Proton-electron focusing in EIC ring cooler

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Proton-electron focusing in EIC Ring Cooler

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1 Introduction

In this note we consider an effect of the proton-electron focusing on an emittance of e-bunches in the ring cooler.

It will be shown that for the optimized parameters of the ring cooler, when the proton and the electron beams are well-matched in the cooling section (CS), there is no significant emittance growth from a proton-electron space charge (SC) kick.

2 EIC Ring Cooler parameters

Table 1 shows parameters of the EIC Ring Cooler, which are important for the following considerations. The design parameters are taken from the pre-conceptual design [1], the optimized parameters are the result of a recent optimization [2, 3].

3 Space charge focusing

The space charge, either the self-SC or the space charge of a co-traveling beam in the cooling section, introduces an additional focusing. Since such a focusing is nonlinear, it can not be fully compensated by the linear optics. Hence, the SC causes filamentation of the transverse phase space of the e-bunches, which can be described as an emittance growth.

Let us consider a Gaussian, circularly symmetric transverse density distributions for both the e-bunch and the p-bunch. The equation of motion of an individual electron in this case is [4]:

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### Table 1: EIC cooler parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>relativistic $\gamma$</td>
<td>293.1</td>
</tr>
<tr>
<td>Number of protons per bunch ($N_i$)</td>
<td>$6.88 \times 10^{10}$</td>
</tr>
<tr>
<td>rms p-bunch length ($\sigma_{zp}$) [cm]</td>
<td>6</td>
</tr>
<tr>
<td>p-bunch peak current ($I_p = \frac{e c N_p}{\sqrt{2 \pi \sigma_{zp}}} )$ [A]</td>
<td>22</td>
</tr>
<tr>
<td>p-bunch rms momentum spread ($\sigma_{\delta p}$)</td>
<td>$6.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>p-bunch emittance ($\varepsilon_{px,y}$) [nm]</td>
<td>9.6, 1.5</td>
</tr>
<tr>
<td>Ring circumference [m]</td>
<td>449.1</td>
</tr>
<tr>
<td>e-ring tunes $Q_{x,y}$</td>
<td>59.92, 59.85</td>
</tr>
<tr>
<td>Cooling section length [m]</td>
<td>170</td>
</tr>
<tr>
<td>rms e-bunch length ($\sigma_{ze}$) [cm]</td>
<td>12</td>
</tr>
<tr>
<td>e-bunch rms momentum spread ($\sigma_{\delta e}$)</td>
<td>$8.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>Number of electrons per bunch ($N_e$)</td>
<td>$3 \cdot 10^{11}$</td>
</tr>
<tr>
<td>e-bunch peak current ($I_e = \frac{e c N_e}{\sqrt{2 \pi \sigma_{ep}}} )$ [A]</td>
<td>48.3, 32.2</td>
</tr>
<tr>
<td>p-bunch CS beta function ($\beta_{px,py}$) [m]</td>
<td>200, 200, 200, 1200</td>
</tr>
<tr>
<td>e-bunch CS beta function ($\beta_{ex,ey}$) [m]</td>
<td>170, 280, 100, 100</td>
</tr>
<tr>
<td>e-bunch emittance ($\varepsilon_{x,y}$) [nm]</td>
<td>21, 18, 15, 15</td>
</tr>
<tr>
<td>cooling times $(\tau_{x,y,z})$ [min]</td>
<td>design optimized</td>
</tr>
<tr>
<td></td>
<td>123, 61, 25, 118, 118, 19</td>
</tr>
</tbody>
</table>

$\tau'' = K_e \frac{1}{r} \left[ \left( 1 - e^{-\frac{r^2}{2\sigma^2}} \right) - \frac{I_p}{I_e} \left( 1 - e^{-\frac{r^2}{2\sigma_{zp}^2}} \right) \right] - \kappa r$  \hfill (1)

Here generalized perviance $K_e = \frac{2I_e}{I_A(\gamma \beta)^3} \approx \frac{2I_e}{I_A \gamma^3}$ (in this formula $\gamma$ and $\beta$ are the relativistic factors, and we will assume $\beta = 1$ for further considerations), $I_A = \frac{4\pi e m c^3}{\varepsilon_0} \approx 17 \cdot 10^3$ A is Alfven current, $\sigma \equiv \sigma_x = \sigma_y$, $I_e$ and $I_p$ are the respective peak currents, and $\kappa$ represents the gradient of the linear focusing elements.

We are interested in a proton-electron focusing in the cooling section, so for this exercise we are dropping the self space charge term:

$\tau'' = -K_e \frac{I_p}{I_e r} \left( 1 - e^{-\frac{r^2}{2\sigma_{zp}^2}} \right) - \kappa r$ \hfill (2)

Notice, that a defocusing caused by the self SC of the e-bunches alleviates the p-e focusing in the cooling section to some extent. Therefore, Eq. (2)

---

1 No redistribution of cooling decrements is assumed
2 Loss of cooling due to a partial bunches’ overlap is taken into account in calculations of $\tau_{x,y,z}$.
overestimates the space charge effect on beam dynamics.

Rewriting Eq. (2) for $x$ and $y$ coordinates and we get:

$$
\begin{align*}
\frac{x''}{K_p x^2+y^2} &= \left(1 - e^{-\frac{x^2+y^2}{2\sigma_p^2}}\right) x - \kappa x \\
\frac{y''}{K_p x^2+y^2} &= \left(1 - e^{-\frac{x^2+y^2}{2\sigma_p^2}}\right) y - \kappa y
\end{align*}
$$

where $K_p = K_e I_p / I_e$.

Expanding Eq. (3) with respect to a small parameter $(x^2+y^2)/\sigma_p^2$ we get linearized equations of motion:

$$
\begin{align*}
\frac{x''}{K_p x^2+y^2} + \left(\frac{\kappa}{2\sigma_p^2} + \frac{K_p}{\sigma_p^2}\right) x &= 0 \\
\frac{y''}{K_p x^2+y^2} + \left(\frac{\kappa}{2\sigma_p^2} + \frac{K_p}{\sigma_p^2}\right) y &= 0
\end{align*}
$$

4 Motion in Courant-Snyder coordinates

It is convenient to consider the effects on the bunch emittance using the Courant-Snyder coordinates $(\xi, \zeta)$:

$$
\begin{align*}
\xi &= \frac{x}{\sqrt{\beta}} \\
\zeta &= \frac{x \alpha}{\sqrt{\beta}} + x' \sqrt{\beta}
\end{align*}
$$

An independent variable in the $(\xi, \zeta)$-coordinates is a betatron phase $\varphi$. A betatron motion in such a phase space is represented by a circle of the radius $\sqrt{2J}$, where an action $J = (\xi^2 + \zeta^2)/2$.

For the Gaussian bunch, the distribution function in the $(J, \varphi)$ variables is given by:

$$
f_{J, \varphi} = \frac{1}{2\pi \varepsilon} e^{-J/\varepsilon}
$$

and the average value of the action is:

$$
\langle J \rangle = \int_0^\infty \int_0^{2\pi} J f_{J, \varphi} d\varphi dJ = \varepsilon
$$

One turn in the storage ring in the Courant-Snyder coordinates is represented by a simple rotation matrix:

$$
\begin{pmatrix}
\xi_{h,v1} \\
\zeta_{h,v1}
\end{pmatrix} = \begin{pmatrix}
\cos(2\pi \nu_{h,v}) & \sin(2\pi \nu_{h,v}) \\
-\sin(2\pi \nu_{h,v}) & \cos(2\pi \nu_{h,v})
\end{pmatrix} \cdot \begin{pmatrix}
\xi_{h,v0} \\
\zeta_{h,v0}
\end{pmatrix}
$$
where $\nu$ is a fractional tune of the ring, and $h, v$ indexes correspond to the horizontal and the vertical motion respectively.

We will assume that the p-e focusing in the CS is weak enough to be represented simply by an instantaneous change in the electron angles ($\delta x'$, $\delta y'$) after each turn in the storage ring. This assumption will be justified later. From Eq. 3 we get:

$$
\begin{align*}
\delta x' &= -\frac{K_pL_{CS}x}{x^2+y^2} \left(1 - e^{-\frac{x^2+y^2}{2\sigma_p^2}}\right) \\
\delta y' &= -\frac{K_pL_{CS}y}{x^2+y^2} \left(1 - e^{-\frac{x^2+y^2}{2\sigma_p^2}}\right)
\end{align*}
$$

(9)

where $L_{CS}$ is the length of the cooling section.

The linear part of the SC focusing kick can be represented by a transfer matrix:

$$
M = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \equiv -\frac{K_pL_{CS}}{2\sigma_p^2}
$$

(10)

Denoting the variables before the kick with an index 1 and after the kick with an index 2 we can write for the Twiss parameters:

$$
\begin{align*}
\beta_2 &= \beta_1 \equiv \beta_{CS} \\
\alpha_2 &= -k\beta_1 + \alpha_1
\end{align*}
$$

(11)

where $\beta_{CS}$ is an average e-beam beta-function in the cooling section.

Then, from Eqs. 5 and 11 we get:

$$
\begin{align*}
\xi_{h2} &= \xi_{h1} \\
\zeta_{h2} &= \zeta_{h1} + \sqrt{\beta_{x1}}(\delta x' - kx_1)
\end{align*}
$$

(12)

An equation for the vertical motion is identical to Eq. 12.

If the SC kick was linear, then we would have $\delta x' = kx_1$, which would mean that $J_2 = J_1$ as it should be for the linear focusing.

Instead, we have a nonlinear kick, which gives us the following expression for $\delta \zeta = \zeta_2 - \zeta_1$:

$$
\delta \zeta_h = \sqrt{\beta_{x1}}(\delta x' - kx_1) = -K_pL_{CS}x\sqrt{\beta_{x1}} \left(1 - e^{-\frac{x^2+y^2}{2\sigma_p^2}}\right) \left(\frac{-x_1}{x^2+y^2}x_1 - \frac{x_1}{2\sigma_p^2}\right)
$$

(13)

Again, an equation for $\delta \zeta_v$ is identical to Eq. 13.

Finally, substituting Eq. 5 into Eq. 13, we get:
Under our assumptions, the motion of each electron in \((\xi, \zeta)\)-phase space is equivalent to the motion of a harmonic oscillator experiencing a weak nonlinear driving force:

\[
\left\{ \begin{array}{l}
\xi'_{h,v} = \zeta_{h,v} \\
\zeta'_{h,v} = -\xi_{h,v} + \delta \zeta_{h,v} \sum_n \delta D(\varphi_{h,v} - 2\pi Q_{h,v} n)
\end{array} \right.
\]  

(15)

where \(\delta D\) is the Dirac delta function, and \(Q_h\) and \(Q_v\) are the full horizontal and vertical tunes of the storage ring.

The easiest and the fastest way to simulate the turn-by-turn motion of an electron is to apply the transformation (8) to coordinates of a particle from the previous turn and to add the kick (14) to the obtained result. Solving Eq. (15) numerically gives the same result, of course, but slows down the simulations.

5 Single pass effect

Before considering dynamics of an ensemble of nonlinear oscillators we will check that the SC effect is indeed small for a single pass through the cooling section.

The change in action due to a single nonlinear kick \((\delta J = J_2 - J_1)\) is:

\[
\delta J = \frac{\xi_1^2 + \zeta_1^2 + 2\zeta_1 \delta \zeta + \delta \zeta^2}{2} - J_1 = J_1 \sin(\varphi) \delta \zeta + \frac{\delta \zeta^2}{2}
\]  

(16)

Assuming that the beam distribution doesn’t change over one pass through the CS, we get the change in the emittance \((\delta \varepsilon)\) by substituting Eq. (16) into Eq. (7):

\[
\delta \varepsilon = \int_0^{2\pi} \int_0^{2\pi} f_{J,\varphi}(J \sin(\varphi) \delta \zeta + \delta \zeta^2/2) d\varphi dJ
\]  

(17)

To simplify further estimates we will drop the x-y coupling terms in Eq. (14):

\[
\delta \zeta \approx -K_p L_{CS} \left( \frac{1 - e^{-\frac{\beta_{CS} \xi^2}{2\sigma_p^2}}}{\xi} - \frac{\beta_{CS} \xi}{2\sigma_p^2} \right)
\]  

(18)
Finally, substituting Eq. (18) into Eq. (17) and noticing that $\xi = \sqrt{2J} \cos(\varphi)$ and that $\forall J$ the integral $\int_{0}^{2\pi} J \sin(\varphi) \delta \zeta d\varphi = 0$, we get the following expression for the fractional emittance change over one pass through the CS:

$$\frac{\delta \varepsilon}{\varepsilon} = \frac{L_{p}^{2} L_{CS}^{2}}{2\pi \varepsilon^{2} I_{A}^{2} \sigma_{p}^{6}} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-J/\varepsilon} \left( 1 - e^{-\frac{J}{\sigma_{e}}} \cos^{2} \varphi - \frac{J}{\sigma_{p}} \cos^{2} \varphi \right)^{2} d\varphi dJ$$

(19)

Introducing a dimensionless variable $j \equiv J/\varepsilon$ we can rewrite Eq. (19) in a little bit more symmetric form:

$$\frac{\delta \varepsilon}{\varepsilon} = \frac{L_{p}^{2} L_{CS}^{2}}{2\pi \varepsilon^{2} I_{A}^{2} \sigma_{p}^{6}} \Xi(\sigma_{e}/\sigma_{p})$$

$$\Xi(d) = \int_{0}^{\infty} \int_{0}^{2\pi} e^{-j} \left( 1 - e^{-d^{2}j \cos^{2} \varphi} - d^{2}j \cos^{2} \varphi \right)^{2} d\varphi dj$$

(20)

Figure 1: An overlap function $\Xi$.

Notice that the integral $\Xi$ is simply a function of the ratio of the two beam sizes ($\sigma_{e}/\sigma_{p}$) in the cooling section. The plot in Fig. 1 demonstrates that this dependence is rather strong.

For the optimized parameters $\delta \varepsilon/\varepsilon \approx 6 \cdot 10^{-3}$. The effect of the proton-electron space charge kick on a single pass through the CS is small. This justifies our approach of small nonlinear excitations.
For the original design parameters we would have $\delta \varepsilon / \varepsilon \approx 1.6$, which would be a very strong effect on the emittance in a single pass through the cooling section.

6 Beam dynamics in presence of p-e focusing

Let us now consider the motion of a single electron in the storage ring. We will follow the simulations’ recipe given at the end of the Section 4. Figure 2 shows a phase space trajectory for a particle with $J = \varepsilon$ tracked in the storage ring for 67k turns (or $\approx 0.1$ s). For clarity, we also show a phase space trajectory of a particle with the same amplitude for the case of an undisturbed motion.

Figure 2: Phase space trajectory of an electron in the presence of the proton-electron focusing in the CS (red trace) and for the undisturbed motion (blue trace).

We can already see from Fig. 2 that for the optimized parameters the p-e focusing in the cooling section must not introduce any noticeable emittance growth. Tracking an ensemble of 10k particles for 20k turns, which is equivalent to $\approx 30$ ms, confirms this conclusion (see Fig. 3).

7 Conclusion

We studied the effect of the proton-electron space charge focusing in the cooling section on the e-beam emittance in the EIC ring cooler.
Figure 3: An ensemble of $10^4$ particles tracked for 20k turns in the electron ring in the presence of the p-e focusing in the CS.

We conclude that for the optimized parameters the emittance growth associated with this effect is negligible.

8 Acknowledgments

We are grateful to Valeri Lebedev for inspiring these studies.

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References

