# Integrated Field of a Finite Wire 

S. Brooks

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## Collider Accelerator Department

## Brookhaven National Laboratory

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## 1 Assumptions

The wire segment in question travels in a straight line from position $\mathbf{a}$ to $\mathbf{b}$ and carries current $I$ in the direction towards $\mathbf{b}$. The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be. The integration will be performed along the entire $z$ axis ( $-\infty$ to $\infty$ ).

## 2 Derivation

The Biot-Savart law is

$$
\mathbf{B}(\mathbf{x})=\frac{\mu_{0}}{4 \pi} \int \frac{\mathbf{I} \times \mathbf{r}}{|\mathbf{r}|^{3}} \mathrm{~d} s
$$

where $\mathbf{r}$ is the vector to $\mathbf{x}$ from the relevant point on the conductor. The parametrisation

$$
\mathbf{r}=\mathbf{x}-(\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})), \quad \mathrm{d} s=|\mathbf{b}-\mathbf{a}| \mathrm{d} \lambda, \quad \mathbf{I}=\frac{I(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|},
$$

for $\lambda \in[0,1]$, is used for the wire segment. $\mathbf{I}$ is constant with $s$ so the cross product can be taken outside the integral. Integrating along the $z$ axis gives

$$
\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \mathrm{d} z=\frac{\mu_{0}}{4 \pi} \mathbf{I} \times \iint_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^{3}} \mathrm{~d} z \mathrm{~d} s,
$$

where $z$ is the third component of $\mathbf{x}$ or of $\mathbf{r}$, equivalently, because those two vectors are related by a translation and the whole $z$ axis is integrated over in both cases.

$$
\int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^{3}} \mathrm{~d} z=\mathbf{f}\left(r_{x}, r_{y}\right) \quad \text { where } \quad \mathbf{f}(x, y)=\int_{-\infty}^{\infty} \frac{(x, y, z)}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}} \mathrm{~d} z .
$$

Since $\int_{-\infty}^{\infty} \frac{1}{\left(k^{2}+z^{2}\right)^{3 / 2}} \mathrm{~d} z=\frac{2}{k^{2}}$ and $\int_{-\infty}^{\infty} \frac{z}{\left(k^{2}+z^{2}\right)^{3 / 2}} \mathrm{~d} z=0$,

$$
\mathbf{f}(x, y)=\frac{2(x, y, 0)}{x^{2}+y^{2}}
$$

and thus

$$
\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \mathrm{d} z=\frac{\mu_{0}}{4 \pi} \mathbf{I} \times \int \frac{2\left(r_{x}, r_{y}, 0\right)}{r_{x}^{2}+r_{y}^{2}} \mathrm{~d} s
$$

$$
\begin{aligned}
& =\frac{\mu_{0}|\mathbf{b}-\mathbf{a}|}{2 \pi} \mathbf{I} \times \int_{0}^{1} \frac{\left(r_{x}, r_{y}, 0\right)}{r_{x}^{2}+r_{y}^{2}} \mathrm{~d} \lambda \\
& =\frac{\mu_{0} I}{2 \pi}(\mathbf{b}-\mathbf{a}) \times \int_{0}^{1} \frac{\left(r_{x}, r_{y}, 0\right)}{r_{x}^{2}+r_{y}^{2}} \mathrm{~d} \lambda \\
& =\frac{\mu_{0} I}{2 \pi}(\mathbf{b}-\mathbf{a}) \times\left(i_{x}, i_{y}, 0\right),
\end{aligned}
$$

where $i_{x}=\int_{0}^{1} \frac{r_{x}}{r_{x}^{2}+r_{y}^{2}} \mathrm{~d} \lambda$ and $i_{y}$ is similar with $x$ and $y$ swapped. Letting $\ell=\mathbf{b}-\mathbf{a}$ and $\mathbf{X}=\mathbf{x}-\mathbf{a}$ so that $\mathbf{r}=\mathbf{X}-\lambda \ell$ gives

$$
i_{x}=\int_{0}^{1} \frac{X-\lambda l_{x}}{\left(X-\lambda l_{x}\right)^{2}+\left(Y-\lambda l_{y}\right)^{2}} \mathrm{~d} \lambda
$$

This integral is of the form

$$
\begin{gathered}
\int_{0}^{1} \frac{p x+q}{(p x+q)^{2}+(r x+s)^{2}} \mathrm{~d} x \\
=\left[\frac{p \ln \left((p x+q)^{2}+(r x+s)^{2}\right)+2 r \arctan \frac{x\left(p^{2}+r^{2}\right)+p q+r s}{q r-p s}}{2\left(p^{2}+r^{2}\right)}\right]_{0}^{x=1} \\
=\frac{1}{2\left(p^{2}+r^{2}\right)}\left(p \ln \left((p+q)^{2}+(r+s)^{2}\right)+2 r \arctan \frac{p^{2}+r^{2}+p q+r s}{q r-p s}-p \ln \left(q^{2}+s^{2}\right)\right. \\
\left.-2 r \arctan \frac{p q+r s}{q r-p s}\right)
\end{gathered}
$$

with the substitutions $p=-l_{x}=a_{x}-b_{x}, q=X=x-a_{x}, r=-l_{y}=a_{y}-b_{y}$ and $s=Y=y-a_{y}$. For $i_{y}$, swap $p$ with $r$ and $q$ with $s$. Finally, evaluate

$$
\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \mathrm{d} z=\frac{\mu_{0} I}{2 \pi} \ell \times\left(i_{x}, i_{y}, 0\right)=\frac{\mu_{0} I}{2 \pi}\left(-l_{z} i_{y}, l_{z} i_{x}, l_{x} i_{y}-l_{y} i_{x}\right)
$$

### 2.1 Special case: wire in $Z$ direction

The final formula for $i_{x}$ in the last section divides by zero if $l_{x}=l_{y}=0(p=r=0)$. In this case, the original expression for $i_{x}$ becomes simply

$$
i_{x}=\int_{0}^{1} \frac{X}{X^{2}+Y^{2}} \mathrm{~d} \lambda=\frac{X}{X^{2}+Y^{2}}
$$

so that $i_{x}=\frac{X}{X^{2}+Y^{2}}$ and $i_{y}=\frac{Y}{X^{2}+Y^{2}}$. The integrated field formula simplifies to

$$
\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \mathrm{d} z=\frac{\mu_{0} I}{2 \pi}\left(-l_{z} i_{y}, l_{z} i_{x}, l_{x} i_{y}-l_{y} i_{x}\right)=\frac{\mu_{0} I l_{z}}{2 \pi\left(X^{2}+Y^{2}\right)}(-Y, X, 0)
$$


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