# Conversion of a Twiss parameter in cooling process 

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## 1 General considerations

It is convenient to consider effects on beam emittance in action-angle variables $(J, \varphi)$, where $\varphi$ is a betatron phase and:

$$
\begin{equation*}
J=\frac{1}{2}\left(\gamma x^{2}+2 \alpha x x^{\prime}+\beta x^{\prime 2}\right) \tag{1}
\end{equation*}
$$

For a Gaussian beam the distribution function $f_{J, \varphi}$ is given by:

$$
\begin{equation*}
f_{J, \varphi}=\frac{1}{2 \pi \varepsilon} e^{-J / \varepsilon} \tag{2}
\end{equation*}
$$

And the beam emittance is:

$$
\begin{equation*}
\varepsilon=\int_{0}^{2 \pi} \int_{0}^{\infty} J f_{J, \varphi} d J d \varphi \tag{3}
\end{equation*}
$$

Let's consider a linear transformation $\left(x_{0}, x_{0}^{\prime}\right) \rightarrow\left(x_{1}, x_{1}^{\prime}\right)$, here we are not making any assumptions about properties of the transfer matrix $M$ :

$$
\binom{x_{1}}{x_{1}^{\prime}}=\left(\begin{array}{ll}
m_{11} & m_{12}  \tag{4}\\
m_{21} & m_{22}
\end{array}\right) \cdot\binom{x_{0}}{x_{0}^{\prime}}
$$

Then we get from Eq. (4):

$$
\binom{x_{0}}{x_{0}^{\prime}}=\frac{1}{\operatorname{det} M}\left(\begin{array}{cc}
m_{22} & -m_{12}  \tag{5}\\
-m_{21} & m_{11}
\end{array}\right) \cdot\binom{x_{1}}{x_{1}^{\prime}}
$$

[^1]Substituting Eq. (5) into Eq. (1) we obtain the following expression for the transformed action:

$$
\begin{equation*}
\operatorname{det} M \cdot J_{0}=\frac{1}{2}\left(\gamma_{1} x_{1}^{2}+2 \alpha_{1} x_{1} x_{1}^{\prime}+\beta_{1} x_{1}^{\prime 2}\right) \tag{6}
\end{equation*}
$$

where Twiss parameters are transformed by the following law:

$$
\left(\begin{array}{c}
\beta_{1}  \tag{7}\\
\alpha_{1} \\
\gamma_{1}
\end{array}\right)=\frac{1}{\operatorname{det} M}\left(\begin{array}{ccc}
m_{11}^{2} & -2 m_{11} m_{12} & m_{12}^{2} \\
-m_{11} m_{21} & m_{11} m_{22}+m_{12} m_{21} & -m_{12} m_{22} \\
m_{21}^{2} & -2 m_{21} m_{22} & m_{22}^{2}
\end{array}\right) \cdot\left(\begin{array}{c}
\beta_{0} \\
\alpha_{0} \\
\gamma_{0}
\end{array}\right)
$$

It is important to notice, that for Twiss parameters, a "canonical" form of an ellipse equation, for which an action on the left-hand side of the equation is an ellipse area divided by $2 \pi$, sets a strict requirement to parameters: $\beta \gamma=1+\alpha^{2}$. This requirement is satisfied for parameters $\beta_{1}, \alpha_{1}, \gamma_{1}$ defined by Eq. (7) as long as it is satisfied for $\beta_{0}, \alpha_{0}, \gamma_{0}$.

For a usual symplectic transformation, $\operatorname{det} M=1$, and the action is the invariant of motion. For a special case when the transformation does not conserve the beam emittance Eq. (6) tells us that:

$$
\begin{equation*}
J_{1}=\operatorname{det} M \cdot J_{0} \tag{8}
\end{equation*}
$$

If this non-conservative disturbance is small enough to assume that the same $f_{J, \varphi}$ can be used after the disturbance is applied to the bunch, then we get from Eqs. (3) and (8):

$$
\begin{equation*}
\varepsilon_{1}=\operatorname{det} M \cdot \varepsilon_{0} \tag{9}
\end{equation*}
$$

## 2 Case of cooling

For a special case of the "linear" cooling we have the transfer matrix $M_{C}$ :

$$
M_{C}=\left(\begin{array}{cc}
1 & 0  \tag{10}\\
0 & (1-c)
\end{array}\right)
$$

where parameter $c$ defines the cooling rate:

$$
\begin{equation*}
\lambda=\frac{1}{T_{\text {rev }}} \frac{x_{1}^{\prime}-x_{0}^{\prime}}{x_{0}^{\prime}}=\frac{c}{T_{\text {rev }}} \tag{11}
\end{equation*}
$$

Here $T_{\text {rev }}$ is the revolution frequency of the hadron storage ring.
Substituting $\operatorname{det} M_{C}=1-c$ into Eqs. (9) we get:

$$
\begin{equation*}
\varepsilon_{1}=(1-c) \varepsilon_{0} \tag{12}
\end{equation*}
$$

Then for the emittance cooling rate we obtain:

$$
\begin{equation*}
\lambda_{\varepsilon}=\frac{1}{T_{\text {rev }}} \frac{\varepsilon_{1}-\varepsilon_{0}}{\varepsilon_{0}}=\frac{c}{T_{\text {rev }}} \tag{13}
\end{equation*}
$$

Next, let us consider the change in the rms beam size $\left(\sigma_{x}\right)$ and the rms angular spread $\left(\sigma_{x^{\prime}}\right)$ on a single pass through the cooling section. By default:

$$
\begin{align*}
\sigma_{x} & =\left(\int_{0}^{2 \pi} \int_{0}^{\infty} x^{2}(J, \varphi) f_{J, \varphi} d J d \varphi\right)^{1 / 2} \\
\sigma_{x^{\prime}} & =\left(\int_{0}^{2 \pi} \int_{0}^{\infty} x^{\prime 2}(J, \varphi) f_{J, \varphi} d J d \varphi\right)^{1 / 2} \tag{14}
\end{align*}
$$

Here, regular coordinates $x, x^{\prime}$ can be expressed through action-angle variables as:

$$
\begin{align*}
x & =\sqrt{2 J \beta} \cos \varphi \\
x^{\prime} & =\sqrt{\frac{2 J}{\beta}}(\sin \varphi-\alpha \cos \varphi) \tag{15}
\end{align*}
$$

According to Eq. (7) for transfer matrix (10) we get:

$$
\begin{align*}
\beta_{1} & =\beta_{0} /(1-c) \\
\alpha_{1} & =\alpha_{0}  \tag{16}\\
\gamma_{1} & =\gamma_{0}(1-c)
\end{align*}
$$

It follows from Eqs. (8), (15) and (16) that, since according to Eq. (10) $x_{1}=x_{0}$, then $\cos \left(\varphi_{1}\right)=\cos (\varphi 0)$. Therefore:

$$
\begin{equation*}
\varphi_{1}=\varphi_{0} \tag{17}
\end{equation*}
$$

Then, substituting Eq. (8) and Eqs. (15.17) into Eq. (14) we obtain:

$$
\begin{align*}
\sigma_{x 1} & =\sigma_{x 0} \\
\sigma_{x 1^{\prime}} & =(1-c) \sigma_{x 0^{\prime}} \tag{18}
\end{align*}
$$

Once again, let us stress that Eq. (18) gives the change in $\sigma_{x}$ and $\sigma_{x^{\prime}}$ for a single pass through the cooling section. The betatron motion in the ring couples $x$ and $x^{\prime}$ resulting in the following size and angular spread cooling rates: $\lambda_{\sigma_{x}}=\lambda_{\sigma_{x^{\prime}}}=\lambda_{\varepsilon} / 2=c /\left(2 T_{\text {rev }}\right)$.

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