BNL-223410-2022-TECH

C-A/AP/664

Conversion of a Twiss parameter in cooling process

S. Seletskiy

September 2022

Collider Accelerator Department

Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

DISCLAIMER

Conversion of Twiss parameters in cooling process

S. Seletskiy[*], D. Kayran

September 12, 2022

1 General considerations

It is convenient to consider effects on beam emittance in action-angle variables (J, φ), where φ is a betatron phase and:

$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x x' + \beta x'^2) \tag{1}$$

For a Gaussian beam the distribution function $f_{J,\varphi}$ is given by:

$$f_{J,\varphi} = \frac{1}{2\pi\varepsilon} e^{-J/\varepsilon} \tag{2}$$

And the beam emittance is:

$$\varepsilon = \int_0^{2\pi} \int_0^{\infty} J f_{J,\varphi} dJ d\varphi \tag{3}$$

Let's consider a linear transformation $(x_0, x_0') \to (x_1, x_1')$, here we are not making any assumptions about properties of the transfer matrix M:

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{4}$$

Then we get from Eq. (4):

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{5}$$

[*]seletskiy@bnl.gov

Substituting Eq. (5) into Eq. (1) we obtain the following expression for the transformed action:

$$\det M \cdot J_0 = \frac{1}{2}(\gamma_1 x_1^2 + 2\alpha_1 x_1 x_1' + \beta_1 x_1'^2) \tag{6}$$

where Twiss parameters are transformed by the following law:

$$\begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} \tag{7}$$

It is important to notice, that for Twiss parameters, a "canonical" form of an ellipse equation, for which an action on the left-hand side of the equation is an ellipse area divided by 2π, sets a strict requirement to parameters: $\beta\gamma = 1 + \alpha^2$. This requirement is satisfied for parameters $\beta_1, \alpha_1, \gamma_1$ defined by Eq. (7) as long as it is satisfied for $\beta_0, \alpha_0, \gamma_0$.

For a usual symplectic transformation, $\det M = 1$, and the action is the invariant of motion. For a special case when the transformation does not conserve the beam emittance Eq. (6) tells us that:

$$J_1 = \det M \cdot J_0 \tag{8}$$

If this non-conservative disturbance is small enough to assume that the same $f_{J,\varphi}$ can be used after the disturbance is applied to the bunch, then we get from Eqs. (3) and (8):

$$\varepsilon_1 = \det M \cdot \varepsilon_0 \tag{9}$$

2 Case of cooling

For a special case of the "linear" cooling we have the transfer matrix M_C:

$$M_C = \begin{pmatrix} 1 & 0 \\ 0 & (1-c) \end{pmatrix} \tag{10}$$

where parameter c defines the cooling rate:

$$\lambda = \frac{1}{T_{rev}} \frac{x_1' - x_0'}{x_0'} = \frac{c}{T_{rev}} \tag{11}$$

Here T_{rev} is the revolution frequency of the hadron storage ring.

Substituting $\det M_C = 1 - c$ into Eqs. (9) we get:

$$\varepsilon_1 = (1-c)\varepsilon_0 \tag{12}$$

Then for the emittance cooling rate we obtain:

$$\lambda_\varepsilon = \frac{1}{T_{rev}}\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0} = \frac{c}{T_{rev}} \tag{13}$$

Next, let us consider the change in the rms beam size (σ_x) and the rms angular spread ($\sigma_{x'}$) on a single pass through the cooling section. By default:

$$\begin{aligned} \sigma_x &= \left(\int_0^{2\pi}\int_0^{\infty} x^2(J,\varphi) f_{J,\varphi} dJ d\varphi\right)^{1/2} \\ \sigma_{x'} &= \left(\int_0^{2\pi}\int_0^{\infty} x'^2(J,\varphi) f_{J,\varphi} dJ d\varphi\right)^{1/2} \end{aligned} \tag{14}$$

Here, regular coordinates x, x' can be expressed through action-angle variables as:

$$\begin{aligned} x &= \sqrt{2J\beta}\cos\varphi \\ x' &= \sqrt{\tfrac{2J}{\beta}}(\sin\varphi - \alpha\cos\varphi) \end{aligned} \tag{15}$$

According to Eq. (7) for transfer matrix (10) we get:

$$\begin{aligned} \beta_1 &= \beta_0/(1-c) \\ \alpha_1 &= \alpha_0 \\ \gamma_1 &= \gamma_0(1-c) \end{aligned} \tag{16}$$

It follows from Eqs. (8), (15) and (16) that, since according to Eq. (10) $x_1 = x_0$, then $\cos(\varphi_1) = \cos(\varphi 0)$. Therefore:

$$\varphi_1 = \varphi_0 \tag{17}$$

Then, substituting Eq. (8) and Eqs. (15-17) into Eq. (14) we obtain:

$$\begin{aligned} \sigma_{x1} &= \sigma_{x0} \\ \sigma_{x1'} &= (1-c)\sigma_{x0'} \end{aligned} \tag{18}$$

Once again, let us stress that Eq. (18) gives the change in σ_x and $\sigma_{x'}$ for a single pass through the cooling section. The betatron motion in the ring couples x and x' resulting in the following size and angular spread cooling rates: $\lambda_{\sigma_x} = \lambda_{\sigma_{x'}} = \lambda_\varepsilon/2 = c/(2T_{rev})$.

3 Acknowledgments

We would like to thank Michael Blaskiewicz for pointing out an error in the initial version of this note.

We are grateful to Alexei Fedotov and Jorg Kewisch for many fruitful discussions.