# Analytic Formulae for Fields in Windowframe Magnets 

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November 2016

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## U.S. Department of Energy <br> USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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November 7, 2016

## 1 Problem

We will consider an idealised magnetostatics problem where the iron of the windowframe has $\mu_{r}=\infty$ so produces a perfect infinite family of image currents (or equivalently the transverse B field component is zero at the surface of the iron). Within the interior of the windowframe, we further assume that $\mu_{r}=1$ i.e. $\mu=\mu_{0}$ and there is no magnetisation, $\mathbf{M}=\mathbf{0}$. Under these assumptions, Maxwell's equations for the $\mathbf{B}$ field become:

$$
\begin{aligned}
\nabla \cdot \mathbf{B} & =0 \\
\nabla \times \mathbf{B} & =\mu_{0} \mathbf{J}
\end{aligned}
$$

where $\mathbf{J}$ is the current density vector.

## 2 2D Version

Expanding these equations into their components gives:

$$
\begin{aligned}
\partial_{x} B_{x}+\partial_{y} B_{y}+\partial_{z} B_{z} & =0 \\
\partial_{y} B_{z}-\partial_{z} B_{y} & =\mu_{0} J_{x} \\
\partial_{z} B_{x}-\partial_{x} B_{z} & =\mu_{0} J_{y} \\
\partial_{x} B_{y}-\partial_{y} B_{x} & =\mu_{0} J_{z} .
\end{aligned}
$$

In a 2 D problem, there is no $z$ variation, so $\partial_{z}=0$, simplifying the equations to:

$$
\begin{aligned}
\partial_{x} B_{x}+\partial_{y} B_{y} & =0 \\
\partial_{y} B_{z} & =\mu_{0} J_{x} \\
-\partial_{x} B_{z} & =\mu_{0} J_{y} \\
\partial_{x} B_{y}-\partial_{y} B_{x} & =\mu_{0} J_{z} .
\end{aligned}
$$

For the windowframe design, we will solve the middle two equations by assuming there is only transverse field ( $B_{z}=0$ ) and longitudinal current ( $J_{x}=J_{y}=0$ ). This leaves:

$$
\begin{aligned}
\partial_{x} B_{x}+\partial_{y} B_{y} & =0 \\
\partial_{x} B_{y}-\partial_{y} B_{x} & =\mu_{0} J_{z} .
\end{aligned}
$$

## 3 Solutions with Separate $x, y$ Dependencies

One way of satisfying $\partial_{x} B_{x}+\partial_{y} B_{y}=0$ is to say $B_{x}$ is only a function of $y$ and $B_{y}$ is only a function of $x$, making both derivatives zero. That leaves the other equation

$$
\partial_{x} B_{y}(x)-\partial_{y} B_{x}(y)=B_{y}^{\prime}(x)-B_{x}^{\prime}(y)=\mu_{0} J_{z}
$$

If we only consider special current distributions that can be expressed in the form $J_{z}(x, y)=$ $f(x)+g(y)$, then separating the $x$ and $y$ dependencies gives that

$$
\begin{aligned}
B_{y}^{\prime}(x) & =\mu_{0} f(x) \\
-B_{x}^{\prime}(y) & =\mu_{0} g(y)
\end{aligned}
$$

is sufficient for a solution. Or more explicitly,

$$
\begin{aligned}
B_{y}(x) & =\mu_{0} \int^{x} f\left(x_{1}\right) \mathrm{d} x_{1} \\
B_{x}(y) & =-\mu_{0} \int^{y} g\left(y_{1}\right) \mathrm{d} y_{1}
\end{aligned}
$$

The condition that the field be perpendicular to the iron surfaces sets the constant of integration to give

$$
B_{y}\left( \pm R_{x}\right)=B_{x}\left( \pm R_{y}\right)=0
$$

where $R_{x}$ and $R_{y}$ are the half-width and half-height of the iron aperture respectively.

## 4 Dipole Field

Assume there are coils of thickness $X$ and longitudinal current density $\pm J$ on the left and righthand sides of the windowframe. Thus $f(x)=J$ when $x \in\left[-R_{x},-R_{x}+X\right]$ and $f(x)=-J$ when $x \in\left[R_{x}-X, R_{x}\right]$. The perfect reflection of the iron makes the images of these currents to extend to infinity vertically, so this representation of the current is valid with $g(y)=0$. The integral for $B_{y}$ trivially gives

$$
B_{y}=\mu_{0} J X
$$

for $x \in\left[-R_{x}+X, R_{x}-X\right]$, i.e. the aperture of the windowframe.
Realistically, with four rectangular coils packed into a square windowframe, the coils do not reach all the way to the top and bottom. Instead, the coils are of half-height $R_{x}-X$, giving a filling factor $\left(R_{x}-X\right) / R_{x}$. As a crude but often quite accurate approximation, this filling factor may be multiplied to give a more realistic dipole field:

$$
B_{y}=\frac{\mu_{0} J X\left(R_{x}-X\right)}{R_{x}}
$$

## 5 Quadrupole Gradient

We have current density of $J$ to the left and right, so $f(x)=J$ for $|x| \geq R_{x}-X$ but an opposite current of $-J$ on the top and bottom, so $g(y)=-J$ for $|y| \geq R_{y}-Y$. The $x$ integral gives that $B_{y}\left(R_{x}\right)=B_{y}\left(-R_{x}\right)+2 \mu_{0} J X$, which is a problem since both $B_{y}\left(R_{x}\right)$ and $B_{y}\left(-R_{x}\right)$ should be
zero by the iron boundary condition. However, there is a modification to the solution that still works:

$$
\begin{aligned}
B_{y}(x) & =\mu_{0} \int^{x} f\left(x_{1}\right) \mathrm{d} x_{1}+G x \\
B_{x}(y) & =-\mu_{0} \int^{y} g\left(y_{1}\right) \mathrm{d} y_{1}+G y
\end{aligned}
$$

The additional terms satisfy $\partial_{x} B_{x}+\partial_{y} B_{y}=0$ trivially, since the functional dependencies are still intact and $\partial_{x} B_{y}-\partial_{y} B_{x}=0$ because $G$ is the same in both equations. This has added the field of a quadrupole gradient $G$ onto the solution.

To cancel our previous problem, we need $2 R_{x} G=-2 \mu_{0} J X$, so $G=-\mu_{0} J X / R_{x}$. In the vertical direction, we require

$$
\begin{aligned}
-2 \mu_{0} J Y & =2 R_{y} G \\
-2 \mu_{0} J Y & =2 R_{y}\left(-\mu_{0} J X / R_{x}\right) \\
Y & =R_{y}\left(X / R_{x}\right)
\end{aligned}
$$

so the coil thicknesses must be in proportion to the aspect ratio of the iron. For a square frame where $R_{x}=R_{y}$, we must have $Y=X$ i.e. equal thickness coils.

In any case, the integral contributes zero within the bore of the windowframe, leaving a field $B_{y}=G x, B_{x}=G y$, which is a quadrupole of gradient

$$
G=-\frac{\mu_{0} J X}{R_{x}}=-\frac{\mu_{0} J Y}{R_{y}}
$$

For a square frame of full bore $B=2\left(R_{x}-X\right)$, we have $R_{x}=B / 2+X$, so

$$
G=-\frac{\mu_{0} J X}{B / 2+X}=-\frac{2 \mu_{0} J X}{B+2 X}=-\frac{2 \mu_{0} J X}{D}
$$

where $D=2 R_{x}$ is the diameter of the iron frame. The gradient will never have magnitude larger than $\mu_{0} J$, regardless of bore or coil thickness. This construction assumes rectangular coil cross-sections with empty corners, so it may be possible to exceed this limit by filling in the corners.


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