

Integrated Field of a Finite Wire

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1 Assumptions

The wire segment in question travels in a straight line from position \mathbf{a} to \mathbf{b} and carries current I in the direction towards \mathbf{b} . The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be. The integration will be performed along the entire z axis ($-\infty$ to ∞).

2 Derivation

The Biot-Savart law is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{r}}{|\mathbf{r}|^3} ds,$$

where \mathbf{r} is the vector to \mathbf{x} from the relevant point on the conductor. The parametrisation

$$\mathbf{r} = \mathbf{x} - (\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})), \quad ds = |\mathbf{b} - \mathbf{a}| d\lambda, \quad \mathbf{I} = \frac{I(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|},$$

for $\lambda \in [0, 1]$, is used for the wire segment. \mathbf{I} is constant with s so the cross product can be taken outside the integral. Integrating along the z axis gives

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) dz = \frac{\mu_0}{4\pi} \mathbf{I} \times \int \int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} dz ds,$$

where z is the third component of \mathbf{x} or of \mathbf{r} , equivalently, because those two vectors are related by a translation and the whole z axis is integrated over in both cases.

$$\int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} dz = \mathbf{f}(r_x, r_y) \quad \text{where} \quad \mathbf{f}(x, y) = \int_{-\infty}^{\infty} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} dz.$$

Since $\int_{-\infty}^{\infty} \frac{1}{(k^2 + z^2)^{3/2}} dz = \frac{2}{k^2}$ and $\int_{-\infty}^{\infty} \frac{z}{(k^2 + z^2)^{3/2}} dz = 0$,

$$\mathbf{f}(x, y) = \frac{2(x, y, 0)}{x^2 + y^2}$$

and thus

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) dz = \frac{\mu_0}{4\pi} \mathbf{I} \times \int \frac{2(r_x, r_y, 0)}{r_x^2 + r_y^2} ds$$

$$\begin{aligned}
&= \frac{\mu_0 |\mathbf{b} - \mathbf{a}|}{2\pi} \mathbf{I} \times \int_0^1 \frac{(r_x, r_y, 0)}{r_x^2 + r_y^2} d\lambda \\
&= \frac{\mu_0 I}{2\pi} (\mathbf{b} - \mathbf{a}) \times \int_0^1 \frac{(r_x, r_y, 0)}{r_x^2 + r_y^2} d\lambda \\
&= \frac{\mu_0 I}{2\pi} (\mathbf{b} - \mathbf{a}) \times (i_x, i_y, 0),
\end{aligned}$$

where $i_x = \int_0^1 \frac{r_x}{r_x^2 + r_y^2} d\lambda$ and i_y is similar with x and y swapped. Letting $\ell = \mathbf{b} - \mathbf{a}$ and $\mathbf{X} = \mathbf{x} - \mathbf{a}$ so that $\mathbf{r} = \mathbf{X} - \lambda\ell$ gives

$$i_x = \int_0^1 \frac{X - \lambda l_x}{(X - \lambda l_x)^2 + (Y - \lambda l_y)^2} d\lambda.$$

This integral is of the form

$$\begin{aligned}
&\int_0^1 \frac{px + q}{(px + q)^2 + (rx + s)^2} dx \\
&= \left[\frac{p \ln((px + q)^2 + (rx + s)^2) + 2r \arctan \frac{x(p^2 + r^2) + pq + rs}{qr - ps}}{2(p^2 + r^2)} \right]_0^{x=1} \\
&= \frac{1}{2(p^2 + r^2)} \left(p \ln((p + q)^2 + (r + s)^2) + 2r \arctan \frac{p^2 + r^2 + pq + rs}{qr - ps} - p \ln(q^2 + s^2) \right. \\
&\quad \left. - 2r \arctan \frac{pq + rs}{qr - ps} \right),
\end{aligned}$$

with the substitutions $p = -l_x = a_x - b_x$, $q = X = x - a_x$, $r = -l_y = a_y - b_y$ and $s = Y = y - a_y$. For i_y , swap p with r and q with s . Finally, evaluate

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) dz = \frac{\mu_0 I}{2\pi} \ell \times (i_x, i_y, 0) = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x).$$

2.1 Special case: wire in Z direction

The final formula for i_x in the last section divides by zero if $l_x = l_y = 0$ ($p = r = 0$). In this case, the original expression for i_x becomes simply

$$i_x = \int_0^1 \frac{X}{X^2 + Y^2} d\lambda = \frac{X}{X^2 + Y^2},$$

so that $i_x = \frac{X}{X^2 + Y^2}$ and $i_y = \frac{Y}{X^2 + Y^2}$. The integrated field formula simplifies to

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) dz = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x) = \frac{\mu_0 I l_z}{2\pi (X^2 + Y^2)} (-Y, X, 0).$$