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1 General considerations

It is convenient to consider effects on beam emittance in action-angle variables (J, φ) , where φ is a betatron phase and:

$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x x' + \beta x'^2) \quad (1)$$

For a Gaussian beam the distribution function $f_{J,\varphi}$ is given by:

$$f_{J,\varphi} = \frac{1}{2\pi\varepsilon} e^{-J/\varepsilon} \quad (2)$$

And the beam emittance is:

$$\varepsilon = \int_0^{2\pi} \int_0^{\infty} J f_{J,\varphi} dJ d\varphi \quad (3)$$

Let's consider a linear transformation $(x_0, x'_0) \rightarrow (x_1, x'_1)$, here we are not making any assumptions about properties of the transfer matrix M :

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (4)$$

Then we get from Eq. (4):

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} \quad (5)$$

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Substituting Eq. (5) into Eq. (1) we obtain the following expression for the transformed action:

$$\det M \cdot J_0 = \frac{1}{2}(\gamma_1 x_1^2 + 2\alpha_1 x_1 x_1' + \beta_1 x_1'^2) \quad (6)$$

where Twiss parameters are transformed by the following law:

$$\begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} \quad (7)$$

It is important to notice, that for Twiss parameters, a “canonical” form of an ellipse equation, for which an action on the left-hand side of the equation is an ellipse area divided by 2π , sets a strict requirement to parameters: $\beta\gamma = 1 + \alpha^2$. This requirement is satisfied for parameters $\beta_1, \alpha_1, \gamma_1$ defined by Eq. (7) as long as it is satisfied for $\beta_0, \alpha_0, \gamma_0$.

For a usual symplectic transformation, $\det M = 1$, and the action is the invariant of motion. For a special case when the transformation does not conserve the beam emittance Eq. (6) tells us that:

$$J_1 = \det M \cdot J_0 \quad (8)$$

If this non-conservative disturbance is small enough to assume that the same $f_{J,\varphi}$ can be used after the disturbance is applied to the bunch, then we get from Eqs. (3) and (8):

$$\varepsilon_1 = \det M \cdot \varepsilon_0 \quad (9)$$

2 Case of cooling

For a special case of the “linear” cooling we have the transfer matrix M_C :

$$M_C = \begin{pmatrix} 1 & 0 \\ 0 & (1 - c) \end{pmatrix} \quad (10)$$

where parameter c defines the cooling rate:

$$\lambda = \frac{1}{T_{rev}} \frac{x_1' - x_0'}{x_0'} = \frac{c}{T_{rev}} \quad (11)$$

Here T_{rev} is the revolution frequency of the hadron storage ring.

Substituting $\det M_C = 1 - c$ into Eqs. (9) we get:

$$\varepsilon_1 = (1 - c)\varepsilon_0 \quad (12)$$

Then for the emittance cooling rate we obtain:

$$\lambda_\varepsilon = \frac{1}{T_{rev}} \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0} = \frac{c}{T_{rev}} \quad (13)$$

Next, let us consider the change in the rms beam size (σ_x) and the rms angular spread ($\sigma_{x'}$) on a single pass through the cooling section. By default:

$$\begin{aligned} \sigma_x &= \left(\int_0^{2\pi} \int_0^\infty x^2(J, \varphi) f_{J,\varphi} dJ d\varphi \right)^{1/2} \\ \sigma_{x'} &= \left(\int_0^{2\pi} \int_0^\infty x'^2(J, \varphi) f_{J,\varphi} dJ d\varphi \right)^{1/2} \end{aligned} \quad (14)$$

Here, regular coordinates x, x' can be expressed through action-angle variables as:

$$\begin{aligned} x &= \sqrt{2J\beta} \cos \varphi \\ x' &= \sqrt{\frac{2J}{\beta}} (\sin \varphi - \alpha \cos \varphi) \end{aligned} \quad (15)$$

According to Eq. (7) for transfer matrix (10) we get:

$$\begin{aligned} \beta_1 &= \beta_0 / (1 - c) \\ \alpha_1 &= \alpha_0 \\ \gamma_1 &= \gamma_0 (1 - c) \end{aligned} \quad (16)$$

It follows from Eqs. (8), (15) and (16) that, since according to Eq. (10) $x_1 = x_0$, then $\cos(\varphi_1) = \cos(\varphi_0)$. Therefore:

$$\varphi_1 = \varphi_0 \quad (17)$$

Then, substituting Eq. (8) and Eqs. (15-17) into Eq. (14) we obtain:

$$\begin{aligned} \sigma_{x1} &= \sigma_{x0} \\ \sigma_{x1'} &= (1 - c)\sigma_{x0'} \end{aligned} \quad (18)$$

Once again, let us stress that Eq. (18) gives the change in σ_x and $\sigma_{x'}$ for a single pass through the cooling section. The betatron motion in the ring couples x and x' resulting in the following size and angular spread cooling rates: $\lambda_{\sigma_x} = \lambda_{\sigma_{x'}} = \lambda_\varepsilon / 2 = c / (2T_{rev})$.

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