

Bounded Approximate Solutions of Linear Systems using SVD

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1 Definitions

The Singular Value Decomposition (SVD) of a complex matrix is conventionally $A = U\Sigma V^*$, where M^* denotes \bar{M}^T . Here, U and V are unitary matrices with $U^{-1} = U^*$ and Σ is diagonal with $\Sigma = \text{diag}[\sigma_n]$. For real matrices this is just $A = U\Sigma V^T$ and unitarity is equivalent to $U^{-1} = U^T$, i.e. orthogonality. In fact, V^T is also orthogonal since $(V^T)^{-1} = (V^{-1})^{-1} = V = (V^T)^T$, which means the simpler definition $A = U\Sigma V$ can be used for the rest of this note.

2 Fundamental Problem

In control systems, one often uses a linear or locally-linear model to determine the required inputs. Suppose an input vector change $\mathbf{x} \in X$ produces an output response $A\mathbf{x} \in Y$ that is meant to achieve some desired change $\mathbf{b} \in Y$. The input and output spaces X and Y may have different dimensionalities and therefore A can be a rectangular matrix. This means that an exact solution may not be possible, particularly if $\dim Y > \dim X$. Thus the ‘best’ solution can be formulated as the minimisation problem of finding $\arg \min |A\mathbf{x} - \mathbf{b}|_Y$.

However, particularly in the case of ill-conditioned matrices, the exact solution may require unacceptably large control inputs. What is required practically is the best approximation that can be achieved while \mathbf{x} is not too large. This suggests casting the fundamental problem as

$$\arg \min_{|\mathbf{x}|_X \leq r} |A\mathbf{x} - \mathbf{b}|_Y$$

with $r > 0$ being chosen depending on how large a solution is acceptable. As $r \rightarrow \infty$, the value will eventually settle at the exact or optimum solution if one exists.

3 Solution using SVD

The SVD decomposition of A gives

$$\arg \min_{|\mathbf{x}|_X \leq r} |A\mathbf{x} - \mathbf{b}|_Y = \arg \min_{|\mathbf{x}|_X \leq r} |U\Sigma V\mathbf{x} - \mathbf{b}|_Y.$$

Here, A and Σ are possibly-rectangular matrices mapping from X to Y , V is a square orthogonal matrix mapping X to itself and U is another mapping Y to itself. Note that any orthogonal

matrix U preserves the norm as $|U\mathbf{x}|^2 = \mathbf{x}^T U^T U \mathbf{x} = \mathbf{x}^T U^{-1} U \mathbf{x} = \mathbf{x}^T \mathbf{x} = |\mathbf{x}|^2$ so $|U\mathbf{x}| = |\mathbf{x}|$ as norms are non-negative. In particular,

$$|\mathbf{x}|_X = |V\mathbf{x}|_X \quad \text{and} \quad |U\Sigma V\mathbf{x} - \mathbf{b}|_Y = |\Sigma V\mathbf{x} - U^{-1}\mathbf{b}|_Y,$$

where the second equality has multiplied by the unitary matrix U^{-1} . This means that

$$\arg \min_{|\mathbf{x}|_X \leq r} |A\mathbf{x} - \mathbf{b}|_Y = \arg \min_{|V\mathbf{x}|_X \leq r} |\Sigma V\mathbf{x} - U^{-1}\mathbf{b}|_Y.$$

Defining vectors $\mathbf{v} = V\mathbf{x}$ and $\mathbf{u} = U^{-1}\mathbf{b}$ this becomes

$$\arg \min_{|\mathbf{x}|_X \leq r} |A\mathbf{x} - \mathbf{b}|_Y = V^{-1} \arg \min_{|\mathbf{v}|_X \leq r} |\Sigma\mathbf{v} - \mathbf{u}|_Y,$$

where the right-hand arg min is now understood to find the value of \mathbf{v} , so the premultiplication for $\mathbf{x} = V^{-1}\mathbf{v}$ is required. The problem has now been simplified into one with a diagonal matrix instead of A .

3.1 Exact Minimum Solution

If the unrestricted arg min also satisfies $|\mathbf{x}|_X \leq r$ then it is the solution. The unrestricted minimum is a fixed point of the norm expression squared:

$$\begin{aligned} 0 &= \frac{\partial}{\partial v_n} |\Sigma\mathbf{v} - \mathbf{u}|_Y^2 = \frac{\partial}{\partial v_n} \sum_{i=1}^{\dim Y} (\Sigma\mathbf{v} - \mathbf{u})_i^2 = \frac{\partial}{\partial v_n} \sum_{i=1}^{\dim Y} (1_{i \leq \dim X} \sigma_i v_i - u_i)^2 \\ &= \frac{\partial}{\partial v_n} (\sigma_n v_n - u_n)^2 = \frac{\partial}{\partial v_n} (\sigma_n^2 v_n^2 - 2\sigma_n v_n u_n + u_n^2) = 2\sigma_n^2 v_n - 2\sigma_n u_n \\ &\Leftrightarrow \sigma_n (\sigma_n v_n - u_n) = 0. \end{aligned}$$

For each n , this is true if either $v_n = u_n/\sigma_n$ or $\sigma_n = 0$. In the latter case, the Σ matrix does not range over the full dimensionality of Y and any value of v_n may be chosen because the minimum is non-unique. It is usually best to choose $v_n = 0$ in all such ambiguous cases, since this corresponds to the minimum with smallest $|\mathbf{v}|_X = |\mathbf{x}|_X$. There is also the case when $\dim Y < \dim X$, where the above equation reduces to $0 = 0$ for $n > \dim Y$, giving no constraint on v_n , which should be set to zero by the same argument. The exact minimum can be written explicitly as

$$\mathbf{x} = V^{-1}[(U^{-1}\mathbf{b})_{n/0} \sigma_n], \quad \text{where} \quad x/0y = \begin{cases} x/y & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

3.2 Constrained Minimum

The function $|\Sigma\mathbf{v} - \mathbf{u}|_Y$ does not have multiple disconnected local minima, so if the exact minimum with smallest norm found in the previous section still has $|\mathbf{x}|_X > r$, the constrained minimum must have $|\mathbf{x}|_X = r$ rather than being an interior point. The local gradient found in the previous section

$$\nabla_{\mathbf{v}} |\Sigma\mathbf{v} - \mathbf{u}|_Y^2 = 2[\sigma_n^2 v_n - \sigma_n u_n]$$

must be a scalar multiple of the position \mathbf{v} because otherwise it has some component parallel to the surface of the radius r hypersphere and the value of the function can be reduced. The

gradient is expected to be negative with increasing r , anti-parallel to \mathbf{v} , so for some $\lambda > 0$,

$$\begin{aligned} \nabla_{\mathbf{v}} |\Sigma \mathbf{v} - \mathbf{u}|_Y^2 &= -2\lambda^2 \mathbf{v} \\ \Leftrightarrow 2(\sigma_n^2 v_n - \sigma_n u_n) &= -2\lambda^2 v_n \\ \Leftrightarrow (\sigma_n^2 + \lambda^2)v_n - \sigma_n u_n &= 0 \\ &\Leftrightarrow v_n = \frac{\sigma_n u_n}{\sigma_n^2 + \lambda^2}. \end{aligned}$$

For the case where $n > \dim Y$, the gradient of that component is zero as before and $0 = -2\lambda^2 v_n$, so $v_n = 0$. The constrained minimum can be written explicitly as

$$\mathbf{x} = V^{-1} \begin{bmatrix} \sigma_n (U^{-1} \mathbf{b})_n \\ \sigma_n^2 + \lambda^2 \end{bmatrix}, \quad \text{where we set } (U^{-1} \mathbf{b})_n = 0 \quad \text{if } n > \dim Y.$$

The norm of \mathbf{x} decreases monotonically with λ because $|\mathbf{x}|_X = |\mathbf{v}|_X$ and every element of \mathbf{v} decreases in magnitude with increasing λ . As $\lambda \rightarrow 0$ the constrained minimum tends towards the exact minimum. As $\lambda \rightarrow \infty$, the constrained minimum tends towards $\mathbf{0}$ but if renormalised, the limit has $v_n = \sigma_n u_n$, which is $-\frac{1}{2}$ times the gradient of $|\Sigma \mathbf{v} - \mathbf{u}|_Y^2$ at $\mathbf{v} = \mathbf{0}$. Thus the large λ limit corresponds to a infinitesimal ‘steepest descent’ step.

The continuity and monotonicity of $|\mathbf{x}|_X = r(\lambda)$ ensures a value of λ can always be found for any value of r between 0 and the norm of the exact solution point. For example, a bisection search or root-finding algorithm can determine λ for a given r , after first checking the exact solution point does not have norm less than r .

3.3 Implementation Note

Using the orthogonal property of U and V , entries $(U^{-1} \mathbf{b})_n$ should be calculated as the much faster equivalent $(U^T \mathbf{b})_n$ and the premultiplication by V^{-1} should be implemented as V^T . Once the SVD is calculated, nothing slower than matrix-vector multiplication is required.

4 Units

Elements of the vector spaces X and Y can be physical quantities with units $[X]$ and $[Y]$ respectively. By definition, A has units $[Y]/[X]$. In the SVD, the entries of U and V have no units as they map within the same space, leaving Σ and its entries σ_n with units $[Y]/[X]$. The parameter λ in the previous section was defined to also have units $[Y]/[X]$ but r has units $[X]$.

5 Identity with the Levenberg–Marquardt Algorithm

The Levenberg–Marquardt algorithm involves a ‘damped’ least squares step, which for a Jacobian matrix J involves solving

$$(J^T J + \lambda_{LM} I) \mathbf{x} = J^T \mathbf{b},$$

where $\lambda_{LM} \geq 0$ is called the damping factor. If the Jacobian is decomposed via SVD as $J = U \Sigma V$, this becomes

$$(V^T \Sigma U^T U \Sigma V + \lambda_{LM} I) \mathbf{x} = V^T \Sigma U^T \mathbf{b}$$

and noting that $U^T U = I$ by orthogonality of U ,

$$(V^T \Sigma^2 V + \lambda_{LM} I) \mathbf{x} = V^T \Sigma U^T \mathbf{b}.$$

Pre-multiplying both sides by V and using its orthogonality $V V^T = I$ gives

$$\begin{aligned} (\Sigma^2 V + \lambda_{LM} V) \mathbf{x} &= \Sigma U^T \mathbf{b} \\ \Rightarrow (\Sigma^2 + \lambda_{LM}) V \mathbf{x} &= \Sigma U^T \mathbf{b}. \end{aligned}$$

This is starting to look vaguely familiar. Inverting the left-hand side to give an expression for \mathbf{x} yields

$$\begin{aligned} \mathbf{x} &= V^{-1} (\Sigma^2 + \lambda_{LM} I)^{-1} \Sigma U^T \mathbf{b} \\ &= V^{-1} (\Sigma^2 + \lambda_{LM} I)^{-1} \Sigma U^{-1} \mathbf{b}. \end{aligned}$$

Comparing this to the constrained minimum formula with parameter λ from a previous section:

$$\mathbf{x} = V^{-1} \left[\frac{\sigma_n (U^{-1} \mathbf{b})_n}{\sigma_n^2 + \lambda^2} \right]$$

and noting that $\Sigma = \text{diag}[\sigma_n]$ reveals that these are the same formulae if $\lambda_{LM} = \lambda^2$.