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# Explicit formulas for redistribution of cooling rates in non-magnetized electron cooling

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## Abstract

In this paper we derive explicit formulas for the friction force and the cooling rates in non-magnetized electron coolers in the presence of redistribution of cooling decrements.

## 1 Introduction

Electron Cooling (EC) [1, 2] is a technique that allows increasing a 6-D phase space density of stored hadron beams.

In EC a beam of “cold” electrons co-propagate with a hadron beam with the same average velocity in a straight section of the storage ring, called a cooling section (CS). A hadron interacts with electrons in a CS via Coulomb force, which introduces dynamical friction [3] acting on each hadron. After every passage through the CS the electrons are either dumped or returned to the electron gun for charge recovery, thus, on each turn the ions interact with fresh electrons. Over many revolutions in the accelerator the average friction reduces both the transverse and the longitudinal momentum spread of the ion bunch.

The friction force (in the beam frame) acting on an ion co-traveling with an electron bunch is given by [4, 5]:

$$\vec{F} = -\frac{4\pi e^4 Z^2}{m_e} \int L_C \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f(r_e, v_e) d^3 v_e \quad (1)$$

Here,  $e$  is the electron charge,  $Z \cdot e$  is the ion charge,  $m_e$  is the mass of electron,  $\vec{v}_i$  and  $\vec{v}_e$  are ion and electron velocities in the beam frame,  $L_C$

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is the Coulomb logarithm, which has a weak dependence on  $v_e$  and can be moved from under the integral, and  $f(r_e, v_e)$  is a six-dimensional distribution function of the electrons.

The cooling rate ( $\lambda_{x,y,z}$ ) in the laboratory frame can be obtained from:

$$\lambda_{x,y,z} = \frac{F_{x,y,z}\eta}{\gamma m_i v_{i(x,y,z)}} \quad (2)$$

where the duty factor  $\eta = L_{CS}/(2\pi R)$ ,  $R$  is the storage ring radius,  $L_{CS}$  is the length of the cooling section,  $m_i = A_i m_p$ ,  $m_p$  is the proton mass,  $A_i$  is the ion mass number, and indexes  $x, y, z$  correspond to the horizontal, vertical and longitudinal components of  $F$ ,  $v$  and  $\lambda$ .

Similarly to Eq. (2), the change in the ion's velocity on a single pass through the CS is given by:

$$\Delta v_{i(x,y,z)} = \frac{F_{x,y,z} L_{CS}}{m_i \gamma \beta c} \quad (3)$$

It is well known [5] that the cooling rates can be “redistributed” between the longitudinal and transverse directions.

The goal of this paper is to derive the explicit expressions for the cooling rates with a  $x - z$  redistribution.

## 2 Basic remarks about redistribution mechanism

The redistribution requires two conditions. The first one is a coupling between the longitudinal and transverse (we will consider the horizontal one) motion of an ion. This is created by the ions' dispersion in the CS. The second condition is dependence of the longitudinal friction force on the horizontal position of an ion in the cooling section, i.e. the longitudinal component of the cooling force must have the transverse gradient. A robust way to create the required gradient is to introduce the electron beam dispersion in the CS.

The cooling force can be well approximated by a linear function of the ion velocity ( $v_i$ ) if  $v_i$  is less than the rms velocity spread of the electrons.

We will assume:

$$\begin{aligned} F_x &= -C_x v_{ix} \\ F_z &= -C_z (v_{iz} - K \cdot x_i) \end{aligned} \quad (4)$$

where  $K$  is the horizontal gradient of the longitudinal cooling force.

Substituting Eq. (4) into Eq. (3) and noticing that an ion's relative momentum  $\delta_i = v_{iz}/(\beta c)$  and an ion's angle  $x'_i = v_{ix}/(\gamma\beta c)$ , we get for a single pass through the CS:

$$\begin{aligned}\Delta x'_i &= -c_x x'_i \\ \Delta \delta_i &= -c_z(\delta_i - k \cdot x_i)\end{aligned}\quad (5)$$

where  $c_x = \frac{C_x L_{CS}}{\gamma\beta c m_i}$ ,  $c_z = \frac{C_z L_{CS}}{\gamma\beta c m_i}$  and  $k = \frac{K}{\beta c}$ .

Let us introduce the action-angle variables  $(J, \varphi)$  for horizontal motion. Here  $\varphi$  is a betatron phase and action  $J$  is given by:

$$J = \frac{1}{2} (\gamma_x x_i^2 + \beta_x x_i'^2) \quad (6)$$

where  $\beta_x$  is a Twiss  $\beta$ -function,  $\gamma_x = 1/\beta_x$  is the Twiss  $\gamma$ -function and we assumed that the ions' Twiss  $\alpha$ -function in the CS is equal to zero.

We will further assume that the ions have a Gaussian distribution with the distribution function  $f_{J,\varphi} = \frac{1}{2\pi\varepsilon} e^{-J/\varepsilon}$ . Here the horizontal beam emittance ( $\varepsilon_x$ ) is an average value of the action:

$$\varepsilon_x = \int_0^{2\pi} \int_0^\infty J f_{J,\varphi} dJ d\varphi \quad (7)$$

Finally we will assume that the cooling kick is weak enough to neglect a change in the distribution function  $f_{J,\varphi}$  on a single pass through the CS and that the cooling section is short enough to represent the horizontal transfer matrix through the CS by:

$$M_{CS} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - c_x \end{pmatrix} \quad (8)$$

Notice, that  $M_{CS}$  is not symplectic, neither its determinant is equal to one, which corresponds to a simple physical fact that the cooling force does not conserve the beam emittance.

Now, we shall notice that on a single pass through the CS the Twiss parameters are getting changed as [6]:

$$\begin{aligned}\beta_{x1} &= \beta_{x0}/(1 - c_x) \\ \gamma_{x1} &= \gamma_{x0}(1 - c_x)\end{aligned}\quad (9)$$

Therefore, on a single pass through the cooling section the action becomes:

$$\begin{aligned}
J_1 &= \frac{1}{2} \left( \frac{(1-c_x)(x_{i0}+\Delta x_i)^2}{\beta_{x0}} + \frac{\beta_{x0}(x'_{i0}+\Delta x'_i)^2}{1-c_x} \right) \\
&\approx J_0 + \frac{x_{i0}\Delta x_i}{\beta_{x0}} - \frac{c_x x_{i0}^2}{2\beta_{x0}} + \frac{\beta_{x0}x'_{i0}(2\Delta x'_i+c_x x'_{i0})}{2} \\
&= J_0 + \frac{x_{i0}\Delta x_i}{\beta_{x0}} - \frac{c_x x_{i0}^2}{2\beta_{x0}} - \frac{\beta_{x0}c_x x_{i0}^2}{2} = J_0 - c_x J_0 + \frac{x_{i0}\Delta x_i}{\beta_{x0}}
\end{aligned} \tag{10}$$

In the cooling section, which incorporates ion dispersion  $x_i = x_{i0} + D_i \delta_{i0}$ . After ion passes the CS and the dispersion is zeroed, we get for the resulting horizontal coordinate of an ion:  $x_{i1} = x_{i0} + D_i \delta_{i0} - D_i \delta_{i1} = x_{i0} - D_i \Delta \delta_i$ . Substituting these relations into Eqs. (5) and (10) we get:

$$\begin{aligned}
\Delta J &= \frac{x_{i0} D_i c_z (\delta_{i0} - k x_{i0} - k D_i \delta_{i0})}{\beta_T} - c_x J_0 \\
\Delta \delta_i &= -c_z (\delta_{i0} - k x_{i0} - k D_i \delta_{i0})
\end{aligned} \tag{11}$$

Noticing that  $x_{i0} = \sqrt{2J_0 \beta_T} \cos(\varphi)$ , substituting Eq. (11) into Eqs. (7) we find:

$$\Delta \varepsilon_x = -\varepsilon_x (c_x + c_z k D_i) \tag{12}$$

Similar to above considerations, we introduce a longitudinal action  $J_z$ :

$$J_z = \frac{1}{2} \left( \frac{z_i^2}{\beta_z} + \beta_z \delta_i^2 \right) \tag{13}$$

where  $\beta_z = \sigma_{zi} / \sigma_{\delta i}$ .

Then, for a single pass through the CS we get:

$$\begin{aligned}
J_{z1} &= \frac{1}{2} \left( \frac{(1-c_z)z_{i0}^2}{\beta_{z0}} + \frac{\beta_{z0}(\delta_{i0}+\Delta \delta_{i0})^2}{1-c_z} \right) = \\
&= \frac{1}{2} \left( \frac{(1-c_z)z_{i0}^2}{\beta_{z0}} + \frac{\beta_{z0}(\delta_{i0}(1-c_z)+c_z k(D_i \delta_{i0}+x_{i0}))^2}{1-c_z} \right) \approx \\
&\approx J_{z0}(1-c_z) + c_z k \beta_{z0} \delta_{i0} (D_i \delta_{i0} + x_{i0})
\end{aligned} \tag{14}$$

Next, we notice that  $\delta = \sqrt{2J_z/\beta_z} \sin(\phi)$ , where  $\phi$  is the synchrotron phase.

Finally, to find a single pass change to the longitudinal emittance  $\varepsilon_z = \sigma_{zi} \cdot \sigma_{\delta i}$ , we convolve  $\Delta J_z = J_{z1} - J_{z0}$  and  $f_{J_z, \phi} = \frac{1}{2\pi \varepsilon_z} e^{-J_z/\varepsilon_z}$ :

$$\Delta \varepsilon_z = -\varepsilon_z (c_z - c_z k D_i) \tag{15}$$

We can get respective cooling rates from Eqs. (12) and (15) by dividing each expression by a revolution period  $T_{rev} = \frac{2\pi R}{\beta c}$ :

$$\begin{aligned}
\lambda_{x(r)} &= -\left( c_x \frac{\beta c}{2\pi R} + k D_i \cdot c_z \frac{\beta c}{2\pi R} \right) \\
\lambda_{z(r)} &= -c_z \frac{\beta c}{2\pi R} \cdot (1 - k D_i)
\end{aligned} \tag{16}$$

Here index ( $r$ ) denotes the redistributed rate.

Finally, recalling the relations between  $c_x, c_z$  and  $C_x, C_z$  and comparing Eq. (16) to Eq. (2) we get the general expression for the cooling rates redistribution:

$$\begin{aligned}\lambda_{x(r)} &= \lambda_x + kD_i\lambda_z \\ \lambda_{z(r)} &= \lambda_z - kD_i\lambda_x\end{aligned}\quad (17)$$

The rates on the right hand side of the equations are the rates at  $D_i = 0$

### 3 Cooling in presence of electron dispersion

We consider an electron bunch with Gaussian 6-D distribution in the presence of e-beam dispersion ( $D_e$ ) in the cooling section. To simplify the resulting formulas we will assume electron  $\alpha_T = 0$  in the cooling section, which is a reasonable approximation for an electron cooler (a treatment of  $\alpha_T \neq 0$  case can be found in [7]). Then the electron bunch distribution is:

$$\begin{aligned}f(r_e, v_e) &= \frac{N_e}{\gamma(2\pi)^3 \Delta_x \Delta_y \Delta_z \sigma_x \sigma_y \sigma_z} f_x f_y f_z \\ f_x &= \exp \left[ -\frac{(x - D_e \delta_e)^2}{2\sigma_x^2} - \frac{v_{ex}^2}{2\Delta_x^2} \right] \\ f_y &= \exp \left( -\frac{y^2}{2\sigma_y^2} - \frac{v_{ey}^2}{2\Delta_y^2} \right) \\ f_z &= \exp \left( -\frac{z^2}{2\sigma_z^2} - \frac{v_{ez}^2}{2\Delta_z^2} \right) = \exp \left( -\frac{z^2}{2\sigma_z^2} - \frac{\delta_e^2}{2\sigma_{\delta_e}^2} \right)\end{aligned}\quad (18)$$

Here  $\Delta_{x,y,z}$  are the electrons' rms velocity spreads and  $\sigma_{x,y,z}$  are the e-bunch rms sizes along horizontal, vertical and longitudinal directions respectively.

With simple algebraic manipulations the distribution can be split in a density and a velocity distribution parts:

$$\begin{aligned}f(r_e, v_e) &= n_e f_{ve} \\ n_e &= \frac{N_e}{\gamma(2\pi)^{3/2} \sigma_{x1} \sigma_y \sigma_z} \exp \left[ -\frac{x^2}{2\sigma_{x1}} - \frac{y^2}{\sigma_y^2} - \frac{z^2}{\sigma_z^2} \right] \\ f_{ve} &= \frac{1}{(2\pi)^{3/2} \Delta_x \Delta_y \Delta_{z1}} \exp \left[ -\frac{v_{ex}^2}{2\Delta_x^2} - \frac{v_{ey}^2}{\Delta_y^2} - \frac{(v_{ez} - \mu_z)^2}{\Delta_{z1}^2} \right] \\ \sigma_{x1} &= \sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta_e}^2} \\ \Delta_{z1} &= \Delta_z \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta_e}^2}} \\ \mu_z &= x \Delta_z \frac{D_e \sigma_{\delta_e}}{\sigma_x^2 + D_e^2 \sigma_{\delta_e}^2}\end{aligned}\quad (19)$$

Substituting Eq. (19) into Eq. (1) we get the expression for the friction force of the form:

$$\vec{F} = -C_0 \int \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f_{ve} d^3 v_e \quad (20)$$

Here,  $C_0 = \frac{4\pi n_e e^4 Z^2 L_C}{m_e}$ .

Equation (20) with distribution  $f_{ve}$  of the form (19) can be reduced to 1-D integrals for each component of the friction force (so called Binney's formulas [8]) by noticing that the friction force in v-space looks like a point-charge Coulomb force in the physical space. Then we can introduce an effective potential in a velocity-space:

$$U = C_0 \int \frac{f_{ve}}{|\vec{v}_i - \vec{v}_e|} d^3 v_e \quad (21)$$

such that

$$F_{x,y,z} = \partial U / \partial v_{i(x,y,z)} \quad (22)$$

The detailed step-by-step derivation of the Binney's formulas for the considered case is given in [9]. Here we simply give the final result for  $\Delta_x = \Delta_y$ . Considerations of unequal v-distributions in all three directions can be found in [10].

For the case of  $\Delta_x = \Delta_y \equiv \Delta_t$  the friction force is:

$$\begin{cases} F_{x,y} &= -C_1 v_{i(x,y)} \int_0^\infty g_t(q) dq \\ F_z &= -C_1 (v_{iz} - \mu_z) \int_0^\infty g_z(q) dq \\ g_t(q) &= \frac{1}{\Delta_t^2 (1+q)^2 \sqrt{\Delta_t^2 q + \Delta_{z1}^2}} \exp \left[ -\frac{v_{ix}^2 + v_{iy}^2}{2\Delta_t^2 (1+q)} - \frac{(v_{iz} - \mu_z)^2}{2(\Delta_t^2 q + \Delta_{z1}^2)} \right] \\ g_z(q) &= \frac{1}{(1+q)(\Delta_t^2 q + \Delta_{z1}^2)^{3/2}} \exp \left[ -\frac{v_{ix}^2 + v_{iy}^2}{2\Delta_t^2 (1+q)} - \frac{(v_{iz} - \mu_z)^2}{2(\Delta_t^2 q + \Delta_{z1}^2)} \right] \end{cases} \quad (23)$$

where  $C_1 = 2\sqrt{2\pi} n_e r_e^2 m_e c^4 Z^2 L_C$ .

The integrals 23 can be taken analytically in the approximation of "small amplitudes":

$$\begin{aligned} F_{x,y} &= -v_{i(x,y)} \frac{C_1}{\Delta_\perp^2 \Delta_{z1}} \Phi(\Delta_{z1}/\Delta_\perp) \\ F_z &= -(v_{iz} - \mu_z) \frac{2C_1}{\Delta_\perp^2 \Delta_{z1}} (1 - \Phi(\Delta_{z1}/\Delta_\perp)) \\ \Phi(d) &= \begin{cases} \frac{d}{1-d^2} \left( \frac{\arccos(d)}{\sqrt{1-d^2}} - d \right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1} \left( \frac{\log(d-\sqrt{d^2-1})}{\sqrt{d^2-1}} + d \right), & d > 1 \end{cases} \end{aligned} \quad (24)$$

where  $d = \Delta_{z1}/\Delta_\perp$ .

It is useful to rewrite Eq. (24) for the peak cooling force (at  $x, y, z = 0$ ) explicitly showing the dependence of  $n_e$  (via  $\sigma_{x1}$ ),  $\mu_z$  and  $\Delta_{z1}$  on  $D_e$ :



$$\begin{aligned}
F_{x,y} &= -v_{i(x,y)} \frac{C}{\Delta_{\perp}^2 \Delta_z} \Phi \left( \frac{\Delta_z}{\Delta_{\perp}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \\
F_z &= - \left( v_{iz} - \Delta_z \frac{D_e \sigma_{\delta e}}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \cdot x_i \right) \frac{2C}{\Delta_{\perp}^2 \Delta_z} \left[ 1 - \Phi \left( \frac{\Delta_z}{\Delta_{\perp}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \right] \\
C &= \frac{N_e r_e^2 m_e c^4 Z^2 L_C}{\pi \gamma \sigma_x \sigma_y \sigma_z} \\
\Phi(d) &= \begin{cases} \frac{d}{1-d^2} \left( \frac{\arccos(d)}{\sqrt{1-d^2}} - d \right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1} \left( \frac{\log(d-\sqrt{d^2-1})}{\sqrt{d^2-1}} + d \right), & d > 1 \end{cases}
\end{aligned} \tag{25}$$

## 4 Explicit formulas for $x - z$ redistribution of nonmagnetized cooling

We notice that Eq. (25) can be presented in form (4):

$$\begin{aligned}
F_x &= -C_x v_{ix} \\
F_z &= -C_z (v_{iz} - K \cdot x_i) \\
C_x &= \frac{N_e r_e^2 m_e c Z^2 L_C}{\pi \gamma^3 \beta^3 \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \Phi \left( \frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \\
C_z &= \frac{2N_e r_e^2 m_e c Z^2 L_C}{\pi \gamma^3 \beta^3 \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \left[ 1 - \Phi \left( \frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \right] \\
K &= \beta c \frac{D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned} \tag{26}$$

Here we used  $\Delta_z = \beta c \sigma_{\delta}$ , and  $\Delta_{\perp} = \gamma \beta c \sigma_{\theta}$ , where  $\sigma_{\theta} \equiv \sigma_{\theta x} = \sigma_{\theta y}$ .

Now we can write the explicit redistribution formulas, using equations derived in section 2.

$$\begin{aligned}
\lambda_{x(r)} &= \lambda_x + \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_z \\
\lambda_{z(r)} &= \lambda_z - \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_x \\
\lambda_x &= \frac{N_e r_e^2 m_e c Z^2 L_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \Phi \left( \frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \\
\lambda_z &= \frac{2N_e r_e^2 m_e c Z^2 L_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \left[ 1 - \Phi \left( \frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2}} \right) \right]
\end{aligned} \tag{27}$$

It is important to notice that the rates  $\lambda_x$  and  $\lambda_z$  are the rates at  $D_i = 0$ , they depend on  $D_e$  and are not equal to the ‘‘undisturbed’’ cooling rates ( $\lambda_{x0} \equiv \lambda_x(D_e = 0)$  and  $\lambda_{z0} \equiv \lambda_z(D_e = 0)$ ). Function  $\Phi$  is strongly nonlinear, as Fig. 1 demonstrates. Therefore, in general case, the redistributed rates

$\lambda_{x(r)}$  and  $\lambda_{z(r)}$  can not be represented as linear combination of the cooling rates ( $\lambda_{x0}$  and  $\lambda_{z0}$ ) in the absence of both the electron and the ion dispersions.

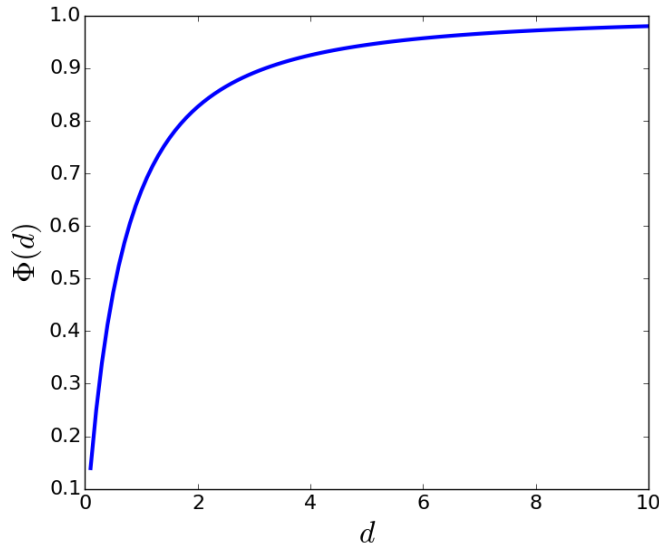


Figure 1: Function  $\Phi(d)$ .

We shall also notice that in the absence of  $D_i$  but with a nonzero  $D_e$  we have some cooling redistribution but it is a redistribution from a horizontal to a vertical direction.

Finally let us consider the two limiting cases.

For the case of the spherically symmetric electron's  $v$ -distribution (in the absence of  $D_e$ )  $\Phi(1) = 2/3$  and:

$$\lambda_{x0} = \lambda_{z0} = \frac{2N_e r_e^2 m_e c Z^2 L_C \eta}{3\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}} \quad (28)$$

Yet, the moment we introduce the electron dispersion the argument of function  $\Phi$  becomes non-unitary and one must use Eq. (27) to calculate rates, since dependence of  $\Phi(d)$  is quite strong near  $d = 1$ , as Fig. 1 shows.

For the case of “flat”  $v$ -distribution ( $\Delta_z \ll \Delta_\perp$ ) we have  $\Phi(d \rightarrow 0) \rightarrow d \cdot \pi/2$ . Introduction of electron dispersion makes the argument of function  $\Phi$  even smaller. Therefore, for the case of flat  $v$ -distribution the redistributed rates can be presented as linear combinations of undisturbed cooling rates, and we get:

$$\begin{aligned}
\lambda_{x(r)} &= \lambda_{x0} + \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{z0} \\
\lambda_{z(r)} &= \lambda_{z0} - \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{z0} \\
\lambda_{x0} &= \frac{N_e r_e^2 m_e c Z^2 L_C \eta}{2\gamma^5 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^3} \\
\lambda_{z0} &= \frac{2N_e r_e^2 m_e c Z^2 L_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_x \sigma_y \sigma_z \sigma_{\theta e}^2 \sigma_{\delta e}}
\end{aligned} \tag{29}$$

## 5 Conclusion

We derived the explicit formulas (Eq. 27) for the cooling rates' redistribution in a non-magnetized electron cooler.

The redistributed rates are the linear combinations of the cooling rates, which, in turn, are strongly nonlinear functions of the electron dispersion in the CS.

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