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Analytic Formulae for Fields in Windowframe Magnets

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1 Problem

We will consider an idealised magnetostatics problem where the iron of the windowframe has $\mu_r = \infty$ so produces a perfect infinite family of image currents (or equivalently the transverse **B** field component is zero at the surface of the iron). Within the interior of the windowframe, we further assume that $\mu_r = 1$ i.e. $\mu = \mu_0$ and there is no magnetisation, $\mathbf{M} = \mathbf{0}$. Under these assumptions, Maxwell's equations for the **B** field become:

$$\nabla \cdot \mathbf{B} = 0 \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

where \mathbf{J} is the current density vector.

2 2D Version

Expanding these equations into their components gives:

 ∂_x

$$B_x + \partial_y B_y + \partial_z B_z = 0$$

$$\partial_y B_z - \partial_z B_y = \mu_0 J_x$$

$$\partial_z B_x - \partial_x B_z = \mu_0 J_y$$

$$\partial_x B_y - \partial_y B_x = \mu_0 J_z$$

In a 2D problem, there is no z variation, so $\partial_z = 0$, simplifying the equations to:

$$\begin{array}{rcl} \partial_x B_x + \partial_y B_y &=& 0\\ & \partial_y B_z &=& \mu_0 J_x\\ & -\partial_x B_z &=& \mu_0 J_y\\ \partial_x B_y - \partial_y B_x &=& \mu_0 J_z. \end{array}$$

For the window frame design, we will solve the middle two equations by assuming there is only transverse field $(B_z = 0)$ and longitudinal current $(J_x = J_y = 0)$. This leaves:

$$\partial_x B_x + \partial_y B_y = 0$$

$$\partial_x B_y - \partial_y B_x = \mu_0 J_z$$

3 Solutions with Separate x, y Dependencies

One way of satisfying $\partial_x B_x + \partial_y B_y = 0$ is to say B_x is only a function of y and B_y is only a function of x, making both derivatives zero. That leaves the other equation

$$\partial_x B_y(x) - \partial_y B_x(y) = B'_y(x) - B'_x(y) = \mu_0 J_z.$$

If we only consider special current distributions that can be expressed in the form $J_z(x, y) = f(x) + g(y)$, then separating the x and y dependencies gives that

$$B'_y(x) = \mu_0 f(x)$$

-B'_x(y) = \mu_0 g(y)

is sufficient for a solution. Or more explicitly,

$$B_y(x) = \mu_0 \int^x f(x_1) \, dx_1$$

$$B_x(y) = -\mu_0 \int^y g(y_1) \, dy_1.$$

The condition that the field be perpendicular to the iron surfaces sets the constant of integration to give

$$B_y(\pm R_x) = B_x(\pm R_y) = 0,$$

where R_x and R_y are the half-width and half-height of the iron aperture respectively.

4 Dipole Field

Assume there are coils of thickness X and longitudinal current density $\pm J$ on the left and righthand sides of the windowframe. Thus f(x) = J when $x \in [-R_x, -R_x + X]$ and f(x) = -J when $x \in [R_x - X, R_x]$. The perfect reflection of the iron makes the images of these currents to extend to infinity vertically, so this representation of the current is valid with g(y) = 0. The integral for B_y trivially gives

$$B_y = \mu_0 J X$$

for $x \in [-R_x + X, R_x - X]$, i.e. the aperture of the window frame.

Realistically, with four rectangular coils packed into a square windowframe, the coils do not reach all the way to the top and bottom. Instead, the coils are of half-height $R_x - X$, giving a filling factor $(R_x - X)/R_x$. As a crude but often quite accurate approximation, this filling factor may be multiplied to give a more realistic dipole field:

$$B_y = \frac{\mu_0 J X (R_x - X)}{R_x}.$$

5 Quadrupole Gradient

We have current density of J to the left and right, so f(x) = J for $|x| \ge R_x - X$ but an opposite current of -J on the top and bottom, so g(y) = -J for $|y| \ge R_y - Y$. The x integral gives that $B_y(R_x) = B_y(-R_x) + 2\mu_0 JX$, which is a problem since both $B_y(R_x)$ and $B_y(-R_x)$ should be zero by the iron boundary condition. However, there is a modification to the solution that still works:

$$B_y(x) = \mu_0 \int^x f(x_1) \, dx_1 + Gx$$

$$B_x(y) = -\mu_0 \int^y g(y_1) \, dy_1 + Gy$$

The additional terms satisfy $\partial_x B_x + \partial_y B_y = 0$ trivially, since the functional dependencies are still intact and $\partial_x B_y - \partial_y B_x = 0$ because G is the same in both equations. This has added the field of a quadrupole gradient G onto the solution.

To cancel our previous problem, we need $2R_xG = -2\mu_0 JX$, so $G = -\mu_0 JX/R_x$. In the vertical direction, we require

$$\begin{array}{rcl} -2\mu_0 JY &=& 2R_y G\\ -2\mu_0 JY &=& 2R_y (-\mu_0 JX/R_x)\\ Y &=& R_y (X/R_x), \end{array}$$

so the coil thicknesses must be in proportion to the aspect ratio of the iron. For a square frame where $R_x = R_y$, we must have Y = X i.e. equal thickness coils.

In any case, the integral contributes zero within the bore of the window frame, leaving a field $B_y = Gx$, $B_x = Gy$, which is a quadrupole of gradient

$$G = -\frac{\mu_0 J X}{R_x} = -\frac{\mu_0 J Y}{R_y}.$$

For a square frame of full bore $B = 2(R_x - X)$, we have $R_x = B/2 + X$, so

$$G = -\frac{\mu_0 J X}{B/2 + X} = -\frac{2\mu_0 J X}{B + 2X} = -\frac{2\mu_0 J X}{D},$$

where $D = 2R_x$ is the diameter of the iron frame. The gradient will never have magnitude larger than $\mu_0 J$, regardless of bore or coil thickness. This construction assumes rectangular coil cross-sections with empty corners, so it may be possible to exceed this limit by filling in the corners.