\delta_e}^2 + \sigma_{x_e}^2}}{\sigma_{x_e} \sqrt{D_i^2 \sigma_{\delta_i}^2 + \sigma_{x_i}^2 + D_e^2 \sigma_{\delta_e}^2 + \sigma_{x_e}^2} \sqrt{\sigma_{y_i}^2 + \sigma_{y_e}^2} \sqrt{\sigma_{z_i}^2 + \sigma_{z_e}^2}} \\
c_0 &= \frac{N_e r_e^2 Z_i^2 c \Lambda_C \eta}{\pi \gamma^4 \beta^3} \frac{m_e}{A_i m_p} \frac{1}{\sigma_{\theta_e}^2 \sigma_{\delta_e}} \\
c_{x0} &= c_0 \cdot \Phi \left(\frac{\sigma_{\delta_e}}{\gamma \sigma_{\theta_e}} \frac{\sigma_{x_e}}{\sqrt{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta_e}^2}} \right) \\
c_{z0} &= 2c_0 \cdot \left[1 - \Phi \left(\frac{\sigma_{\delta_e}}{\gamma \sigma_{\theta_e}} \frac{\sigma_{x_e}}{\sqrt{\sigma_{x_e}^2 + D_e^2 \sigma_{\delta_e}^2}} \right) \right] \\
\Phi(d) &= \begin{cases} \frac{d}{1-d^2} \left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d \right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1} \left(\frac{\log(d-\sqrt{d^2-1})}{\sqrt{d^2-1}} + d \right), & d > 1 \end{cases}
\end{aligned} \tag{32}$$

Equations (32) give explicit expressions for the cooling rates for the case of $x - z$ redistribution. The plot of function Φ is shown in Fig. 1.

It is important to notice that function Φ is strongly nonlinear, as Fig. 1 demonstrates. Therefore, generally speaking, the rates λ_x and λ_z can not be

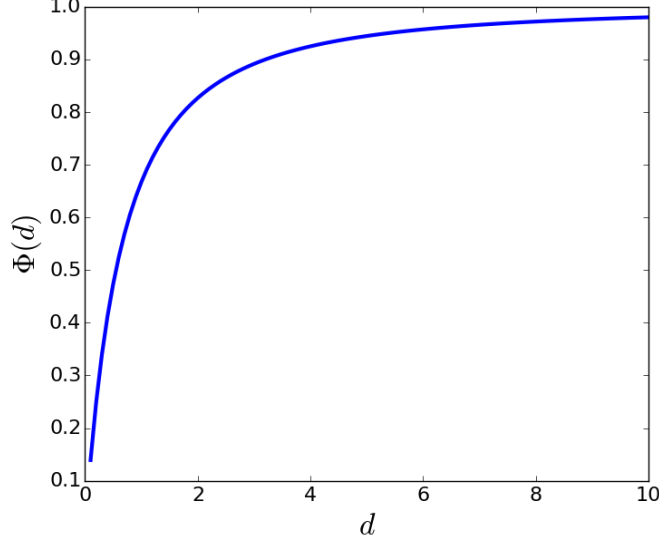


Figure 1: Function $\Phi(d)$.

represented as linear combinations of “undisturbed” cooling rates at $D_e = 0$ and $D_i = 0$ ($\lambda_{x0} \equiv \lambda_x(D_e = 0, D_i = 0)$, $\lambda_{z0} \equiv \lambda_z(D_e = 0, D_i = 0)$).

Setting $D_e = 0$ and $D_i = 0$ we get a well expected result for λ_{x0} and λ_{y0} :

$$\begin{aligned}\lambda_{x0} &= -\frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta e}^2 \sigma_{\delta e} \sqrt{(\sigma_{xi}^2 + \sigma_{xe}^2)(\sigma_{yi}^2 + \sigma_{ye}^2)(\sigma_{zi}^2 + \sigma_{ze}^2)}} \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right) \\ \lambda_{z0} &= -\frac{2N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta e}^2 \sigma_{\delta e} \sqrt{(\sigma_{xi}^2 + \sigma_{xe}^2)(\sigma_{yi}^2 + \sigma_{ye}^2)(\sigma_{zi}^2 + \sigma_{ze}^2)}} \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)\right]\end{aligned}\quad (33)$$

Finally, for the case of a small ion bunch placed at the center of an e-bunch (i.e. when $\sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2 \ll \sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2$, $\sigma_{yi} \ll \sigma_{ye}$, $\sigma_{zi} \ll \sigma_{ze}$) Eqs. (32) are getting reduced to the form derived in [10]:

$$\begin{aligned}\lambda_x &= \lambda_{1x} + \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{1z} \\ \lambda_z &= \lambda_{1z} - \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{1x} \\ \lambda_{1x} &= \frac{N_e r_e^2 m_e c Z^2 L_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}}\right) \\ \lambda_{1z} &= \frac{2N_e r_e^2 m_e c Z^2 L_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}}\right)\right]\end{aligned}\quad (34)$$

It is worth reminding our readers that formulas (33) and (34) give proper expressions (first derived in [11]) for the two classical asymptotic cases of the spherically symmetric ($\Delta_{\perp} = \Delta_z$) and the flat ($\Delta_z \ll \Delta_{\perp}$) e-velocity

distributions. For farther discussions of the asymptotic expressions please see [9] and [10].

3 Conclusion

We derived the explicit formulas (Eqs. 32) for the cooling rates redistribution in a non-magnetized electron cooler taking into account the non-uniformity of the density of the electron bunch.

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