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Magnetic Field from an Infinite Array of Wires

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1 Single Wire

The magnetic field produced by an infinite (in the z direction) wire carrying current I is

$$\mathbf{B}(x,y) = \frac{\mu_0 I}{2\pi}\frac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix},$$

which has magnitude $\frac{\mu_0 I}{2\pi r}$, where $r = \sqrt{x^2+y^2}$.

2 Infinite Regular Array of Wires

Suppose the wires are spaced by distance a and repeat at points $(na, 0)$ for all integer n. The magnetic field is then

$$\mathbf{B}(x,y) = \frac{\mu_0 I}{2\pi}\begin{bmatrix} \sum_{n=-\infty}^{\infty} \frac{-y}{(x-na)^2+y^2} \\ \sum_{n=-\infty}^{\infty} \frac{x-na}{(x-na)^2+y^2} \end{bmatrix}.$$

The infinite sums look troublesome but there is a well-known formula from analysis that can help:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+z} = \pi\cot(\pi z).$$

2.1 x Component

The summand can be re-expressed using partial fractions in the form

$$\frac{-y}{(x-na)^2+y^2} = \frac{b}{n+c} + \frac{d}{n+e}$$

for some constants b, c, d, e:

$$\begin{aligned} \frac{-y}{x^2-2axn+a^2n^2+y^2} &= \frac{bn+be+dn+cd}{(n+c)(n+e)} \\ \frac{\frac{-y}{a^2}}{n^2-\frac{2x}{a}n+\frac{x^2+y^2}{a^2}} &= \frac{(b+d)n+(be+cd)}{n^2+(c+e)n+ce}. \end{aligned}$$

Set $c = -\frac{x}{a} - f$ and $e = -\frac{x}{a} + f$ for some constant f. We then have

$$ce = \left(-\frac{x}{a} - f\right)\left(-\frac{x}{a} + f\right) = \frac{x^2}{a^2} - f^2 = \frac{x^2+y^2}{a^2}$$

therefore $-f^2 = \frac{y^2}{a^2}$ and $f = \pm i\frac{y}{a}$ are solutions. Thus $c = -\frac{x}{a} - i\frac{y}{a}$ and $e = -\frac{x}{a} + i\frac{y}{a}$.

On the numerator, note that $b+d=0$, so $be+cd = b(e-c) = b(2i\frac{y}{a}) = \frac{-y}{a^2}$. Cancelling gives $b = \frac{i}{2a}$. Using all the above values yields

$$\frac{-y}{(x-na)^2+y^2} = \frac{\frac{i}{2a}}{n+\frac{-x-iy}{a}} + \frac{-\frac{i}{2a}}{n+\frac{-x+iy}{a}}.$$

Using the analytic formula for the infinite sum (twice) turns this into

$$\sum_{n=-\infty}^{\infty} \frac{-y}{(x-na)^2+y^2} = \frac{i}{2a}\pi \cot\left(\pi\frac{-x-iy}{a}\right) - \frac{i}{2a}\pi \cot\left(\pi\frac{-x+iy}{a}\right).$$

The right-hand side is not obviously a real number, although it should be. The formula for a complex cotangent is

$$\cot(x+iy) = \frac{\sin(2x) - i\sinh(2y)}{\cosh(2y)-\cos(2x)},$$

which can expand the formula as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{-y}{(x-na)^2+y^2} &= \frac{i}{2a}\pi\frac{\sin(-2\frac{\pi}{a}x) - i\sinh(-2\frac{\pi}{a}y)}{\cosh(-2\frac{\pi}{a}y)-\cos(-2\frac{\pi}{a}x)} - \frac{i}{2a}\pi\frac{\sin(-2\frac{\pi}{a}x) - i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y)-\cos(-2\frac{\pi}{a}x)} \\
&= \frac{i}{2a}\pi\frac{-\sin(2\frac{\pi}{a}x) + i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y)-\cos(2\frac{\pi}{a}x)} - \frac{i}{2a}\pi\frac{-\sin(2\frac{\pi}{a}x) - i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y)-\cos(2\frac{\pi}{a}x)} \\
&= \frac{i}{2a}\pi\frac{-\sin(2\frac{\pi}{a}x) + i\sinh(2\frac{\pi}{a}y) + \sin(2\frac{\pi}{a}x) + i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y)-\cos(2\frac{\pi}{a}x)} \\
&= \frac{i}{2a}\pi\frac{2i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y)-\cos(2\frac{\pi}{a}x)} \\
&= \frac{\pi}{a}\frac{\sinh(2\frac{\pi}{a}y)}{\cos(2\frac{\pi}{a}x)-\cosh(2\frac{\pi}{a}y)}.
\end{aligned}$$

2.2 y Component

The summand can be re-expressed using partial fractions in the form

$$\frac{x-na}{(x-na)^2+y^2} = \frac{b}{n+c} + \frac{d}{n+e}$$

for some constants b, c, d, e:

$$\begin{aligned}
\frac{x-na}{x^2-2axn+a^2n^2+y^2} &= \frac{bn+be+dn+cd}{(n+c)(n+e)} \\
\frac{-\frac{1}{a}n+\frac{x}{a^2}}{n^2-\frac{2x}{a}n+\frac{x^2+y^2}{a^2}} &= \frac{(b+d)n+(be+cd)}{n^2+(c+e)n+ce}.
\end{aligned}$$

As before, the denominator gives $c = -\frac{x}{a} - i\frac{y}{a}$ and $e = -\frac{x}{a} + i\frac{y}{a}$.

Equating coefficients on the numerator gives $-\frac{1}{a} = b+d$ and

$$\frac{x}{a^2} = be+cd = \frac{x}{a}(-b-d) + i\frac{y}{a}(b-d) = \frac{x}{a^2} + i\frac{y}{a}(b-d),$$

therefore $b = d = -\frac{1}{2a}$. Using all the above values yields

$$\frac{x - na}{(x - na)^2 + y^2} = \frac{-\frac{1}{2a}}{n + \frac{-x-iy}{a}} + \frac{-\frac{1}{2a}}{n + \frac{-x+iy}{a}}.$$

The calculation continues as before with the $i, -i$ numerators replaced by $-1, -1$ until

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{x - na}{(x - na)^2 + y^2} &= \frac{1}{2a}\pi\frac{\sin(2\frac{\pi}{a}x) - i\sinh(2\frac{\pi}{a}y) + \sin(2\frac{\pi}{a}x) + i\sinh(2\frac{\pi}{a}y)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{1}{2a}\pi\frac{2\sin(2\frac{\pi}{a}x)}{\cosh(2\frac{\pi}{a}y) - \cos(2\frac{\pi}{a}x)} \\ &= \frac{\pi}{a}\frac{\sin(2\frac{\pi}{a}x)}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)}.\end{aligned}$$

2.3 Conclusion

Combining the two results in the previous sections gives

$$\begin{aligned}\mathbf{B}(x, y) &= \frac{\mu_0 I}{2\pi}\frac{\pi}{a}\frac{1}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)}\begin{bmatrix}\sinh(2\frac{\pi}{a}y) \\ \sin(2\frac{\pi}{a}x)\end{bmatrix} \\ &= \frac{\mu_0 I}{2a}\frac{1}{\cos(2\frac{\pi}{a}x) - \cosh(2\frac{\pi}{a}y)}\begin{bmatrix}\sinh(2\frac{\pi}{a}y) \\ \sin(2\frac{\pi}{a}x)\end{bmatrix}.\end{aligned}$$

3 Infinite Regular Array of Current Sheets

The following integrals from Mathematica online

$$\begin{aligned}\int \frac{\sin x}{\cos x - k}\,\mathrm{d}x &= -\log(\cos x - k) + \mathit{constant} \\ \int \frac{1}{\cos x - \cosh k}\,\mathrm{d}x &= \frac{-2\tan^{-1}(\tan(x/2)/\tanh(k/2))}{\sinh k} + \mathit{constant} \\ \int \frac{\sinh x}{k - \cosh x}\,\mathrm{d}x &= -\log(k - \cosh x) + \mathit{constant} \\ \int \frac{1}{\cos k - \cosh x}\,\mathrm{d}x &= \frac{2\tan^{-1}(\tan(k/2)/\tanh(x/2))}{\sin k} + \mathit{constant},\end{aligned}$$

which can be checked by differentiating the right-hand side, show the way to extend the magnetic formula to current sheets.