

BNL-223627-2022-TECH C-A/AP/674

Integrated Field of a Finite Wire

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January 2019

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U.S. Department of Energy

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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January 29, 2019

1 Assumptions

The wire segment in question travels in a straight line from position **a** to **b** and carries current I in the direction towards **b**. The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be. The integration will be performed along the entire z axis $(-\infty \text{ to } \infty)$.

2 Derivation

The Biot-Savart law is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{r}}{|\mathbf{r}|^3} \,\mathrm{d}s,$$

where \mathbf{r} is the vector to \mathbf{x} from the relevant point on the conductor. The parametrisation

$$\mathbf{r} = \mathbf{x} - (\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})), \qquad ds = |\mathbf{b} - \mathbf{a}| d\lambda, \qquad \mathbf{I} = \frac{I(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|},$$

for $\lambda \in [0, 1]$, is used for the wire segment. I is constant with s so the cross product can be taken outside the integral. Integrating along the z axis gives

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z = \frac{\mu_0}{4\pi} \mathbf{I} \times \int \int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}z \, \mathrm{d}s,$$

where z is the third component of \mathbf{x} or of \mathbf{r} , equivalently, because those two vectors are related by a translation and the whole z axis is integrated over in both cases.

$$\int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}z = \mathbf{f}(r_x, r_y) \qquad \text{where} \qquad \mathbf{f}(x, y) = \int_{-\infty}^{\infty} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \, \mathrm{d}z$$

Since $\int_{-\infty}^{\infty} \frac{1}{(k^2+z^2)^{3/2}} dz = \frac{2}{k^2}$ and $\int_{-\infty}^{\infty} \frac{z}{(k^2+z^2)^{3/2}} dz = 0$,

$$\mathbf{f}(x,y) = \frac{2(x,y,0)}{x^2 + y^2}$$

and thus

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z \quad = \quad \frac{\mu_0}{4\pi} \mathbf{I} \times \int \frac{2(r_x, r_y, 0)}{r_x^2 + r_y^2} \, \mathrm{d}s$$

$$= \frac{\mu_0 |\mathbf{b} - \mathbf{a}|}{2\pi} \mathbf{I} \times \int_0^1 \frac{(r_x, r_y, 0)}{r_x^2 + r_y^2} d\lambda$$
$$= \frac{\mu_0 I}{2\pi} (\mathbf{b} - \mathbf{a}) \times \int_0^1 \frac{(r_x, r_y, 0)}{r_x^2 + r_y^2} d\lambda$$
$$= \frac{\mu_0 I}{2\pi} (\mathbf{b} - \mathbf{a}) \times (i_x, i_y, 0),$$

where $i_x = \int_0^1 \frac{r_x}{r_x^2 + r_y^2} d\lambda$ and i_y is similar with x and y swapped. Letting $\ell = \mathbf{b} - \mathbf{a}$ and $\mathbf{X} = \mathbf{x} - \mathbf{a}$ so that $\mathbf{r} = \mathbf{X} - \lambda \ell$ gives

$$i_x = \int_0^1 \frac{X - \lambda l_x}{(X - \lambda l_x)^2 + (Y - \lambda l_y)^2} \,\mathrm{d}\lambda.$$

This integral is of the form

$$\begin{split} &\int_{0}^{1} \frac{px+q}{(px+q)^{2}+(rx+s)^{2}} \,\mathrm{d}x \\ &= \left[\frac{p\ln((px+q)^{2}+(rx+s)^{2})+2r\arctan\frac{x(p^{2}+r^{2})+pq+rs}{qr-ps}}{2(p^{2}+r^{2})} \right]_{0}^{x=1} \\ &= \frac{1}{2(p^{2}+r^{2})} \left(p\ln((p+q)^{2}+(r+s)^{2})+2r\arctan\frac{p^{2}+r^{2}+pq+rs}{qr-ps}-p\ln(q^{2}+s^{2}) -2r\arctan\frac{pq+rs}{qr-ps} \right), \end{split}$$

with the substitutions $p = -l_x = a_x - b_x$, $q = X = x - a_x$, $r = -l_y = a_y - b_y$ and $s = Y = y - a_y$. For i_y , swap p with r and q with s. Finally, evaluate

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z = \frac{\mu_0 I}{2\pi} \ell \times (i_x, i_y, 0) = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x).$$

2.1 Special case: wire in Z direction

The final formula for i_x in the last section divides by zero if $l_x = l_y = 0$ (p = r = 0). In this case, the original expression for i_x becomes simply

$$i_x = \int_0^1 \frac{X}{X^2 + Y^2} \,\mathrm{d}\lambda = \frac{X}{X^2 + Y^2},$$

so that $i_x = \frac{X}{X^2 + Y^2}$ and $i_y = \frac{Y}{X^2 + Y^2}$. The integrated field formula simplifies to

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x) = \frac{\mu_0 I l_z}{2\pi (X^2 + Y^2)} (-Y, X, 0)$$