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# Off-Axis Magnetic Fields Extrapolated from On-Axis Multipoles

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## 1 Potential Term

In cylindrical polar coordinates $(r, \theta, z)$, consider the magnetic scalar potential

$$\phi = \sin(n\theta + \psi) r^k f(z)$$

for some integers $n, k$, angle $\psi$ and function $f$. The associated magnetic field is

$$\mathbf{B} = \nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z = \begin{bmatrix} \sin(n\theta+\psi) k r^{k-1} f(z) \\ n\cos(n\theta+\psi) r^{k-1} f(z) \\ \sin(n\theta+\psi) r^k f'(z) \end{bmatrix}_{r\theta z},$$

which automatically satisfies $\nabla \times \mathbf{B} = \nabla \times \nabla\phi = \mathbf{0}$. The only remaining condition on $\phi$ is

$$\begin{aligned} 0 = \nabla\cdot\mathbf{B} = \nabla\cdot\nabla\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \\ &= \sin(n\theta+\psi)\left(k(k-1)r^{k-2}f(z) + kr^{k-2}f(z) - n^2 r^{k-2} f(z) + r^k f''(z)\right) \\ &= \sin(n\theta+\psi) r^{k-2}\left((k^2-n^2)f(z) + r^2 f''(z)\right). \end{aligned}$$

### 1.1 Long Multipole

If there is no $z$ behaviour ($f = 1$, say), then for this to hold for all $r, \theta$ requires $k^2 = n^2$. Since $k \leq 0$ cases have a singularity at the origin, put $k = n \geq 1$. This gives the field for an infinitely long multipole:

$$\phi = \sin(n\theta+\psi) r^n, \qquad \mathbf{B} = \begin{bmatrix} n\sin(n\theta+\psi) r^{n-1} \\ n\cos(n\theta+\psi) r^{n-1} \\ 0 \end{bmatrix}_{r\theta z} = n r^{n-1} \begin{bmatrix} \sin((n-1)\theta+\psi) \\ \cos((n-1)\theta+\psi) \\ 0 \end{bmatrix}.$$

$n = 1$ corresponds to a dipole, $n = 2$ to a quadrupole, etc. Also $\psi = 0$ gives these in their normal orientation and $\psi = \frac{\pi}{2}$ in their skew orientation.

### 1.2 Normalisation

Although the rest of the paper will evaluate fields for the potential containing $r^n$ above, this produces fields with magnitude $|\mathbf{B}| = nr^{n-1}$. If multipole strengths are defined as the values of polynomial coefficients of the field function, then $|\mathbf{B}| = k_n r^n$ may be obtained from the potential $\phi = \sin(n\theta+\psi)\frac{k_n}{n+1}r^{n+1}$. If strengths are defined as repeated derivatives of the field function, then $|\mathrm{d}^n\mathbf{B}/\mathrm{d}x^n| = d_n$ (i.e. $|\mathbf{B}| = \frac{d_n}{n!}r^n$) may be obtained from $\phi = \sin(n\theta+\psi)\frac{d_n}{(n+1)!}r^{n+1}$.

## 2 Series Solution

If $f''$ is nonzero, the $\nabla \cdot \nabla \phi = 0$ equation cannot be satisfied by a single term for all $r$. However, consider a sum of such terms with the same $n$ but different $k$:

$$\phi = \sin(n\theta + \psi) \sum_{k=n}^{\infty} r^k f_k(z).$$

Note the lowest term with $k = n$, which dominates the sum near the $r = 0$ axis, corresponds to a long $2n$-pole modulated by $f_n(z)$. The potential must satisfy

$$0 = \nabla \cdot \nabla \phi = \sin(n\theta + \psi) \sum_{k=n}^{\infty} r^{k-2} \left( (k^2 - n^2) f_k(z) + r^2 f_k''(z) \right)$$

for all $\theta$, so the RHS sum must be zero (for all $r, z$). Equating coefficients of $r^k$ gives

$$((k+2)^2 - n^2) f_{k+2}(z) + f_k''(z) = 0$$

for $k \geq n$ and the remaining coefficients of $r^{n-2}$ and $r^{n-1}$ are 0 and $((n+1)^2 - n^2) f_{n+1}(z)$ respectively, so $f_{n+1}(z) = 0$. The above relation gives $f_{k+2}$ as a scaled second derivative of $f_k$, so $f_{n+2j+1}(z) = 0$ for all $j \geq 0$ and

$$f_{n+2j}(z) = \left( \prod_{i=1}^{j} \frac{-1}{(n+2i)^2 - n^2} \right) f_n^{(2j)}(z).$$

The coefficients will be written as

$$C_{nj} = \prod_{i=1}^{j} \frac{-1}{(n+2i)^2 - n^2} = \prod_{i=1}^{j} \frac{-1}{4(n+i)i} = \frac{(-\frac{1}{4})^j n!}{(n+j)!j!}.$$

Setting $f_n = f$, the full potential satisfying Maxwell's equations in free space is:

$$\phi = \sin(n\theta + \psi) \sum_{j=0}^{\infty} r^{n+2j} f_{n+2j}(z) = \sin(n\theta + \psi) \sum_{j=0}^{\infty} C_{nj} r^{n+2j} f^{(2j)}(z).$$

### 2.1 Magnetic Field

Using the formula for the gradient of a single term $(\sin(n\theta + \psi) r^k f(z))$ given at the start of this note, the magnetic field associated with the series solution is

$$\mathbf{B} = \nabla \phi = \sum_{j=0}^{\infty} C_{nj} \begin{bmatrix} \sin(n\theta + \psi)(n+2j) r^{n+2j-1} f^{(2j)}(z) \\ n \cos(n\theta + \psi) r^{n+2j-1} f^{(2j)}(z) \\ \sin(n\theta + \psi) r^{n+2j} f^{(2j+1)}(z) \end{bmatrix}_{r\theta z}$$

$$= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} (n+2j) \sin(n\theta + \psi) f^{(2j)}(z) \\ n \cos(n\theta + \psi) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}_{r\theta z}$$

$$= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \left( \begin{bmatrix} n \sin(n\theta + \psi) f^{(2j)}(z) \\ n \cos(n\theta + \psi) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}_{r\theta z} + \begin{bmatrix} 2j \sin(n\theta + \psi) f^{(2j)}(z) \\ 0 \\ 0 \end{bmatrix}_{r\theta z} \right)$$

$$= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \left( \begin{bmatrix} n\sin((n-1)\theta+\psi) f^{(2j)}(z) \\ n\cos((n-1)\theta+\psi) f^{(2j)}(z) \\ r\sin(n\theta+\psi) f^{(2j+1)}(z) \end{bmatrix} + \begin{bmatrix} 2j\sin(n\theta+\psi) f^{(2j)}(z)\cos\theta \\ 2j\sin(n\theta+\psi) f^{(2j)}(z)\sin\theta \\ 0 \end{bmatrix} \right)$$

$$= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} (n\sin((n-1)\theta+\psi) + 2j\sin(n\theta+\psi)\cos\theta) f^{(2j)}(z) \\ (n\cos((n-1)\theta+\psi) + 2j\sin(n\theta+\psi)\sin\theta) f^{(2j)}(z) \\ r\sin(n\theta+\psi) f^{(2j+1)}(z) \end{bmatrix}.$$

## 2.2 Computation Without Trigonometric Formulae

The product formulae for sin and cos give

$$\sin(n\theta+\psi)\cos\theta = \tfrac{1}{2}\sin((n-1)\theta+\psi) + \tfrac{1}{2}\sin((n+1)\theta+\psi)$$
$$\sin(n\theta+\psi)\sin\theta = \tfrac{1}{2}\cos((n-1)\theta+\psi) - \tfrac{1}{2}\cos((n+1)\theta+\psi),$$

which enables the Cartesian field formula to be written as

$$\mathbf{B} = \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} ((n+j)\sin((n-1)\theta+\psi) + j\sin((n+1)\theta+\psi)) f^{(2j)}(z) \\ ((n+j)\cos((n-1)\theta+\psi) - j\cos((n+1)\theta+\psi)) f^{(2j)}(z) \\ r\sin(n\theta+\psi) f^{(2j+1)}(z) \end{bmatrix}.$$

Note that every instance of cos or $\sin(n\theta+\psi)$ is multiplied by a large power of $r$ usually including $r^n$. This can be used to convert fully to Cartesian coordinates using the complex formulae

$$re^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta) = x + \mathrm{i}y$$
$$\Rightarrow \quad (re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta} = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = (x+\mathrm{i}y)^n,$$

whence setting $c_n = \mathrm{Re}\,(x+\mathrm{i}y)^n$ and $s_n = \mathrm{Im}\,(x+\mathrm{i}y)^n$ gives

$$r^n \cos n\theta = c_n \qquad \text{and} \qquad r^n \sin n\theta = s_n.$$

These values can be calculated without trigonometry from the recurrence relation

$$c_0 = 1; \qquad s_0 = 0; \qquad c_{n+1} = xc_n - ys_n; \qquad s_{n+1} = yc_n + xs_n,$$

which comes from the definition of complex multiplication $(x+\mathrm{i}y)^{n+1} = (x+\mathrm{i}y)(x+\mathrm{i}y)^n$ in terms of components. The formula requires a slightly more general form

$$\hat{c}_n = r^n\cos(n\theta+\psi) = \mathrm{Re}\, r^n e^{\mathrm{i}n\theta} e^{\mathrm{i}\psi}, \qquad \hat{s}_n = r^n\sin(n\theta+\psi) = \mathrm{Im}\, r^n e^{\mathrm{i}n\theta} e^{\mathrm{i}\psi}.$$

These satisfy a similar recurrence because the repeated multiplication by $re^{\mathrm{i}\theta} = x + \mathrm{i}y$ starts with $e^{\mathrm{i}\psi}$ rather than 1:

$$\hat{c}_0 = \cos\psi; \qquad \hat{s}_0 = \sin\psi; \qquad \hat{c}_{n+1} = x\hat{c}_n - y\hat{s}_n; \qquad \hat{s}_{n+1} = y\hat{c}_n + x\hat{s}_n.$$

The evaluations of $\cos\psi$ and $\sin\psi$ are not a problem because they can be precalculated for each multipole, unlike $\hat{c}_n$ and $\hat{s}_n$ that depend on $x, y$. The field formula can now be rewritten as

$$\mathbf{B} = \sum_{j=0}^{\infty} C_{nj} r^{2j} \begin{bmatrix} ((n+j)\hat{s}_{n-1} + j\frac{1}{r^2}\hat{s}_{n+1}) f^{(2j)}(z) \\ ((n+j)\hat{c}_{n-1} - j\frac{1}{r^2}\hat{c}_{n+1}) f^{(2j)}(z) \\ \hat{s}_n f^{(2j+1)}(z) \end{bmatrix}$$

$$= \sum_{j=0}^{\infty} C_{nj} \begin{bmatrix} ((n+j)(r^2)^j\hat{s}_{n-1} + j(r^2)^{j-1}\hat{s}_{n+1}) f^{(2j)}(z) \\ ((n+j)(r^2)^j\hat{c}_{n-1} - j(r^2)^{j-1}\hat{c}_{n+1}) f^{(2j)}(z) \\ (r^2)^j\hat{s}_n f^{(2j+1)}(z) \end{bmatrix},$$

where powers of $r^2 = x^2 + y^2$ have been emphasised as they can be calculated without a square root. Note that negative powers never arise from $(r^2)^{j-1}$ when $j = 0$ because it is multiplied by $j$, so disappears.

## 2.3 Computation Using a 2D Field Map

The only parts of the above formula that depend on $\theta$ are the various $\hat{c}_n$ and $\hat{s}_n$, which are relatively quick (and unavoidable) to calculate for each evaluation point. Calculating the repeated derivatives of $f$ and waiting for the infinite sum to converge are, on the other hand, quite slow but may be precalculated as three functions of just $r$ and $z$ in the following way:

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}F(r,z) + \hat{s}_{n+1}G(r,z) \\ \hat{c}_{n-1}F(r,z) - \hat{c}_{n+1}G(r,z) \\ \hat{s}_n H(r,z) \end{bmatrix},$$

where

$$F(r,z) = \sum_{j=0}^{\infty} C_{nj}(n+j)(r^2)^j f^{(2j)}(z); \qquad G(r,z) = \sum_{j=1}^{\infty} C_{nj} j (r^2)^{j-1} f^{(2j)}(z);$$

$$H(r,z) = \sum_{j=0}^{\infty} C_{nj}(r^2)^j f^{(2j+1)}(z).$$

### 2.3.1 Universal Field Map

Considering just one end of the magnet, if the fringe field is related to a universal fringe field function via $f(z) = \hat{f}(z/l)$, then $f^{(n)}(z) = l^{-n}\hat{f}^{(n)}(z/l)$ and

$$F(r,z) = \sum_{j=0}^{\infty} C_{nj}(n+j)\left(\frac{r}{l}\right)^{2j} \hat{f}^{(2j)}\left(\frac{z}{l}\right) = \hat{F}_n\left(\frac{r}{l}, \frac{z}{l}\right);$$

$$G(r,z) = \frac{1}{r^2}\sum_{j=1}^{\infty} C_{nj} j \left(\frac{r}{l}\right)^{2j} \hat{f}^{(2j)}\left(\frac{z}{l}\right) = \frac{1}{r^2}\hat{G}_n\left(\frac{r}{l}, \frac{z}{l}\right);$$

$$H(r,z) = \frac{1}{r}\sum_{j=0}^{\infty} C_{nj} \left(\frac{r}{l}\right)^{2j+1} \hat{f}^{(2j+1)}\left(\frac{z}{l}\right) = \frac{1}{r}\hat{H}_n\left(\frac{r}{l}, \frac{z}{l}\right),$$

where the new functions $\hat{F}_n$, $\hat{G}_n$, $\hat{H}_n$ depend only on $n$ and the form of $\hat{f}$, not $l$. In terms of these new functions,

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}\hat{F}_n(r/l, z/l) + \hat{s}_{n+1}\frac{1}{r^2}\hat{G}_n(r/l, z/l) \\ \hat{c}_{n-1}\hat{F}_n(r/l, z/l) - \hat{c}_{n+1}\frac{1}{r^2}\hat{G}_n(r/l, z/l) \\ \hat{s}_n \frac{1}{r}\hat{H}_n(r/l, z/l) \end{bmatrix}.$$

Unfortunately use of the universal function $\hat{H}_n$ incurs a square root to calculate $\frac{1}{r}$, which the non-universal forms do not have, assuming they are stored as functions of $r^2$ and $z$.

### 2.3.2 Field Map for Magnet Ends Only

In the interior of a long magnet, the field tends towards the expression given earlier:

$$\mathbf{B} = nr^{n-1}\begin{bmatrix} \sin((n-1)\theta + \psi) \\ \cos((n-1)\theta + \psi) \\ 0 \end{bmatrix} = \begin{bmatrix} n\hat{s}_{n-1} \\ n\hat{c}_{n-1} \\ 0 \end{bmatrix}.$$

This corresponds to $F(r,z) = n$ and $G(r,z) = H(r,z) = 0$. The lowest-order terms up to $r^2$ in these sums are:

$$F(r,z) = C_{n0} n f(z) + C_{n1}(n+1) r^2 f''(z) + \ldots = n f(z) - \tfrac{1}{4} r^2 f''(z) + \ldots;$$

$$G(r,z) = C_{n1} f''(z) + C_{n2} 2 r^2 f''''(z) + \ldots = \frac{-f''(z)}{4(n+1)} + \frac{r^2 f''''(z)}{32(n+1)(n+2)} + \ldots;$$

$$H(r,z) = C_{n0} f'(z) + C_{n1} r^2 f'''(z) + \ldots = f'(z) - \frac{r^2 f'''(z)}{4(n+1)} + \ldots,$$

so when $f(z) = 1$ and $f'(z) = f''(z) = 0$ to some precision, the field calculation may be replaced by the simple multipole expression. Similarly, when $f = f' = f'' = 0$ to a good approximation, the field may be given as zero. This means $F$, $G$, $H$ only need to be calculated for the transition region where $0 < f(z) < 1$ for each magnet (or end of magnet).

### 2.3.3 Magnets with Symmetrical Ends

It is common to use the same fringe field for both ends of a magnet of length $L$, so that $f(z) = g(z) + g(L-z)$, where $g$ is a sigmoid function going from $-\frac{1}{2}$ to $\frac{1}{2}$, representing one end of the magnet (e.g. $g(z) = \frac{1}{2}\tanh(z/l)$). If $F$, $G$, $H$ are now calculated using $g$ in place of $f$, the magnetic field is given by

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}(F(r,z) + F(r,L-z)) + \hat{s}_{n+1}(G(r,z) + G(r,L-z)) \\ \hat{c}_{n-1}(F(r,z) + F(r,L-z)) - \hat{c}_{n+1}(G(r,z) + G(r,L-z)) \\ \hat{s}_n(H(r,z) - H(r,L-z)) \end{bmatrix},$$

noting the sign change for $H(r,L-z)$ because it contains odd derivatives of $g$.

For the regions where $g(z)$ is almost constant at $-\frac{1}{2}$ or $\frac{1}{2}$, $F(r,z) = -\frac{1}{2}n$ or $\frac{1}{2}n$ may be used respectively and $G(r,z) = H(r,z) = 0$. If both $g(z)$ and $g(L-z)$ are in a constant region and they have opposite sign (i.e. far outside the magnet), the $\mathbf{B}$ field will be zero and the calculation should be aborted before calculating $\hat{s}_{n-1}$ etc.