# Magnetic Field from an Infinite Array of Wires 

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September 2017

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## U.S. Department of Energy <br> USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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September 19, 2017

## 1 Single Wire

The magnetic field produced by an infinite (in the $z$ direction) wire carrying current $I$ is

$$
\mathbf{B}(x, y)=\frac{\mu_{0} I}{2 \pi} \frac{1}{x^{2}+y^{2}}\left[\begin{array}{c}
-y \\
x
\end{array}\right]
$$

which has magnitude $\frac{\mu_{0} I}{2 \pi r}$, where $r=\sqrt{x^{2}+y^{2}}$.

## 2 Infinite Regular Array of Wires

Suppose the wires are spaced by distance $a$ and repeat at points $(n a, 0)$ for all integer $n$. The magnetic field is then

$$
\mathbf{B}(x, y)=\frac{\mu_{0} I}{2 \pi}\left[\begin{array}{l}
\sum_{n=-\infty}^{\infty} \frac{-y}{(x-n a)^{2}+y^{2}} \\
\sum_{n=-\infty}^{\infty} \frac{x-n a}{(x-n a)^{2}+y^{2}}
\end{array}\right]
$$

The infinite sums look troublesome but there is a well-known formula from analysis that can help:

$$
\sum_{n=-\infty}^{\infty} \frac{1}{n+z}=\pi \cot (\pi z)
$$

## $2.1 x$ Component

The summand can be re-expressed using partial fractions in the form

$$
\frac{-y}{(x-n a)^{2}+y^{2}}=\frac{b}{n+c}+\frac{d}{n+e}
$$

for some constants $b, c, d, e$ :

$$
\begin{aligned}
\frac{-y}{x^{2}-2 a x n+a^{2} n^{2}+y^{2}} & =\frac{b n+b e+d n+c d}{(n+c)(n+e)} \\
\frac{\frac{-y}{a^{2}}}{n^{2}-\frac{2 x}{a} n+\frac{x^{2}+y^{2}}{a^{2}}} & =\frac{(b+d) n+(b e+c d)}{n^{2}+(c+e) n+c e} .
\end{aligned}
$$

Set $c=-\frac{x}{a}-f$ and $e=-\frac{x}{a}+f$ for some constant $f$. We then have

$$
c e=\left(-\frac{x}{a}-f\right)\left(-\frac{x}{a}+f\right)=\frac{x^{2}}{a^{2}}-f^{2}=\frac{x^{2}+y^{2}}{a^{2}}
$$

therefore $-f^{2}=\frac{y^{2}}{a^{2}}$ and $f= \pm i \frac{y}{a}$ are solutions. Thus $c=-\frac{x}{a}-i \frac{y}{a}$ and $e=-\frac{x}{a}+i \frac{y}{a}$.
On the numerator, note that $b+d=0$, so $b e+c d=b(e-c)=b\left(2 i \frac{y}{a}\right)=\frac{-y}{a^{2}}$. Cancelling gives $b=\frac{i}{2 a}$. Using all the above values yields

$$
\frac{-y}{(x-n a)^{2}+y^{2}}=\frac{\frac{i}{2 a}}{n+\frac{-x-i y}{a}}+\frac{-\frac{i}{2 a}}{n+\frac{-x+i y}{a}} .
$$

Using the analytic formula for the infinite sum (twice) turns this into

$$
\sum_{n=-\infty}^{\infty} \frac{-y}{(x-n a)^{2}+y^{2}}=\frac{i}{2 a} \pi \cot \left(\pi \frac{-x-i y}{a}\right)-\frac{i}{2 a} \pi \cot \left(\pi \frac{-x+i y}{a}\right)
$$

The right-hand side is not obviously a real number, although it should be. The formula for a complex cotangent is

$$
\cot (x+i y)=\frac{\sin (2 x)-i \sinh (2 y)}{\cosh (2 y)-\cos (2 x)}
$$

which can expand the formula as follows:

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{-y}{(x-n a)^{2}+y^{2}} & =\frac{i}{2 a} \pi \frac{\sin \left(-2 \frac{\pi}{a} x\right)-i \sinh \left(-2 \frac{\pi}{a} y\right)}{\cosh \left(-2 \frac{\pi}{a} y\right)-\cos \left(-2 \frac{\pi}{a} x\right)}-\frac{i}{2 a} \pi \frac{\sin \left(-2 \frac{\pi}{a} x\right)-i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(-2 \frac{\pi}{a} x\right)} \\
& =\frac{i}{2 a} \pi \frac{-\sin \left(2 \frac{\pi}{a} x\right)+i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)}-\frac{i}{2 a} \pi \frac{-\sin \left(2 \frac{\pi}{a} x\right)-i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)} \\
& =\frac{i}{2 a} \pi \frac{-\sin \left(2 \frac{\pi}{a} x\right)+i \sinh \left(2 \frac{\pi}{a} y\right)+\sin \left(2 \frac{\pi}{a} x\right)+i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)} \\
& =\frac{i}{2 a} \pi \frac{2 i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)} \\
& =\frac{\pi}{a} \frac{\sinh \left(2 \frac{\pi}{a} y\right)}{\cos \left(2 \frac{\pi}{a} x\right)-\cosh \left(2 \frac{\pi}{a} y\right)}
\end{aligned}
$$

## 2.2 y Component

The summand can be re-expressed using partial fractions in the form

$$
\frac{x-n a}{(x-n a)^{2}+y^{2}}=\frac{b}{n+c}+\frac{d}{n+e}
$$

for some constants $b, c, d, e$ :

$$
\begin{aligned}
\frac{x-n a}{x^{2}-2 a x n+a^{2} n^{2}+y^{2}} & =\frac{b n+b e+d n+c d}{(n+c)(n+e)} \\
\frac{-\frac{1}{a} n+\frac{x}{a^{2}}}{n^{2}-\frac{2 x}{a} n+\frac{x^{2}+y^{2}}{a^{2}}} & =\frac{(b+d) n+(b e+c d)}{n^{2}+(c+e) n+c e} .
\end{aligned}
$$

As before, the denominator gives $c=-\frac{x}{a}-i \frac{y}{a}$ and $e=-\frac{x}{a}+i \frac{y}{a}$.
Equating coefficients on the numerator gives $-\frac{1}{a}=b+d$ and

$$
\frac{x}{a^{2}}=b e+c d=\frac{x}{a}(-b-d)+i \frac{y}{a}(b-d)=\frac{x}{a^{2}}+i \frac{y}{a}(b-d)
$$

therefore $b=d=-\frac{1}{2 a}$. Using all the above values yields

$$
\frac{x-n a}{(x-n a)^{2}+y^{2}}=\frac{-\frac{1}{2 a}}{n+\frac{-x-i y}{a}}+\frac{-\frac{1}{2 a}}{n+\frac{-x+i y}{a}} .
$$

The calculation continues as before with the $i,-i$ numerators replaced by $-1,-1$ until

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{x-n a}{(x-n a)^{2}+y^{2}} & =\frac{1}{2 a} \pi \frac{\sin \left(2 \frac{\pi}{a} x\right)-i \sinh \left(2 \frac{\pi}{a} y\right)+\sin \left(2 \frac{\pi}{a} x\right)+i \sinh \left(2 \frac{\pi}{a} y\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)} \\
& =\frac{1}{2 a} \pi \frac{2 \sin \left(2 \frac{\pi}{a} x\right)}{\cosh \left(2 \frac{\pi}{a} y\right)-\cos \left(2 \frac{\pi}{a} x\right)} \\
& =\frac{\pi}{a} \frac{\sin \left(2 \frac{\pi}{a} x\right)}{\cos \left(2 \frac{\pi}{a} x\right)-\cosh \left(2 \frac{\pi}{a} y\right)} .
\end{aligned}
$$

### 2.3 Conclusion

Combining the two results in the previous sections gives

$$
\begin{aligned}
\mathbf{B}(x, y) & =\frac{\mu_{0} I}{2 \pi} \frac{\pi}{a} \frac{1}{\cos \left(2 \frac{\pi}{a} x\right)-\cosh \left(2 \frac{\pi}{a} y\right)}\left[\begin{array}{c}
\sinh \left(2 \frac{\pi}{a} y\right) \\
\sin \left(2 \frac{\pi}{a} x\right)
\end{array}\right] \\
& =\frac{\mu_{0} I}{2 a} \frac{1}{\cos \left(2 \frac{\pi}{a} x\right)-\cosh \left(2 \frac{\pi}{a} y\right)}\left[\begin{array}{c}
\sinh \left(2 \frac{\pi}{a} y\right) \\
\sin \left(2 \frac{\pi}{a} x\right)
\end{array}\right] .
\end{aligned}
$$

## 3 Infinite Regular Array of Current Sheets

The following integrals from Mathematica online

$$
\begin{aligned}
\int \frac{\sin x}{\cos x-k} \mathrm{~d} x & =-\log (\cos x-k)+\text { constant } \\
\int \frac{1}{\cos x-\cosh k} \mathrm{~d} x & =\frac{-2 \tan ^{-1}(\tan (x / 2) / \tanh (k / 2))}{\sinh k}+\text { constant } \\
\int \frac{\sinh x}{k-\cosh x} \mathrm{~d} x & =-\log (k-\cosh x)+\text { constant } \\
\int \frac{1}{\cos k-\cosh x} \mathrm{~d} x & =\frac{2 \tan ^{-1}(\tan (k / 2) / \tanh (x / 2))}{\sin k}+\text { constant },
\end{aligned}
$$

which can be checked by differentiating the right-hand side, show the way to extend the magnetic formula to current sheets.


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