

Analytic Formulae for Fields in Windowframe Magnets

S. Brooks

November 2016

Collider Accelerator Department
Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No. DE-SC0012704 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Analytic Formulae for Fields in Windowframe Magnets

Stephen Brooks

November 7, 2016

1 Problem

We will consider an idealised magnetostatics problem where the iron of the windowframe has $\mu_r = \infty$ so produces a perfect infinite family of image currents (or equivalently the transverse \mathbf{B} field component is zero at the surface of the iron). Within the interior of the windowframe, we further assume that $\mu_r = 1$ i.e. $\mu = \mu_0$ and there is no magnetisation, $\mathbf{M} = \mathbf{0}$. Under these assumptions, Maxwell's equations for the \mathbf{B} field become:

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J},\end{aligned}$$

where \mathbf{J} is the current density vector.

2 2D Version

Expanding these equations into their components gives:

$$\begin{aligned}\partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \\ \partial_y B_z - \partial_z B_y &= \mu_0 J_x \\ \partial_z B_x - \partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

In a 2D problem, there is no z variation, so $\partial_z = 0$, simplifying the equations to:

$$\begin{aligned}\partial_x B_x + \partial_y B_y &= 0 \\ \partial_y B_z &= \mu_0 J_x \\ -\partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

For the windowframe design, we will solve the middle two equations by assuming there is only transverse field ($B_z = 0$) and longitudinal current ($J_x = J_y = 0$). This leaves:

$$\begin{aligned}\partial_x B_x + \partial_y B_y &= 0 \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z.\end{aligned}$$

3 Solutions with Separate x , y Dependencies

One way of satisfying $\partial_x B_x + \partial_y B_y = 0$ is to say B_x is only a function of y and B_y is only a function of x , making both derivatives zero. That leaves the other equation

$$\partial_x B_y(x) - \partial_y B_x(y) = B'_y(x) - B'_x(y) = \mu_0 J_z.$$

If we only consider special current distributions that can be expressed in the form $J_z(x, y) = f(x) + g(y)$, then separating the x and y dependencies gives that

$$\begin{aligned} B'_y(x) &= \mu_0 f(x) \\ -B'_x(y) &= \mu_0 g(y) \end{aligned}$$

is sufficient for a solution. Or more explicitly,

$$\begin{aligned} B_y(x) &= \mu_0 \int^x f(x_1) dx_1 \\ B_x(y) &= -\mu_0 \int^y g(y_1) dy_1. \end{aligned}$$

The condition that the field be perpendicular to the iron surfaces sets the constant of integration to give

$$B_y(\pm R_x) = B_x(\pm R_y) = 0,$$

where R_x and R_y are the half-width and half-height of the iron aperture respectively.

4 Dipole Field

Assume there are coils of thickness X and longitudinal current density $\pm J$ on the left and right-hand sides of the windowframe. Thus $f(x) = J$ when $x \in [-R_x, -R_x + X]$ and $f(x) = -J$ when $x \in [R_x - X, R_x]$. The perfect reflection of the iron makes the images of these currents to extend to infinity vertically, so this representation of the current is valid with $g(y) = 0$. The integral for B_y trivially gives

$$B_y = \mu_0 J X$$

for $x \in [-R_x + X, R_x - X]$, i.e. the aperture of the windowframe.

Realistically, with four rectangular coils packed into a square windowframe, the coils do not reach all the way to the top and bottom. Instead, the coils are of half-height $R_x - X$, giving a filling factor $(R_x - X)/R_x$. As a crude but often quite accurate approximation, this filling factor may be multiplied to give a more realistic dipole field:

$$B_y = \frac{\mu_0 J X (R_x - X)}{R_x}.$$

5 Quadrupole Gradient

We have current density of J to the left and right, so $f(x) = J$ for $|x| \geq R_x - X$ but an opposite current of $-J$ on the top and bottom, so $g(y) = -J$ for $|y| \geq R_y - Y$. The x integral gives that $B_y(R_x) = B_y(-R_x) + 2\mu_0 J X$, which is a problem since both $B_y(R_x)$ and $B_y(-R_x)$ should be

zero by the iron boundary condition. However, there is a modification to the solution that still works:

$$\begin{aligned} B_y(x) &= \mu_0 \int^x f(x_1) dx_1 + Gx \\ B_x(y) &= -\mu_0 \int^y g(y_1) dy_1 + Gy. \end{aligned}$$

The additional terms satisfy $\partial_x B_x + \partial_y B_y = 0$ trivially, since the functional dependencies are still intact and $\partial_x B_y - \partial_y B_x = 0$ because G is the same in both equations. This has added the field of a quadrupole gradient G onto the solution.

To cancel our previous problem, we need $2R_x G = -2\mu_0 JX$, so $G = -\mu_0 JX/R_x$. In the vertical direction, we require

$$\begin{aligned} -2\mu_0 JY &= 2R_y G \\ -2\mu_0 JY &= 2R_y(-\mu_0 JX/R_x) \\ Y &= R_y(X/R_x), \end{aligned}$$

so the coil thicknesses must be in proportion to the aspect ratio of the iron. For a square frame where $R_x = R_y$, we must have $Y = X$ i.e. equal thickness coils.

In any case, the integral contributes zero within the bore of the windowframe, leaving a field $B_y = Gx$, $B_x = Gy$, which is a quadrupole of gradient

$$G = -\frac{\mu_0 JX}{R_x} = -\frac{\mu_0 JY}{R_y}.$$

For a square frame of full bore $B = 2(R_x - X)$, we have $R_x = B/2 + X$, so

$$G = -\frac{\mu_0 JX}{B/2 + X} = -\frac{2\mu_0 JX}{B + 2X} = -\frac{2\mu_0 JX}{D},$$

where $D = 2R_x$ is the diameter of the iron frame. The gradient will never have magnitude larger than $\mu_0 J$, regardless of bore or coil thickness. This construction assumes rectangular coil cross-sections with empty corners, so it may be possible to exceed this limit by filling in the corners.