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M. Blaskiewicz

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Noise driven emittance growth with coherent forces

M. Blaskiewicz
BNL 911B, Upton, NY 11973, USA

Noise driven emittance growth is modified by the presence of coherent forces. Closed form asymptoic growth rates are given for two frequency distributions. The analytic results are compared with tracking, showing good agreement. Purely numerical results for a parabolic frequncy distribution are also presented. If a small antidamping force is present then additional, real frequency shifts increase growth rates.

INTRODUCTION

Crab cavities are needed to compensate the crossing angle in the LHC and the EIC. Noise on these cavities will lead to emittance growth [1]. The required noise limits on the cavities are at, or slightly exceed, the state of the art. Any additional complications are a serious concern. Recent work has called into question the importance of coherent forces on this emittance growth [2–4]. The simplest model of this is to consider a resonator impedance with wavelength long compared to the bunch and it's coupled bunch modes. For a given mode, the kick from the resonator on a bunch has components proportional to the average offset of the bunch, and the average angle of the bunch. In a time domain model this corresponds to terms proportional to the offset and the velocity. We go on to consider a simple model of tune spread which yields a simple closed form answer. In the second part a more realistic model is considered. A rather complicated formula results but it agrees well with simulations. The final section presents purely numerical results for parabolic tune spread

A SIMPLE MODEL

A simple model that captures the physics can be obtained analytically. Consider a collection of one dimensional oscillators with coordinates $x_1, x_2, \dots x_N$. The equation of motion for particle j is

$$\ddot{x}_j + \omega_j^2 x_j = n(t) - 2\bar{\omega}\Omega\bar{x}(t) + 2\alpha\dot{\bar{x}}(t), \tag{1}$$

where the oscillator has frequency ω_j and $n(t)$ is random noise. In the absence of frequency spread Ω is the coherent tune shift and α is the growth rate. The central frequency is $\bar{\omega}$ and

$$\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k. \tag{2}$$

Take $x_j = a_j(t)\exp(-i\bar{\omega}t)$ [5] and $\omega_j = \bar{\omega} + \delta\omega_j$. Differentiating one finds

$$\dot{x}_j = -i\bar{\omega}a_j\exp(-i\bar{\omega}t) + \dot{a}_j\exp(-i\bar{\omega}t), \tag{3}$$
$$\ddot{x}_j = -\bar{\omega}^2 a_j\exp(-i\bar{\omega}t) - 2i\bar{\omega}\dot{a}_j\exp(-i\bar{\omega}t) + \ddot{a}_j\exp(-i\bar{\omega}t). \tag{4}$$

Make the single sideband approximation and neglect the last terms on the right in (3) and (4). Cancelling the fast term, equation (1) becomes

$$\dot{a}_j + i\delta\omega_j a_j \approx \frac{in(t)\exp(i\bar{\omega}t)}{2\bar{\omega}} + (\alpha - i\Omega)\bar{a} \approx \tilde{n}(t) + (\alpha - i\Omega)\bar{a}, \tag{5}$$

where $\tilde{n}$ is the noise in the sideband. These approximations are very good for any frequency shifts and spreads obtainable in a synchrotron. Multiply by the integrating factor $\exp(i\delta\omega_j t)$, making the left side a perfect differential, and integrate. This results in

$$a_j(t) - a_j(0)e^{-i\delta\omega_j t} = \int_0^t dt_1 e^{i\delta\omega_j(t_1 - t)}\left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right]. \tag{6}$$

Next average the equation over j

$$\bar{a}(t) - \frac{1}{N}\sum_{j=1}^{N} a_j(0)e^{-i\delta\omega_j t} = \int_0^t dt_1 \left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right] \frac{1}{N}\sum_{j=1}^{N} e^{i\delta\omega_j(t_1 - t)}. \tag{7}$$

Approximate the finite average on the right with an expectation value over a continuous distribution. Consider long term behavior so the second term on the left is negligible. Assume a Lorentzian frequency distribution

$$\frac{1}{N}\sum_{j=1}^{N} e^{i\delta\omega_j\tau} \approx \exp(-\beta|\tau|). \tag{8}$$

Inserting this one finds

$$\bar{a}(t) = \int_0^t dt_1 e^{\beta(t_1 - t)} \left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right]. \tag{9}$$

Differentiating (9) with respect to time

$$\dot{\bar{a}} = \tilde{n} + (\alpha - i\Omega)\bar{a} - \beta\int_0^t dt_1 e^{\beta(t_1 - t)} \left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right] = \tilde{n} + (\alpha - i\Omega)\bar{a} - \beta\bar{a}, \tag{10}$$

where the last equality follows from substituting (9). Using similar techniques

$$\bar{a}(t) = \bar{a}(0)e^{(\alpha - \beta - i\Omega)t} + \int_0^t dt_1 e + (\alpha - \beta - i\Omega)(t - t_1)\tilde{n}(t_1). \tag{11}$$

This expression for $\bar{a}(t)$ can now be used in equation (6). Terms that do not affect leading order behavior are dropped. To calculate emittance growth consider the expectation value

$$\langle|a_j(t)|^2\rangle = \left\langle \int_0^t dt_1 e^{i\delta\omega_j(t_1 - t)} \left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right] \int_0^t dt_2 e^{-i\delta\omega_j(t_2 - t)} \left[\tilde{n}^*(t_2) + (\alpha + i\Omega)\bar{a}^*(t_2)\right] \right\rangle \tag{12}$$

$$= \int_0^t dt_1 \int_0^t dt_2 \left[\langle\tilde{n}(t_1)\tilde{n}^*(t_2)\rangle + (\alpha^2 + \Omega^2)\langle\bar{a}(t_1)\bar{a}^*(t_2)\rangle + 2Re\langle(\alpha - i\Omega)\bar{a}(t_1)\tilde{n}^*(t_2)\rangle\right] e^{-\beta|t_1 - t_2|} \tag{13}$$

Now assume white noise so that

$$\langle\tilde{n}(t_1)\tilde{n}^*(t_2)\rangle = \sigma^2\delta(t_1 - t_2). \tag{14}$$

Then

$$\langle\bar{a}(t_1)\bar{a}^*(t_2)\rangle = \int_0^{t_1} dt_1' e^{(\alpha - \beta - i\Omega)(t_1 - t_1')} \int_0^{t_2} dt_2' e^{(\alpha - \beta + i\Omega)(t_2 - t_2')}\sigma^2\delta(t_1' - t_2') \tag{15}$$

$$= \int_0^{\min(t_1, t_2)} dt_1' e^{(\alpha - \beta - i\Omega)(t_1 - t_1')} e^{(\alpha - \beta + i\Omega)(t_2 - t_1')}\sigma^2 \tag{16}$$

$$= \frac{\sigma^2}{2(\beta - \alpha)} \exp[(\alpha - \beta)|t_1 - t_2| - i\Omega(t_1 - t_2)], \tag{17}$$

where we assumed $(\beta - \alpha)\min(t_1, t_2) >> 1$ so that the terms at $t_1' = 0$ can be neglected.

Finally

$$\langle \bar{a}(t_1)\tilde{n}^*(t_2)\rangle = \sigma^2 \int_0^{t_1} dt_1' e^{(\alpha - 2\beta - i\Omega)(t_1 - t_1')}\delta(t_2 - t_1') \tag{18}$$

$$= \sigma^2 H(t_1 - t_2)\exp[(\alpha - \beta - i\Omega)(t_1 - t_2)] \tag{19}$$

where $H(t)$ is the Heaviside function. Inserting (14), (17) and (19) in (13) results in an integral of the form

$$\int_0^t dt_1 \int_0^t dt_2 f(t_1 - t_2) = \int_{-t}^t (t - |\tau|)f(\tau)d\tau \approx t\int_{-\infty}^{\infty} f(\tau).$$

Using the expression on the right and doing some algebra one finds

$$\langle |a_j(t)|^2\rangle = \sigma^2 t\frac{\beta}{\beta - \alpha}. \tag{20}$$

The beam goes unstable as α approaches β and the growth rate diverges as the threshold is approached. Direct simulations have been performed which confirm this result.

A BETTER MODEL

The lack of dependence on Ω in equation (20) is due to the long tails of the Lorentzian tune distribution. This can be partially remedied by going back to equation (7) and taking

$$\frac{1}{N}\sum_{j=1}^{N} e^{i\delta\omega_j\tau} \approx W_1\exp(-\beta_1|\tau|) - W_2\exp(-\beta_2|\tau|), \tag{21}$$

with $W_1 - W_2 = 1$ and $W_1\beta_1 = W_2\beta_2$. This corresponds to a frequency distribution of

$$p(\delta\omega) = \frac{\beta_1\beta_2(\beta_1 + \beta_2)}{\pi(\beta_1^2 + \delta\omega^2)(\beta_2^2 + \delta\omega^2)}. \tag{22}$$

This distribution has a finite standard deviation and should be more physical. Again the focus will be on long term behavior so we will set all initial amplitudes to zero. Equation (9) becomes

$$\bar{a}(t) = \int_0^t dt_1 \left(W_1 e^{\beta_1(t_1 - t)} - W_2 e^{\beta_2(t_1 - t)}\right)\left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right]. \tag{23}$$

Differentiating with respect to t

$$\dot{\bar{a}}(t) = (W_1 - W_2)\left[\tilde{n}(t) + (\alpha - i\Omega)\bar{a}(t)\right] - \beta_1 W_1\int_0^t dt_1 \left(e^{\beta_1(t_1 - t)} - e^{\beta_2(t_1 - t)}\right)\left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right]. \tag{24}$$

Where we have used $\beta_1 W_1 = \beta_2 W_2$. Differentiate again

$$\ddot{\bar{a}}(t) = (W_1 - W_2)\left[\dot{\tilde{n}}(t) + (\alpha - i\Omega)\dot{\bar{a}}(t)\right] + \beta_1 W_1\int_0^t dt_1 \left(\beta_1 e^{\beta_1(t_1 - t)} - \beta_2 e^{\beta_2(t_1 - t)}\right)\left[\tilde{n}(t_1) + (\alpha - i\Omega)\bar{a}(t_1)\right]. \tag{25}$$

Use combinations of $\ddot{\bar{a}} + \beta_1\dot{\bar{a}}$ and $\ddot{\bar{a}} + \beta_2\dot{\bar{a}}$ to eliminate the instances of $\bar{a}$ within the integrals to obtain

$$\dddot{\bar{a}} + (\beta_2 + \beta_1 - \alpha + i\Omega)\ddot{\bar{a}} + \left[\beta_1\beta_2 - (\alpha - i\Omega)(\beta_1 + \beta_2)\right]\dot{\bar{a}} = (\beta_1 + \beta_2)\dot{\tilde{n}} + \ddot{\tilde{n}} \tag{26}$$

While time derivatives of the noise may look singular, remember we are working within a single betatron sideband so everything is smooth. The white noise correlation in equation (14) was an approximation.

Let the homogeneous solutions of equation (26) be $\exp(\lambda_1 t)$ and $\exp(\lambda_2 t)$. The Green's function is then

$$G(t) = H(t)\frac{\exp(\lambda_1 t) - \exp(\lambda_2 t)}{\lambda_1 - \lambda_2}. \tag{27}$$

And

$$\begin{aligned}\bar{a}(t) &= \int_0^t G(t-t_1)\left[(\beta_1+\beta_2)\tilde{n}(t_1) + \dot{\tilde{n}}(t_1)\right] dt_1 \\ &= \int_0^t \tilde{n}(t_1)\left[(\beta_1+\beta_2)G(t-t_1) + \dot{G}(t-t_1)\right] dt_1 + \tilde{n}(0)G(t).\end{aligned} \tag{28}$$

The surface term $\tilde{n}(0)G(t)$ will not influence long term behavior and is dropped. Equation (12) becomes

$$\begin{aligned}\langle|a_j(t)|^2\rangle &= \int_0^t dt_1 \int_0^t dt_2 \left[\langle\tilde{n}(t_1)\tilde{n}^*(t_2)\rangle + (\alpha^2+\Omega^2)\langle\bar{a}(t_1)\bar{a}^*(t_2)\rangle + 2Re\langle(\alpha - i\Omega)\bar{a}(t_1)\tilde{n}^*(t_2)\rangle\right] \\ &\left(W_1 e^{-\beta_1|t_1-t_2|} - W_2 e^{-\beta_2|t_1-t_2|}\right) \\ &= \int_0^t dt_1 \int_0^t dt_2 \left[I_{nn}(t_1,t_2) + I_{aa}(t_1,t_2) + I_{na}(t_1,t_2)\right]\left(W_1 e^{-\beta_1|t_1-t_2|} - W_2 e^{-\beta_2|t_1-t_2|}\right) \\ &= \langle|a_j(t)|^2\rangle_{nn} + \langle|a_j(t)|^2\rangle_{aa} + \langle|a_j(t)|^2\rangle_{na}\end{aligned} \tag{29}$$

As before we take

$$I_{nn}(t_1,t_2) = \sigma^2\delta(t_1-t_2). \tag{30}$$

So $\langle|a_j(t)|^2\rangle_{nn} = \sigma^2 t$.

Proceeding one finds

$$\begin{aligned}I_{aa}(t_1,t_2) &= (\alpha^2+\Omega^2) < \bar{a}(t_1)\bar{a}^*(t_2) > \\ &\approx (\alpha^2+\Omega^2)\sigma^2 \int_{-\infty}^{\infty} ds\dot{G}(t_1-t_2+s)\dot{G}^*(s) + (\beta_1+\beta_2)^2 G(t_1-t_2+s)G^*(s)\end{aligned} \tag{31}$$

where the approximation is $Re(\lambda(t_1+t_2) \gg 1$, which always holds asymptotically. The integrals are elementary but tedious. The final result is

$$\begin{aligned}\frac{\langle|a_j(t)|^2\rangle_{aa}}{(\alpha^2+\Omega^2)\sigma^2 t} &= \frac{|\lambda_1|^2+(\beta_1+\beta_2)^2}{|\lambda_1-\lambda_2|^2}\left[\frac{W_1}{\beta_1-\lambda_1} + \frac{W_1}{\beta_1-\lambda_1^*} - \frac{W_2}{\beta_2-\lambda_1} - \frac{W_2}{\beta_2-\lambda_1^*}\right] \\ &-\frac{\lambda_1^*\lambda_2+(\beta_1+\beta_2)^2}{|\lambda_1-\lambda_2|^2}\left[\frac{W_1}{\beta_1-\lambda_2} + \frac{W_1}{\beta_1-\lambda_1^*} - \frac{W_2}{\beta_2-\lambda_2} - \frac{W_2}{\beta_2-\lambda_1^*}\right] \\ &-\frac{\lambda_1\lambda_2^*+(\beta_1+\beta_2)^2}{|\lambda_1-\lambda_2|^2}\left[\frac{W_1}{\beta_1-\lambda_2^*} + \frac{W_1}{\beta_1-\lambda_1} - \frac{W_2}{\beta_2-\lambda_2^*} - \frac{W_2}{\beta_2-\lambda_1}\right] \\ &+\frac{|\lambda_2|^2+(\beta_1+\beta_2)^2}{|\lambda_1-\lambda_2|^2}\left[\frac{W_1}{\beta_1-\lambda_2} + \frac{W_1}{\beta_1-\lambda_2^*} - \frac{W_2}{\beta_2-\lambda_2} - \frac{W_2}{\beta_2-\lambda_2^*}\right].\end{aligned} \tag{32}$$

The last term is $\langle|a_j(t)|^2\rangle_{na} = U + U^*$ with

$$U = \frac{(\alpha-i\Omega)\sigma^2 t}{\lambda_1-\lambda_2}\left[\frac{W_1(\beta_1+\beta_2+\lambda_1)}{\beta_1-\lambda_1} - \frac{W_1(\beta_1+\beta_2+\lambda_2)}{\beta_1-\lambda_2} - \frac{W_2(\beta_1+\beta_2+\lambda_1)}{\beta_2-\lambda_1} + \frac{W_2(\beta_1+\beta_2+\lambda_2)}{\beta_2-\lambda_2}\right]. \tag{33}$$

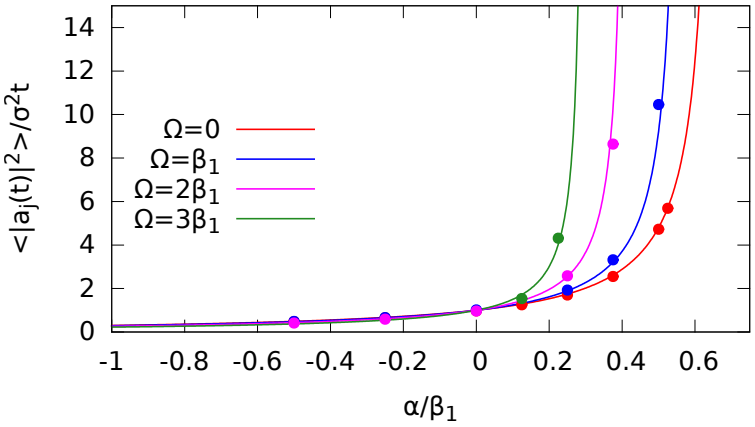


FIG. 1: Relative emittance growth rate versus resistive part of the coherent tune shift for $\beta_2 = 2\beta_1$ and various values of Ω. The solid lines are from equation 29 normalized to the single particle growth rate. The dots are direct simulations for the same parameters.

PARABOLIC DISTRIBUTION

Consider the parabolic frequency distribution

$$p(\delta\omega) = \frac{3(\beta^2 - \delta\omega^2)}{4\beta^3} H(\beta - |\delta\omega|). \tag{34}$$

Equation (1) has been simulated using this distribution and results are shown in Table 1. The statistical errors in the table are estimated at a few percent. Consider column 1, with $\alpha/\beta = -0.25$. For this case increasing Ω reduces the growth rate. This also holds for column 2. For column 3, with $\alpha/\beta = +0.1$, increasing Ω causes the growth rate to increase.

TABLE I: Values of $< |a_j(t)|^2 > /\sigma^2 t$ for various collective coeffcients.

$\alpha/\beta \rightarrow$	-0.25	-0.1	0.1	0.25
$\Omega/\beta \downarrow$				
0.0	0.607	0.950	1.35	2.52
0.5	0.465	0.720	3.44	∞
1.0	0.193	0.295	∞	∞
1.5	0.109	0.112	∞	∞

CONCLUSIONS

The effect of coherent forces on noise driven emittance growth has been considered for 3 cases. For the Lorentzian tune distribution Ω had no effect. For the dual Lorentizian distribution Ω was destabilizing for positive α. When $\alpha < 0$ the analytic formula shows some reduction in growth rate with Ω. For a parabolic distribution with $\alpha < 0$,

increasing Ω reduced growth rates. When $\alpha > 0$ the growth rate increased with Ω. In summary, an adequate damper leads to growth rate reduction. The impact of Ω is somewhat weaker but can be beneficial when $\alpha < 0$.

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[4] X. Buffat, Suppression of Emittance Growth by a Collective Force: Van Kampen Approach, wepotk059, IPAC22

[5] Taking $x_j = a_{r,j}(t)\cos(\bar{\omega}t) + a_{i,j}(t)\sin(\bar{\omega}t)$ with everything real and proceeding similarly gives identical results.