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Conversion of Twiss parameters in cooling process

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1 General considerations

It is convenient to consider effects on beam emittance in action-angle variables (J, φ) , where φ is a betatron phase and:

$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$
 (1)

For a Gaussian beam the distribution function $f_{J,\varphi}$ is given by:

$$f_{J,\varphi} = \frac{1}{2\pi\varepsilon} e^{-J/\varepsilon} \tag{2}$$

And the beam emittance is:

$$\varepsilon = \int_{0}^{2\pi} \int_{0}^{\infty} Jf_{J,\varphi} dJ d\varphi$$
(3)

Let's consider a linear transformation $(x_0, x'_0) \to (x_1, x'_1)$, here we are not making any assumptions about properties of the transfer matrix M:

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$
(4)

Then we get from Eq. (4):

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$
(5)

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Substituting Eq. (5) into Eq. (1) we obtain the following expression for the transformed action:

$$\det M \cdot J_0 = \frac{1}{2} (\gamma_1 x_1^2 + 2\alpha_1 x_1 x_1' + \beta_1 x_1'^2)$$
(6)

where Twiss parameters are transformed by the following law:

$$\begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$
(7)

It is important to notice, that for Twiss parameters, a "canonical" form of an ellipse equation, for which an action on the left-hand side of the equation is an ellipse area divided by 2π , sets a strict requirement to parameters: $\beta\gamma = 1 + \alpha^2$. This requirement is satisfied for parameters $\beta_1, \alpha_1, \gamma_1$ defined by Eq. (7) as long as it is satisfied for $\beta_0, \alpha_0, \gamma_0$.

For a usual symplectic transformation, $\det M = 1$, and the action is the invariant of motion. For a special case when the transformation does not conserve the beam emittance Eq. (6) tells us that:

$$J_1 = \det M \cdot J_0 \tag{8}$$

If this non-conservative disturbance is small enough to assume that the same $f_{J,\varphi}$ can be used after the disturbance is applied to the bunch, then we get from Eqs. (3) and (8):

$$\varepsilon_1 = \det M \cdot \varepsilon_0 \tag{9}$$

2 Case of cooling

For a special case of the "linear" cooling we have the transfer matrix M_C :

$$M_C = \left(\begin{array}{cc} 1 & 0\\ 0 & (1-c) \end{array}\right) \tag{10}$$

where parameter c defines the cooling rate:

$$\lambda = \frac{1}{T_{rev}} \frac{x_1' - x_0'}{x_0'} = \frac{c}{T_{rev}}$$
(11)

Here T_{rev} is the revolution frequency of the hadron storage ring.

Substituting det $M_C = 1 - c$ into Eqs. (9) we get:

$$\varepsilon_1 = (1-c)\varepsilon_0 \tag{12}$$

Then for the emittance cooling rate we obtain:

$$\lambda_{\varepsilon} = \frac{1}{T_{rev}} \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0} = \frac{c}{T_{rev}}$$
(13)

Next, let us consider the change in the rms beam size (σ_x) and the rms angular spread $(\sigma_{x'})$ on a single pass through the cooling section. By default:

$$\sigma_{x} = \left(\int_{0}^{2\pi} \int_{0}^{\infty} x^{2}(J,\varphi) f_{J,\varphi} dJ d\varphi \right)^{1/2}$$

$$\sigma_{x'} = \left(\int_{0}^{2\pi} \int_{0}^{\infty} x'^{2}(J,\varphi) f_{J,\varphi} dJ d\varphi \right)^{1/2}$$
(14)

Here, regular coordinates x, x' can be expressed through action-angle variables as:

$$\begin{aligned}
x &= \sqrt{2J\beta}\cos\varphi \\
x' &= \sqrt{\frac{2J}{\beta}}(\sin\varphi - \alpha\cos\varphi)
\end{aligned}$$
(15)

According to Eq. (7) for transfer matrix (10) we get:

$$\beta_1 = \beta_0 / (1 - c)$$

$$\alpha_1 = \alpha_0$$

$$\gamma_1 = \gamma_0 (1 - c)$$
(16)

It follows from Eqs. (8), (15) and (16) that, since according to Eq. (10) $x_1 = x_0$, then $\cos(\varphi_1) = \cos(\varphi_0)$. Therefore:

$$\varphi_1 = \varphi_0 \tag{17}$$

Then, substituting Eq. (8) and Eqs. (15-17) into Eq. (14) we obtain:

$$\begin{aligned}
\sigma_{x1} &= \sigma_{x0} \\
\sigma_{x1'} &= (1-c)\sigma_{x0'}
\end{aligned} (18)$$

Once again, let us stress that Eq. (18) gives the change in σ_x and $\sigma_{x'}$ for a single pass through the cooling section. The betatron motion in the ring couples x and x' resulting in the following size and angular spread cooling rates: $\lambda_{\sigma_x} = \lambda_{\sigma_{x'}} = \lambda_{\varepsilon}/2 = c/(2T_{rev})$.

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