# Integrals of Distorted Gaussian Functions 

S. Brooks

May 2020

# Collider Accelerator Department Brookhaven National Laboratory 

U.S. Department of Energy<br>USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

[^0]
## DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

# Integrals of Distorted Gaussian Functions 

Stephen Brooks

May 1, 2020

## 1 Integral of Gaussian times Polynomial

Series solutions involving Gaussian-smoothed sources may involve integrating a slightly-distorted Gaussian function, which can be expressed as a Gaussian multiplied by a polynomial or series. The 1D Gaussian function centred at $x=0$ is $f_{1, \sigma}(x)=\frac{1}{\sigma \sqrt{2 \pi}} e^{-x^{2} / 2 \sigma^{2}}$ and obeys $\int_{-\infty}^{\infty} f_{1, \sigma}(x) \mathrm{d} x=$ 1. First, calculate some derivatives:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f_{1, \sigma}=-\frac{1}{\sigma^{2}} x f_{1, \sigma} \quad \Rightarrow \quad \frac{\mathrm{~d}}{\mathrm{~d} x}\left[x^{n} f_{1, \sigma}\right]=n x^{n-1} f_{1, \sigma}-\frac{1}{\sigma^{2}} x^{n+1} f_{1, \sigma} .
$$

Let

$$
I_{n}=\int_{-\infty}^{\infty} x^{n} f_{1, \sigma}(x) \mathrm{d} x,
$$

so that $I_{0}=1$. Integrating both sides of the formula for $\frac{\mathrm{d}}{\mathrm{d} x}\left[x^{n} f_{1, \sigma}\right]$ gives

$$
\left[x^{n} f_{1, \sigma}\right]_{-\infty}^{\infty}=n I_{n-1}-\frac{1}{\sigma^{2}} I_{n+1} .
$$

The left hand side is zero because a Gaussian decreases faster than any polynomial can increase as $x \rightarrow \pm \infty$. The $n=0$ case gives $I_{1}=0$, which can also be seen because $x^{1}$ is an odd function. The $n+1$ cases give a recurrence

$$
I_{n+2}=\sigma^{2}(n+1) I_{n},
$$

which can be solved for the even numbers to give the general solution

$$
I_{2 n}=\frac{(2 n)!}{n!2^{n}} \sigma^{2 n}, \quad I_{2 n+1}=0
$$

For the 3D spherical Gaussian $f_{3, \sigma}(\mathbf{x})=\frac{1}{(\sigma \sqrt{2 \pi})^{3}} e^{-|\mathbf{x}|^{2} / 2 \sigma^{2}}$, the relationship $f_{3, \sigma}(x, y, z)=$ $f_{1, \sigma}(x) f_{1, \sigma}(y) f_{1, \sigma}(z)$ can be used to evaluate it multiplied by 3 -variable polynomial terms:

$$
\int x^{i} y^{j} z^{k} f_{3, \sigma}(\mathbf{x}) \mathrm{d}^{3} \mathbf{x}=\int_{-\infty}^{\infty} x^{i} f_{1, \sigma}(x) \mathrm{d} x \int_{-\infty}^{\infty} y^{j} f_{1, \sigma}(y) \mathrm{d} y \int_{-\infty}^{\infty} z^{k} f_{1, \sigma}(z) \mathrm{d} z=I_{i} I_{j} I_{k} .
$$

This is only nonzero when all of $i, j, k$ are even.

## 2 Integral of Gaussian of Another Function

If the argument of the Gaussian function is another function of position, integration by substitution may be used:

$$
\int_{-\infty}^{\infty} f_{1, \sigma}(h(x)) \mathrm{d} x=\int_{-\infty}^{\infty} f_{1, \sigma}(h) \frac{\mathrm{d} x}{\mathrm{~d} h} \mathrm{~d} h
$$

which works best if $h(x)$ is invertible, otherwise multiple solutions for $h$ will have to be summed to cover all $x$ in the original integral. As the Gaussian $f_{1, \sigma}$ peaks near $h=0$, a series expansion of $\frac{\mathrm{d} x}{\mathrm{~d} h}$ may be used, which makes each term an integral of the form in the previous section.

More explicitly, suppose $h(x)$ is a bijection and $h\left(x_{0}\right)=0$. The Taylor expansion of $h$ is

$$
h(x)=h\left(x_{0}\right)+\sum_{n=1}^{\infty} \frac{1}{n!} h^{(n)}\left(x_{0}\right)\left(x-x_{0}\right)^{n} .
$$

Since the first nonzero term is in $\left(x-x_{0}\right)^{1}$, the $p^{\text {th }}$ power of $h(x)$ will have a series starting at $\left(x-x_{0}\right)^{p}$ for some coefficients:

$$
h(x)^{p}=\sum_{n=p}^{\infty} c_{p n}\left(x-x_{0}\right)^{n}, \quad \text { where } \quad c_{1 n}=\frac{1}{n!} h^{(n)}\left(x_{0}\right), \quad c_{p+1, n}=\sum_{i=p}^{n-1} c_{p i} c_{1, n-i} .
$$

This helps construct the inverse of this series that satsifies

$$
x-x_{0}=\sum_{n=1}^{\infty} a_{n} h(x)^{n}=\sum_{n=1}^{\infty} a_{n} \sum_{i=n}^{\infty} c_{n i}\left(x-x_{0}\right)^{i}=\sum_{i=1}^{\infty}\left(\sum_{n=1}^{i} a_{n} c_{n i}\right)\left(x-x_{0}\right)^{i} .
$$

The coefficients $a_{n}$ may be found by 'long division', where each successive $a_{n}$ is chosen to make the coefficient of $\left(x-x_{0}\right)^{n}$ correct, starting with

$$
a_{1} c_{11}=1 \quad \Rightarrow \quad a_{1}=\frac{1}{c_{11}}
$$

and continuing for $n>1$

$$
\sum_{i=1}^{n} a_{i} c_{i n}=0 \quad \Rightarrow \quad a_{n}=\frac{-\sum_{i=1}^{n-1} a_{i} c_{i n}}{c_{n n}}
$$

Taking the derivative of the resulting series with respect to $h(x)$ gives

$$
\frac{\mathrm{d} x}{\mathrm{~d} h}=\sum_{n=1}^{\infty} n a_{n} h^{n-1}
$$

and now the integral may be evaluated using results from the previous section:

$$
\int_{-\infty}^{\infty} f_{1, \sigma}(h(x)) \mathrm{d} x=\int_{-\infty}^{\infty} f_{1, \sigma}(h) \sum_{n=1}^{\infty} n a_{n} h^{n-1} \mathrm{~d} h=\sum_{n=1}^{\infty} n a_{n} I_{n-1}
$$


[^0]:    Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No.DE-SC0012704 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

