

# Magnetic Field from an Infinite Array of Wires

S. Brooks

September 2017

Collider Accelerator Department  
**Brookhaven National Laboratory**

**U.S. Department of Energy**

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No. DE-SC0012704 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

## **DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

# Magnetic Field from an Infinite Array of Wires

Stephen Brooks

September 19, 2017

## 1 Single Wire

The magnetic field produced by an infinite (in the  $z$  direction) wire carrying current  $I$  is

$$\mathbf{B}(x, y) = \frac{\mu_0 I}{2\pi} \frac{1}{x^2 + y^2} \begin{bmatrix} -y \\ x \end{bmatrix},$$

which has magnitude  $\frac{\mu_0 I}{2\pi r}$ , where  $r = \sqrt{x^2 + y^2}$ .

## 2 Infinite Regular Array of Wires

Suppose the wires are spaced by distance  $a$  and repeat at points  $(na, 0)$  for all integer  $n$ . The magnetic field is then

$$\mathbf{B}(x, y) = \frac{\mu_0 I}{2\pi} \begin{bmatrix} \sum_{n=-\infty}^{\infty} \frac{-y}{(x-na)^2 + y^2} \\ \sum_{n=-\infty}^{\infty} \frac{x-na}{(x-na)^2 + y^2} \end{bmatrix}.$$

The infinite sums look troublesome but there is a well-known formula from analysis that can help:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n + z} = \pi \cot(\pi z).$$

### 2.1 $x$ Component

The summand can be re-expressed using partial fractions in the form

$$\frac{-y}{(x - na)^2 + y^2} = \frac{b}{n + c} + \frac{d}{n + e}$$

for some constants  $b, c, d, e$ :

$$\begin{aligned} \frac{-y}{x^2 - 2axn + a^2n^2 + y^2} &= \frac{bn + be + dn + cd}{(n + c)(n + e)} \\ \frac{\frac{-y}{a^2}}{n^2 - \frac{2x}{a}n + \frac{x^2 + y^2}{a^2}} &= \frac{(b + d)n + (be + cd)}{n^2 + (c + e)n + ce}. \end{aligned}$$

Set  $c = -\frac{x}{a} - f$  and  $e = -\frac{x}{a} + f$  for some constant  $f$ . We then have

$$ce = \left(-\frac{x}{a} - f\right) \left(-\frac{x}{a} + f\right) = \frac{x^2}{a^2} - f^2 = \frac{x^2 + y^2}{a^2}$$

therefore  $-f^2 = \frac{y^2}{a^2}$  and  $f = \pm i\frac{y}{a}$  are solutions. Thus  $c = -\frac{x}{a} - i\frac{y}{a}$  and  $e = -\frac{x}{a} + i\frac{y}{a}$ .

On the numerator, note that  $b + d = 0$ , so  $be + cd = b(e - c) = b(2i\frac{y}{a}) = \frac{-y}{a^2}$ . Cancelling gives  $b = \frac{i}{2a}$ . Using all the above values yields

$$\frac{-y}{(x - na)^2 + y^2} = \frac{\frac{i}{2a}}{n + \frac{-x - iy}{a}} + \frac{-\frac{i}{2a}}{n + \frac{-x + iy}{a}}.$$

Using the analytic formula for the infinite sum (twice) turns this into

$$\sum_{n=-\infty}^{\infty} \frac{-y}{(x - na)^2 + y^2} = \frac{i}{2a} \pi \cot\left(\pi \frac{-x - iy}{a}\right) - \frac{i}{2a} \pi \cot\left(\pi \frac{-x + iy}{a}\right).$$

The right-hand side is not obviously a real number, although it should be. The formula for a complex cotangent is

$$\cot(x + iy) = \frac{\sin(2x) - i \sinh(2y)}{\cosh(2y) - \cos(2x)},$$

which can expand the formula as follows:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{-y}{(x - na)^2 + y^2} &= \frac{i}{2a} \pi \frac{\sin(-2\frac{\pi}{a}x) - i \sinh(-2\frac{\pi}{a}y)}{\cosh(-2\frac{\pi}{a}y) - \cos(-2\frac{\pi}{a}x)} - \frac{i}{2a} \pi \frac{\sin(-2\frac{\pi}{a}x) - i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(-2\frac{\pi}{a}x)} \\ &= \frac{i}{2a} \pi \frac{-\sin(2\frac{\pi}{a}x) + i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} - \frac{i}{2a} \pi \frac{-\sin(2\frac{\pi}{a}x) - i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{i}{2a} \pi \frac{-\sin(2\frac{\pi}{a}x) + i \sinh(2\frac{\pi}{a}y) + \sin(2\frac{\pi}{a}x) + i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{i}{2a} \pi \frac{2i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{\pi}{a} \frac{\sinh(2\frac{\pi}{a}y)}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)}. \end{aligned}$$

## 2.2 y Component

The summand can be re-expressed using partial fractions in the form

$$\frac{x - na}{(x - na)^2 + y^2} = \frac{b}{n + c} + \frac{d}{n + e}$$

for some constants  $b, c, d, e$ :

$$\begin{aligned} \frac{x - na}{x^2 - 2axn + a^2n^2 + y^2} &= \frac{bn + be + dn + cd}{(n + c)(n + e)} \\ \frac{-\frac{1}{a}n + \frac{x}{a^2}}{n^2 - \frac{2x}{a}n + \frac{x^2 + y^2}{a^2}} &= \frac{(b + d)n + (be + cd)}{n^2 + (c + e)n + ce}. \end{aligned}$$

As before, the denominator gives  $c = -\frac{x}{a} - i\frac{y}{a}$  and  $e = -\frac{x}{a} + i\frac{y}{a}$ .

Equating coefficients on the numerator gives  $-\frac{1}{a} = b + d$  and

$$\frac{x}{a^2} = be + cd = \frac{x}{a}(-b - d) + i\frac{y}{a}(b - d) = \frac{x}{a^2} + i\frac{y}{a}(b - d),$$

therefore  $b = d = -\frac{1}{2a}$ . Using all the above values yields

$$\frac{x - na}{(x - na)^2 + y^2} = \frac{-\frac{1}{2a}}{n + \frac{-x-iy}{a}} + \frac{-\frac{1}{2a}}{n + \frac{-x+iy}{a}}.$$

The calculation continues as before with the  $i, -i$  numerators replaced by  $-1, -1$  until

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{x - na}{(x - na)^2 + y^2} &= \frac{1}{2a} \pi \frac{\sin(2\frac{\pi}{a}x) - i \sinh(2\frac{\pi}{a}y) + \sin(2\frac{\pi}{a}x) + i \sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{1}{2a} \pi \frac{2 \sin(2\frac{\pi}{a}x)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{\pi}{a} \frac{\sin(2\frac{\pi}{a}x)}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)}. \end{aligned}$$

### 2.3 Conclusion

Combining the two results in the previous sections gives

$$\begin{aligned} \mathbf{B}(x, y) &= \frac{\mu_0 I}{2\pi} \frac{\pi}{a} \frac{1}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)} \begin{bmatrix} \sinh(2\frac{\pi}{a}y) \\ \sin(2\frac{\pi}{a}x) \end{bmatrix} \\ &= \frac{\mu_0 I}{2a} \frac{1}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)} \begin{bmatrix} \sinh(2\frac{\pi}{a}y) \\ \sin(2\frac{\pi}{a}x) \end{bmatrix}. \end{aligned}$$

## 3 Infinite Regular Array of Current Sheets

The following integrals from Mathematica online

$$\begin{aligned} \int \frac{\sin x}{\cos x - k} dx &= -\log(\cos x - k) + \text{constant} \\ \int \frac{1}{\cos x - \cosh k} dx &= \frac{-2 \tan^{-1}(\tan(x/2)/\tanh(k/2))}{\sinh k} + \text{constant} \\ \int \frac{\sinh x}{k - \cosh x} dx &= -\log(k - \cosh x) + \text{constant} \\ \int \frac{1}{\cos k - \cosh x} dx &= \frac{2 \tan^{-1}(\tan(k/2)/\tanh(x/2))}{\sin k} + \text{constant}, \end{aligned}$$

which can be checked by differentiating the right-hand side, show the way to extend the magnetic formula to current sheets.