

## Analytic Formulae for Fields in Windowframe Magnets

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# Analytic Formulae for Fields in Windowframe Magnets

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## 1 Problem

We will consider an idealised magnetostatics problem where the iron of the windowframe has  $\mu_r = \infty$  so produces a perfect infinite family of image currents (or equivalently the transverse  $\mathbf{B}$  field component is zero at the surface of the iron). Within the interior of the windowframe, we further assume that  $\mu_r = 1$  i.e.  $\mu = \mu_0$  and there is no magnetisation,  $\mathbf{M} = \mathbf{0}$ . Under these assumptions, Maxwell's equations for the  $\mathbf{B}$  field become:

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J},\end{aligned}$$

where  $\mathbf{J}$  is the current density vector.

## 2 2D Version

Expanding these equations into their components gives:

$$\begin{aligned}\partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \\ \partial_y B_z - \partial_z B_y &= \mu_0 J_x \\ \partial_z B_x - \partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

In a 2D problem, there is no  $z$  variation, so  $\partial_z = 0$ , simplifying the equations to:

$$\begin{aligned}\partial_x B_x + \partial_y B_y &= 0 \\ \partial_y B_z &= \mu_0 J_x \\ -\partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

For the windowframe design, we will solve the middle two equations by assuming there is only transverse field ( $B_z = 0$ ) and longitudinal current ( $J_x = J_y = 0$ ). This leaves:

$$\begin{aligned}\partial_x B_x + \partial_y B_y &= 0 \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

### 3 Solutions with Separate $x$ , $y$ Dependencies

One way of satisfying  $\partial_x B_x + \partial_y B_y = 0$  is to say  $B_x$  is only a function of  $y$  and  $B_y$  is only a function of  $x$ , making both derivatives zero. That leaves the other equation

$$\partial_x B_y(x) - \partial_y B_x(y) = B'_y(x) - B'_x(y) = \mu_0 J_z.$$

If we only consider special current distributions that can be expressed in the form  $J_z(x, y) = f(x) + g(y)$ , then separating the  $x$  and  $y$  dependencies gives that

$$\begin{aligned} B'_y(x) &= \mu_0 f(x) \\ -B'_x(y) &= \mu_0 g(y) \end{aligned}$$

is sufficient for a solution. Or more explicitly,

$$\begin{aligned} B_y(x) &= \mu_0 \int^x f(x_1) dx_1 \\ B_x(y) &= -\mu_0 \int^y g(y_1) dy_1. \end{aligned}$$

The condition that the field be perpendicular to the iron surfaces sets the constant of integration to give

$$B_y(\pm R_x) = B_x(\pm R_y) = 0,$$

where  $R_x$  and  $R_y$  are the half-width and half-height of the iron aperture respectively.

### 4 Dipole Field

Assume there are coils of thickness  $X$  and longitudinal current density  $\pm J$  on the left and right-hand sides of the windowframe. Thus  $f(x) = J$  when  $x \in [-R_x, -R_x + X]$  and  $f(x) = -J$  when  $x \in [R_x - X, R_x]$ . The perfect reflection of the iron makes the images of these currents to extend to infinity vertically, so this representation of the current is valid with  $g(y) = 0$ . The integral for  $B_y$  trivially gives

$$B_y = \mu_0 J X$$

for  $x \in [-R_x + X, R_x - X]$ , i.e. the aperture of the windowframe.

Realistically, with four rectangular coils packed into a square windowframe, the coils do not reach all the way to the top and bottom. Instead, the coils are of half-height  $R_x - X$ , giving a filling factor  $(R_x - X)/R_x$ . As a crude but often quite accurate approximation, this filling factor may be multiplied to give a more realistic dipole field:

$$B_y = \frac{\mu_0 J X (R_x - X)}{R_x}.$$

### 5 Quadrupole Gradient

We have current density of  $J$  to the left and right, so  $f(x) = J$  for  $|x| \geq R_x - X$  but an opposite current of  $-J$  on the top and bottom, so  $g(y) = -J$  for  $|y| \geq R_y - Y$ . The  $x$  integral gives that  $B_y(R_x) = B_y(-R_x) + 2\mu_0 J X$ , which is a problem since both  $B_y(R_x)$  and  $B_y(-R_x)$  should be

zero by the iron boundary condition. However, there is a modification to the solution that still works:

$$\begin{aligned} B_y(x) &= \mu_0 \int^x f(x_1) dx_1 + Gx \\ B_x(y) &= -\mu_0 \int^y g(y_1) dy_1 + Gy. \end{aligned}$$

The additional terms satisfy  $\partial_x B_x + \partial_y B_y = 0$  trivially, since the functional dependencies are still intact and  $\partial_x B_y - \partial_y B_x = 0$  because  $G$  is the same in both equations. This has added the field of a quadrupole gradient  $G$  onto the solution.

To cancel our previous problem, we need  $2R_x G = -2\mu_0 JX$ , so  $G = -\mu_0 JX/R_x$ . In the vertical direction, we require

$$\begin{aligned} -2\mu_0 JY &= 2R_y G \\ -2\mu_0 JY &= 2R_y(-\mu_0 JX/R_x) \\ Y &= R_y(X/R_x), \end{aligned}$$

so the coil thicknesses must be in proportion to the aspect ratio of the iron. For a square frame where  $R_x = R_y$ , we must have  $Y = X$  i.e. equal thickness coils.

In any case, the integral contributes zero within the bore of the windowframe, leaving a field  $B_y = Gx$ ,  $B_x = Gy$ , which is a quadrupole of gradient

$$G = -\frac{\mu_0 JX}{R_x} = -\frac{\mu_0 JY}{R_y}.$$

For a square frame of full bore  $B = 2(R_x - X)$ , we have  $R_x = B/2 + X$ , so

$$G = -\frac{\mu_0 JX}{B/2 + X} = -\frac{2\mu_0 JX}{B + 2X} = -\frac{2\mu_0 JX}{D},$$

where  $D = 2R_x$  is the diameter of the iron frame. The gradient will never have magnitude larger than  $\mu_0 J$ , regardless of bore or coil thickness. This construction assumes rectangular coil cross-sections with empty corners, so it may be possible to exceed this limit by filling in the corners.