Third Integer Resonant Extraction: Ripple Compensation and Bunched Beams

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In this note, I describe the basic slow extraction process based on a driven resonance, slow extraction spill structure and correction, extracting bunched beams using resonant extraction, and how to extract a burst of bunched beams using resonant extraction.

Contents

I. INTRODUCTION

Given,

$$\bar{S} = \frac{L_S}{B\rho}\left(\frac{\partial^2 B_S}{\partial x^2}\right)_0 \tag{1}$$

the classical Hamiltonian in the presense of a sextupole is,

$$H_0(x, x', s) = \frac{1}{2}(x'^2 + Kx^2) + \frac{\bar{S}}{6}x^3, \tag{2}$$

where K is the linear focusing from lattice quadrupoles. Nominally, we rewrite this, expanding into the set of azimuthal harmonics and transformed (Floquet) to action angle variables (J, ϕ, θ) [1].

$$H_1(\phi, J, \theta) = \nu \cdot J + (2J)^{3/2}\sum_n [G_{3,n}\cos(3\phi - n\theta + \xi_{3,n}) + 3G_{1,n}\cos(\phi - n\theta + \xi_{1,n})] \tag{3}$$

where ν is the horizontal tune, $G_{k,n}$ and $\xi_{k,n}$ are the modules and phases, respectively, of the complex azimuthal harmonics $\hat{G}_{k,n}$ of the sextupole field in the ring:

$$\hat{G}_{k,n} = G_{k,n} \cdot e^{i\xi_{k,n}} = \frac{1}{48\pi} \oint \bar{S}(s)\beta(s)^{3/2} e^{i[k\chi(s)-(k\nu-n)\theta(s)]} ds \tag{4}$$

with $k = 1$ or 3 and $\chi(s) = \int \frac{ds}{\beta(s)}$, $\theta = \frac{2\pi s}{L_{Ring}}$, and $\beta(s)$ is the horizontal beta function. Equation 3 is a summation over an infinite number of oscillating harmonics. At slow extraction time, though, the betatron tune is brought very close to a strong resonance, so there is a harmonic number N, such that a detune value

$$\delta = (\nu - N/3) \tag{5}$$

is very small. The harmonic number, N, is the only one that is persistent over many turns, and all others can be ignored. So,

$$H_1(\phi, J) = \nu \cdot J + (2J)^{3/2} G_{3,N} \cos(3\Phi + \xi_{3,N}), \tag{6}$$

where $\Phi = \phi - \frac{N}{3}\theta$.

Particles are confined within a stable area of phase space, as long as δ is not too small. As δ is made smaller, particles motion become unstable and their trajectory amplitudes grow nonlinearly. The maximum stable area is,

$$2 \cdot J_* = \left(\frac{\delta}{3G_{3,N}}\right)^2. \tag{7}$$

4 particles tracked through 13/3 resonance in Booster. D3 location.
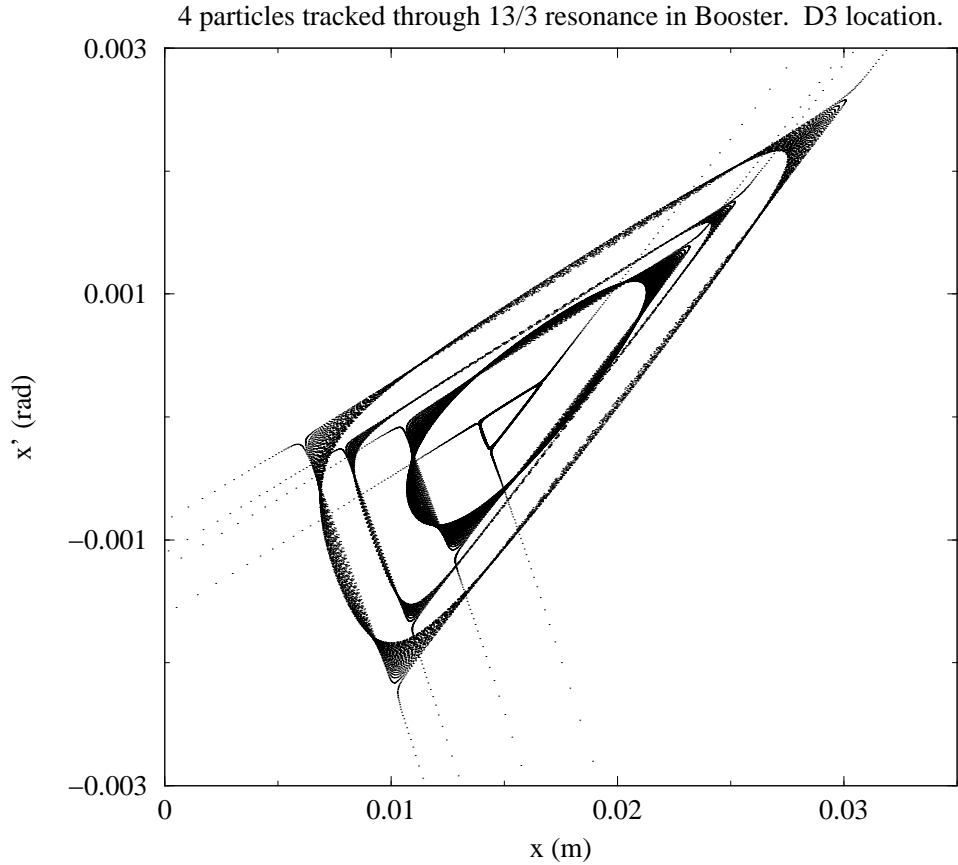
FIG. 1: Four particles starting out inside the stable region, then passing into nonlinear growth as betatron tune is decreased, driving them into the resonance. This example is for the AGS Booster D3 phase space.

This description of slow extraction is useful for understanding the harmonics sextupoles generate, especially if we want to study higher order effects. However, to understand the basic extraction process, we can go to a simplier

description, keeping in mind that the Hamiltonians, Eq. 2 and 6, are essentially desribing the same process. Redefining the sextupole strength as,

$$S = \frac{1}{2}\beta^{3/2}\bar{S},$$

(8)

then, in terms of tune and sextupole strength parameters, the size of the stable area can be written as [4],

$$E_s = A_s^2\pi = \frac{48\sqrt{3}\pi^2}{S^2}\left(\nu_s - \frac{N}{3}\right)^2$$

(9)

This is the classic equation, written in terms of tune and sextupole strength, that describes the stable area triangle. For this discussion, since E_s is the stable area of the separatrix, ν_s has a special meaning. Only a small fraction of the particles in the internal beam are close to resonance. The small band of particles in resonance are from $Q_s = N/3$ to $Q_s = \nu_s$, with corresponding amplitudes from $A_s = 0$ to $A_s = A_{max}$, where Q_s is the single particle betatron tune, A_s is the single particle amplitude, and A_{max} is the maximum particle amplitude in the distribution of particles close enough to be in resonance. We can think of A_{max} as a particle that is at 99% of the full internal beam emittance, for example. Beyond this small fraction of particles, the internal beam is unaware of the resonance, although the entire beam is slowly going to be driven into the resonance over a long time period. These principles can be seen in Figure 1. In this figure the particles begin inside the stable region following mostly elliptical trajectories in phase space. As the betatron tune (Q) is decreased, the particles step into the unstable region and their amplitudes begin to grow nonlinearly. Eventually, these particles will arrive at a septum device to either be extracted or they may hit the septum, with a probability proportional to the particle step size at the septum location. What is not evident from this picture is that the betatron tune value where particles enter into nonlinear growth is higher for the larger amplitude particles and smaller for the lower amplitude particles. This is what Eq. 9 describes. This will be discussed more, below.

The beam is dynamically driven into the resonance by moving the betatron tune closer to $N/3$ in the accelerator. The rate that this is done determines how long the extracted beam pulse will be. But the distribution of the particles in tune space is not uniform. So the rate will follow some error-like function to transform the Gaussian-like beam distribution into a uniform distribution in time.

A. Extraction methods: debunched beam with betatron tune ramp

Consider that we have a synchrotron setup for slow extraction using a single set of quadrupoles that will be ramped to drive the particles into the resonance as described above. In this case the beam is debunched (so the momenta of the particles are unchanged during extraction), the lattice chromaticity is non-zero, negative, and constant, and the main dipole field is held constant. In addition, the lattice dispersion in the sextupoles is non-zero. Recall that chromaticity is defined as,

$$\xi = \frac{dQ}{Q}\bigg/\frac{dp}{p}$$

(10)

The only thing that is changing is the current in the betatron tune quadrupoles and thus the betatron tune of the machine. In this situation, the particle distribution has a tune spread that correlates linearly with the momentum distribution through the chromaticity. If we are driving the tune down, from above the N/3 resonance, then particles with a low tune, and therefore a higher momentum given a non-zero negative chromaticity, will be extracted first. Particles with high tunes will have lower momentum and are extracted last. Equation 9 applies to particles that are on-momentum and on the equalibrium orbit, passing through the center of the sextupoles. It also applies if the dispersion in the sextupoles is zero. For non-zero dispersion, off-momentum particle trajectories will be shifted inside the sextupoles causing a tune shift. So the particle trajectories in phase space are shifted over by $\Delta X = D_n dp/p$ and the size of the stable triangle is changed due to the chromatic effect of the sextupoles (D_n is the dispersion in a sextupole at location n).

$$E_s = A_s^2\pi = \frac{48\sqrt{3}\pi^2}{S^2}\left(\nu_s - \frac{N}{3} - \frac{3}{6\pi}SD_n\frac{dp}{p}\right)^2$$

(11)

This is a consequence of having non-zero dispersion in the sextupoles, where a momentum displacement translates into a position displacement, causing a tune shift,

$$Q' = \left(\frac{dQ}{dp/p}\right) = \frac{-1}{4\pi}\beta\bar{S}D$$
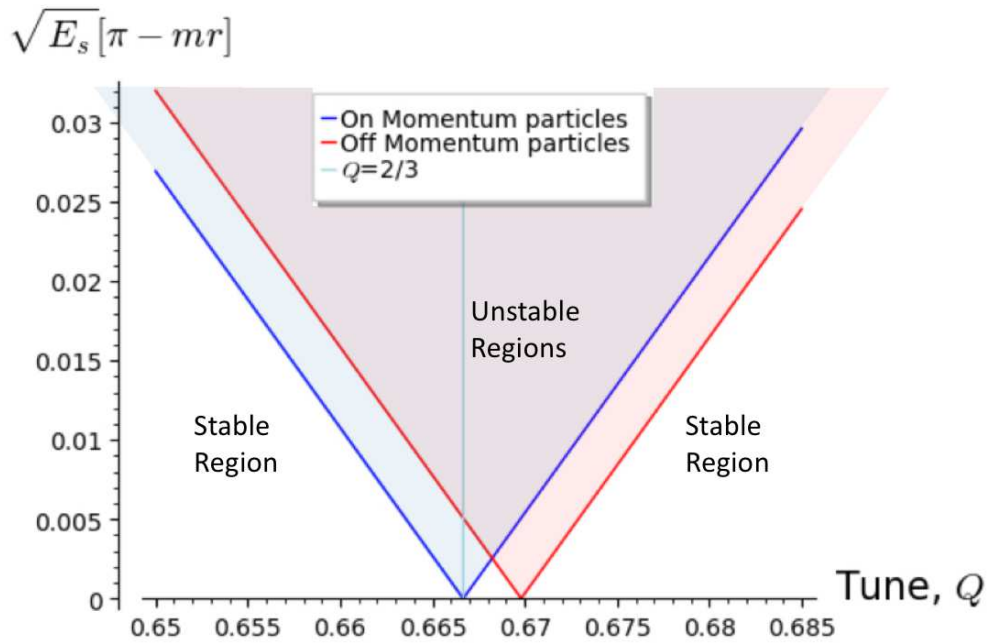
(12)

FIG. 2: Particle amplitude ($\sqrt{E_S}$) vs betatron tune. Two cases are shown. First, is the on-momentum case (Eq. 9) where particles enter into unstable motion as their tune approaches $N/3$ and second, the off-momentum case (Eq. 11) where off-momentum particles enter into unstable motion above $N/3$.

The situation is shown in figure 2, where we have plotted values of $\sqrt{E_s}$ for on-momentum and off-momentum particles. In this figure, stable particle motion is in the regions where the tunes are above and below the pair of lines. Unstable (resonant) motion takes place between each pair of lines. If the non-resonant beam has a tune distribution above $N/3$ and is driven down by shifting the lattice betatron tune down, we can see that off-momentum particles will encounter the resonance condition before on-momentum particles. The off-momentum particles will also travel a longer distance to get to the extraction septum, since the trajectories were also shifted. This is a problem, since it results in higher losses and a larger extracted beam emittance. As seen in the figure, the effect is quite large. Being off-axis in the sextupoles leads to significant distortion in the overlap of the separatrices. Actually, this is not quite a correct statement. What is important is that all particles encounter the resonance at the same tune, for the same particle amplitude, independent of momentum. The spread shown in this figure is really a consequence of the beam not being radially shifted to match the momentum spread of the beam. So this simple method of extraction is inefficient and not practical. One solution is to design the lattice to have zero dispersion at the sextupoles. This is not always possible, especially for an existing machine being repurposed to perform slow extraction.

B. Extraction methods: debunched beam chromatic extraction

To get around this problem we would like to keep the beam centered, or close to centered, in the sextupoles as it is being extracted. The lattice has a non-zero chromaticity. The dispersion in the sextupoles is non-zero. And, the particle momenta are frozen (RF voltages are zero). We will want to place the beam such that particles are nearly centered in the sextupoles as they encounter the resonance condition. This will maximize the size of the stable triangle, but more importantly, all phase space trajectories depend only on their amplitudes and not on their momentum and all particles will encounter the resonance at the same tune, independent of their momentum.

Going back to Equation 2, there are two things happening in a normal chromatic slow extraction. First, in order for the tune to change, quadrupole gradients $(dB/dx)_0$ need to be changed adiabatically. Since $K \propto (1/B\rho)(dB/dx)_0$, changing the quadrupole currents only changes the gradient, $(dB/dx)_0$. Second, the main dipole field needs to also be changed adiabatically, to track $B\rho$ with the lowering of the quadrupole gradients. We want to ramp K not the quadrupole gradient. This is the problem described above. It has the consequence that the radial location corresponding to the resonant condition will shift, if we only ramp the gradients. This will then move the resonance relative to the location of the extraction septa. The overlap of the seperatrices, as seen in Figure 1 will move, spreading

them over a larger range of X. One can imagine a benefit of this, if the seperatrices were shifted in such a way that they happened to line up on the lower edge (sepertrix arm). This could reduce the angular spread in the extracted beam. This is not the best way to establish such a condition, since at any moment in time, many momenta are being extracted simultaneously. A better way to establish such a condition is to adjust the dispersion in the septa locations to establish the so-called Hardt condition [2]. This is done by establishing the condition,

$$\frac{8\pi}{S}Q_o\xi + D_{sept}\cos\phi_1 - D'_{sept}\sin\phi_1 = 0 \tag{13}$$

where, D_{sept} is the dispersion at the septum magnet, ϕ_1 is the phase angle between the 'virtual' sextupole and the septum magnet, and ξ is the normalized chromaticity. The 'virtual' sextupole is defined as the location where $\hat{S}$ in Eq. 2 is defined given the actual resonant sextupole configuration in the lattice.

Ideally, we would want to ramp the main dipole field to keep the beam nearly centered in the sextupoles during the extraction process, by an amount that would shift the tune by,

$$d\nu_D = \frac{3}{6\pi}SD_n\frac{dp}{p} \tag{14}$$

The amount we need to change the field to keep the beam nearly centered in the sextupoles is then,

$$\Delta B = -\gamma_{tr}^2\frac{B}{R}D_n\frac{dp}{p} \tag{15}$$

In a machine with chromaticity, a given momentum range is being selected to be extracted first and then as the tune is shifted closer to $N/3$, the rest of the momenta are extracted in sequence. There will always be a range of $\delta = \xi Q_0(p_A - p_{res})/p_0$ that are entering the resonance condition, where p_A corresponds to the particle momentum at amplitude A and p_{res} corresponds to the momentum of a zero amplitude particle.

In machines like the AGS Booster and the AGS at BNL, the main dipole and quadrupole bus is used to ramp both the quadrupole gradients as well as the main dipole fields, in order to keep the separatrices well centered. So these machines ramp K and not just the quadrupole gradients. What this is doing is keeping the average momentum of the beam in the stable region at a fixed position, as particles stream out into the unstable region and are extracted.

In a machine, such as the JPARC main ring where the chromaticity is set to zero, we have the situation where all momenta share the same betatron tune value. There is no spread in tune. But there is a spread in momenta and if there is lattice dispersion that could cause the orbit in the sextupoles to shift, this will still change tune and will shift the particle trajectories. What is described by Eq. 11 remains true. However, there is now a direct linear correspondance between a particles betatron tune and a given amplitude for entering into resonance. The resonace condition is still linearly dependant on amplitude. In this case, at any moment in time only particles of a given amplitude and tune are being extracted. Since all particles have the same tune, but different amplitudes, large amplitude particles are extracted first and zero amplitude particles are extracted last. Since chromaticity is zero, there is no longer any correspondance between a radial position of the resonance and a particles tune. We can imagine that as chromaticity goes to zero, $dQ/dp = \xi Q_0/p_0 \to 0$. Since dp cannot be zero, this just means that a particles tune is no longer correlated with a given momentum or radius. For a constant field the momentum remains correlated with radius through the momentum compaction of the lattice, $dp/p = \gamma_{tr}^2 dR/R$. It remains important that the beam stay nearly centered in the sextupoles during extraction. Even better, the lattice should be designed with zero dispersion in the sextupoles. However, even with non-zero dispersion, since the tune spread is nearly zero in such a lattice, making the momentum spread as small as possible will help reduce the spreading of separatrices with momentum.

In the case of a zero chromaticity lattice, what is now relevant is transverse emittance. We can imagine that as the tune is shifted into the resonance, particles with large single particle emittance values will enter into non-linear motion first. So, we no longer see many seperatrices from a zero amplitude to some large amplitude being extracted simultaneously, but see only a single seperatrix and all particles with that amplitude will be getting extracted. Since emittance can correlate with a position - the projection of the particles in phase space at a moment in time will show only large amplitude particles will have large X. At a given moment in time they may have small X, but we will never see small amplitude particles at large X. If we turn on an orbit deformation and ramp the orbit so that over time the central angular position of the distrubtion moves, we can imagine lining up the seperatrices on a single edge and all particles will stream out with the same angular range. The one detail here is we also need to remove any position dependence on average momentum by reducing the dispersion at the septum location to zero. This is bascially how the dynamic bump in the JPARC slow extraction system works [3].
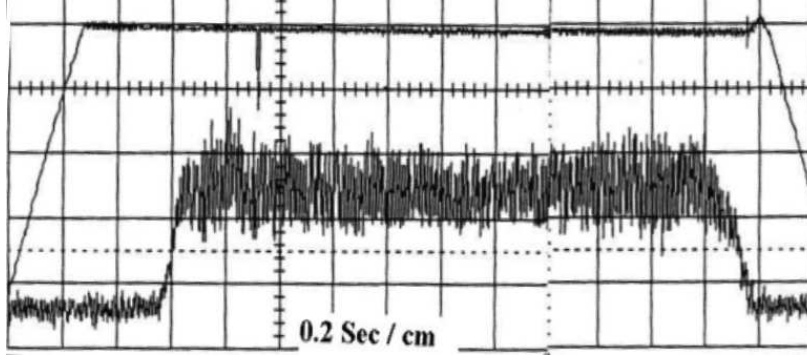
FIG. 3: Example of a spill from the AGS slow extraction of high intensity protons (during the 1990's.)

II.   RIPPLE AND SPILL STRUCTURE

'Spill' is the term used to describe the stream of particles being extracted over time, as beam is driven into the resonance. Spill structure is a consequence of ripple in power supplies causing small B-field, quadrupole gradient, and sextupole gradient variations that change the betatron tune and the resonant condition.

$$Q(t) = \frac{\partial Q}{\partial t} t + \sum_n \Delta Q_n \sin\left(\frac{n\pi}{6} t + \phi_n\right) \tag{16}$$

where the rate that particles are driven into the resonance is $\frac{\partial Q}{\partial t}$ and the variations due to 60 Hz harmonics on tune and B-field power supplies are in the second term. There are also variations in $S(s)$, which translate into variations in $G_{3,N}$.

We can look at this in terms of the rate of particles being extracted as,

$$\frac{dN}{dt} = \frac{dN}{dQ}\frac{dQ}{dt} = \Lambda_0\left(\frac{dQ_0(t)}{dt} + \frac{dQ_R}{dt}\right) \tag{17}$$

where the initial distribution of particles in tune is $\Lambda_0 = \frac{dN}{dQ}$ and the rate that the beam is driven into the resonance is $\frac{dQ_0(t)}{dt}$, where we have expressed this rate as a function of time, since it may not be just a simple linear change of tune in time. The variations due to ripple are, then,

$$\frac{dQ_R}{dt} = \sum_n (n\omega_0)\delta Q_{R,n}\cos(n\omega_o t + \phi_n) \tag{18}$$

This is still not an exact representation of the spill structure, since this only includes the variations due to betatron tune. A modulation of the orbit inside the sextupoles, for example, would cause trajectory shifts and tune shifts. Note that the 60 Hz term has been generalized into an angular frequency ω_0.

The variations in tune can be expressed in terms of the variations in K, as,

$$\Delta Q_K(t) = \frac{1}{4\pi}\sum_{i=1}^{N_K} \beta_i \cdot \Delta K_i(t), \tag{19}$$

where N_K is the number of quadrupole magnets and $\Delta K_i(t)$ and β_i are the gradient ripple and beta function at the given quadrupole.

The variations in the dipole field can move the centroid of the beam, which does not change the tune, directly, but does modulate the $B\rho$, and if the accelerating rf is in a closed loop mode, may modulate the energy slightly, causing tune variations through the chromaticity. The orbit variations will cause motion in the sextupoles, which will cause a tune variation as,

$$\Delta Q_B(t) = \frac{1}{8\pi\sin(\pi Q)}\sum_{i=1}^{N_B}\sum_{j=1}^{N_S}\Delta\theta_{B,i}(t)\cdot S_j\cdot\beta_j^{3/2}\beta_i^{1/2}\cdot\cos(-\pi Q + |\phi_i - \phi_j|), \tag{20}$$

The variations in bend angle and focusing are proportional to the actual variations in current in the magnet coils. To first order these are,

$$\frac{\Delta\theta_B}{\theta_B} = \delta I_B, \quad \frac{\Delta K}{K} = \delta I_Q \tag{21}$$

The amplitudes in these variations will be attenuated by the eddy currents in the vacuum chambers and magnet coils [6]. For a round vacuum chamber,

$$\Delta\theta_B = \theta\Delta I \sum_{n=1} \frac{\sin\left(n\omega_0 t - \tan^{-1}(n\omega_0\tau)\right)}{\sqrt{(1 + n^2\omega_0^2\tau^2}} \tag{22}$$

where,

$$\tau = \frac{1}{2}\mu_0\sigma_c b \cdot d \tag{23}$$

where b is the radius of the vacuum chamber, d is the wall thickness, and σ_c is the conductivity of the chamber material.

A simplified way of looking at this is to say that the eddy currents induced by the field ripple will attenuate the higher frequency harmonics seen by the beam inside the vacuum chamber by some factor, as

$$\Delta\theta_B = \theta_B\left(\delta I_B - C_{eddy} \cdot \frac{d\delta I_B}{dt}\right), \tag{24}$$

where C_{eddy} is a function of the vacuum chamber shape and materials and grows larger with frequency. This factor could be measured by inducing harmonics at different frequencies and measuring the response on the spill.

III.   CORRECTING SPILL STRUCTURE

A.   Chromatic Extraction

The simpliest way to correct the spill structure, after making all other possible improvements on reducing the power supply ripple, is to modulate the betatron tune using the measured harmonics.

$$Q(t) = \frac{\partial Q(t)}{\partial t}t + \sum_n \Delta Q_n \sin\left(\frac{n\pi}{6}t + \phi_n\right) - \sum_n \Delta Q_{c,n} \sin\left(\frac{n\pi}{6}t + \phi_{c,n}\right) \tag{25}$$

where $dQ(t)/dt$ is the tune function shaped to match the distribution of particles in tune to give a uniform spill versus time, ΔQ_n are the amplitude of the tune variations for harmonic n , and $\Delta Q_{c,n}$ are the required tune corrections needed to cancel those variations. To measure the values of $\Delta Q_{c,n}$ and $\phi_{c,n}$ is the challenge. What is typically done is the average amplitude of a given harmonic multiplied by some heuristic gain, perhaps derived from a Fourier spectrum of the previous spill(s), is applied with a phase correction tuned into the feedback system. A realtime feedforward system could be used, in which the spill signal is combined, phase shifted, into a tune quadrupole system with gains and offsets applied to each harmonic. This turns out to be much more difficult to implement since it requires a realtime filter to measure each harmonic amplitude, as well as to contend with noise and drifts.

Figure 4 shows a typical, simplified, feedback system employed for slow extraction harmonic correction. The reference, $r(t)$, is basically zero, or no harmonics. There may be some function, $f(t)$, which is meant to null out any offsets. Noise, δ_n will enter the system affecting the sensitivity. This system would control a high frequency quadrupole to modulate out the spill harmonics. The measurement, $M(s)$, is the critical piece, which could be the spill signal, AC coupled (to remove DC part) and normalized by beam intensity. An improved version might use feedforward of the power supply harmonics to reduce the range of correction needed by the spill feedback system. In principle, $G(s)$, δ_n, and $M(s)$, are time-independent, but will have a frequency response.

There are other ways to measure the field ripple. Monitoring coils in a reference magnets can be used to directly measure the ripple, however, the ripple is usually extremely small; on the order of 1 part in 10^5 or less. So a magnet with 1000 amps may have only a ripple of 0.01 amp. In the AGS reference magnet the reference coil samples the field to measure about 5 volts per 3 T/sec. For a 0.01 amp field ripple, the measured voltage would be as small as 0.15 mV. To get good fidelity in this measurement, the voltage resolution would need to be 1 μV or less. Since this signal
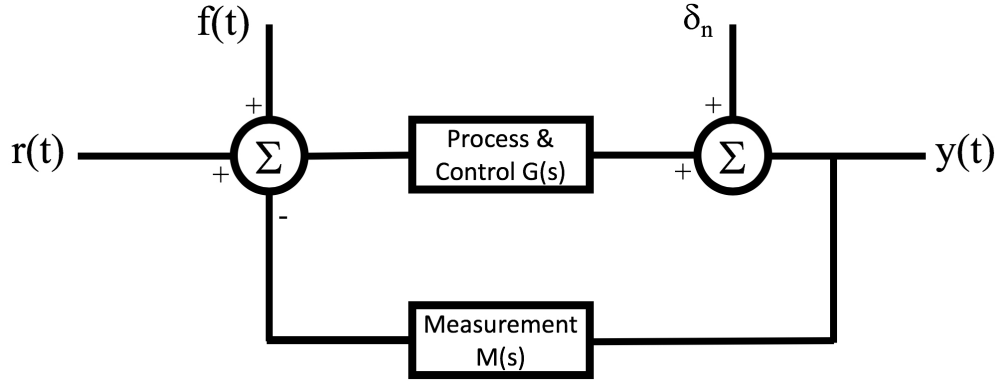
FIG. 4: Typical feedback system for spill harmonic controls (simplified greatly). The Process model, $G(s)$, is basically the function needed to make $M(s)$ to equal the desired $r(t)$, which for a harmonic correction is zero. The transfer function is then very sensitive to the measurement $M(s)$. The measurement, in this case, could be the actual spill signal, decomposed into its harmonics. $G(s)$ would modulate slightly, once a function was found that gave the optimal correction.

is measured inside the vacuum chamber (as the reference magnet is configured with a beam pipe), it could be directly fed into a high frequency quadrupole, as long as the signal propogation time is small compared to the highest 60 Hz harmonics (at or below 1440 Hz), it could be phase shifted and a small gain applied to correct the spill. This will still be imperfect, since the AC Quadrupole would have to have a flat frequency response up to the highest harmonics and it's vacuum chamber (which might be ceramic with rf strips to pass image currents), may still attenuate the higher frequencies.

In the case of a feedback loop, the only information available to the control module is the current measured spill state plus the current set of corrections needed to match the desired reference. Another approach would be to use a reinforcement learning algorithm, to build memory of the corrections and past spills to allow an algorithm to make more informed decisions on how to adjust the current correction. Figure 5, shows a classic reinforcement learning approach.
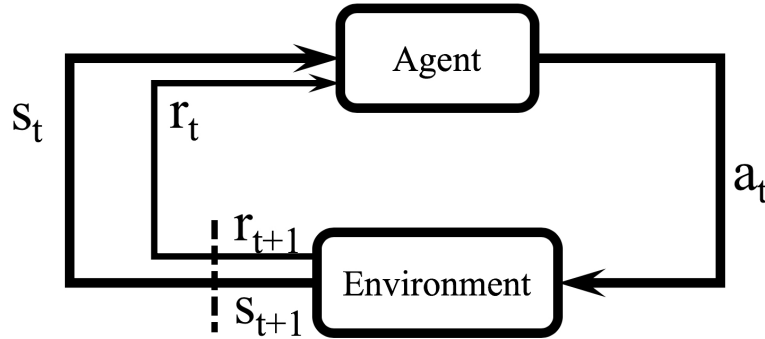
FIG. 5: Basic model for a reinforcement learning system.

Let's say we are going to keep a table of values, called $Q(s_t, a_t)$, which we learn over time given a learning rate, α, a discount factor, γ, and an estimate of the future value, $maxQ(s', a')$. Every iteration, or time step, we lookup up the value of Q in our table. Depending on how well we estimated the future value, we recieve a reward, r_t. This quality of the state-action of the system is updated at every time step, as

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t(s_t, a_t)(r_t + \gamma maxQ_t(s', a') - Q_t(s_t, a_t)) \tag{26}$$

where this states that the quality yielded from being at state s, performing action a is the immediate reward plus the highest Q-value possible from the next state. A deep learning version would just replace the table of values with a

neural network.

To use this to correct our spill harmonics, we would replace our process model with the Q-learning agent. We effectively get a feedback system with deep memory that weighs the measured response based on how well we can make the measurement of all harmonics to be zero.

B. Extraction with zero Chromaticity

At J-PARC, they took a slighly different approach, and attempt to predict the level of ripple in the betatron tune, based on the measured (real-time) ripple on the power supplies. The system can even try to compensate for a predicted eddy current attenuation factor. However, given they operate with zero chromaticity, this is a sensitive process. It is possible to improve the spill structure, but it is difficult to maintain stability.

Since emittance is more relevent for extraction with zero chromaticity, the problem can be re-expressed in term of amplitude control instead of tune control.

$$\frac{dN}{dt} = \frac{dN}{dA}\frac{dA}{dt} = \Lambda_A \left(\frac{dA_r}{dt} - \frac{dA_c}{dt}\right) \tag{27}$$

The idea here is, ripple in tune translates into a modulation of the amplitude being extracted at any given moment. If we imagine the tune not being ramped, but remaining fixed, we would still see the particles driving into and out of the resonance from the tune ripple dependance on the amplitude. So we return to Eq. 9, and rewite it as,

$$A = F_s \left(\nu - \frac{N}{3}\right) \tag{28}$$

where F_s is all the fixed terms in Eq. 9. Then,

$$\frac{dA_r}{dt} = F_s \frac{d\nu_r}{dt} \tag{29}$$

We are simply re-expressing the effect of the variations in tune as variations in amplitude, relative the resonance condition.

A correction then is to modulate the amplitude of the particles, instead of the tune, as they are being extracted. In which case we need,

$$A(t) = F_s \frac{dQ}{dt}T + F_s \sum_n \Delta Q_n \sin\left(\frac{n\pi}{6}t + \phi_n\right) - \sum_m \Delta A_m \sin\left(\frac{n\pi}{6}t + \phi_m\right) \tag{30}$$

Experimentally obtaining the values of ΔA_m and ϕ_m is now the problem. But these translate directly into a modulation of particle amplitudes, which can be thought of as either a heating or cooling of the beam as it is extracting. However, it may be possible to modulate an ac-dipole to compensate, since the amplitude modulations are not actually increasing the emittance, just modulating the apparent amplitude seen by the resonance due to modulations in the tune.

IV.   SPILL MANIPULATIONS AND CONTROL

The addition of an AC Quadrupole, with response up to 10 kHz, enables the spill structure to be reduced in significant ways. The strength of such a magnet would need to compensate for a ± 0.05 amp modulation on a 5000 amp signal from 240, 2 m long magnets. This means a ripple power of roughly $24kW$ needs to be compensated. Roughly speaking the magnet would have to modulate around ± 50 amp up to 1000 Hz or so. This implies a voltage around ± 500 volts (giving $25kW$ compensation).

To visualize how such a system works, recall Eq. 7 and Eq. 9. A particle is stable if its single particle emittance is smaller than the area of the separatrix. Inverting Eq. 9 and solving with respect to δ, shows the tune interval for the particles that will be unstable. In particular, it shows the amplitude dependance.

$$|\delta| < A_s \sqrt{\frac{1}{48\sqrt{3}\pi}|S|} \tag{31}$$

which is written a little awkwardly on purpose. In terms of absolute tune, where $\delta = \nu_{particle} - \nu_{resonance}$ then,

$$\nu_{res} - \sqrt{\frac{1}{48\sqrt{3}\pi}|S|}A_s < \nu_{part} < \nu_{res} + \sqrt{\frac{1}{48\sqrt{3}\pi}|S|}A_s \tag{32}$$

This gives two straight lines with equal and opposite slopes (e.g, see figure 2), in a space where we plot amplitude, A_s, versus betatron tune. The region between the two lines is the unstable region and the regions below and above that region, in tune, are regions of stable motion.
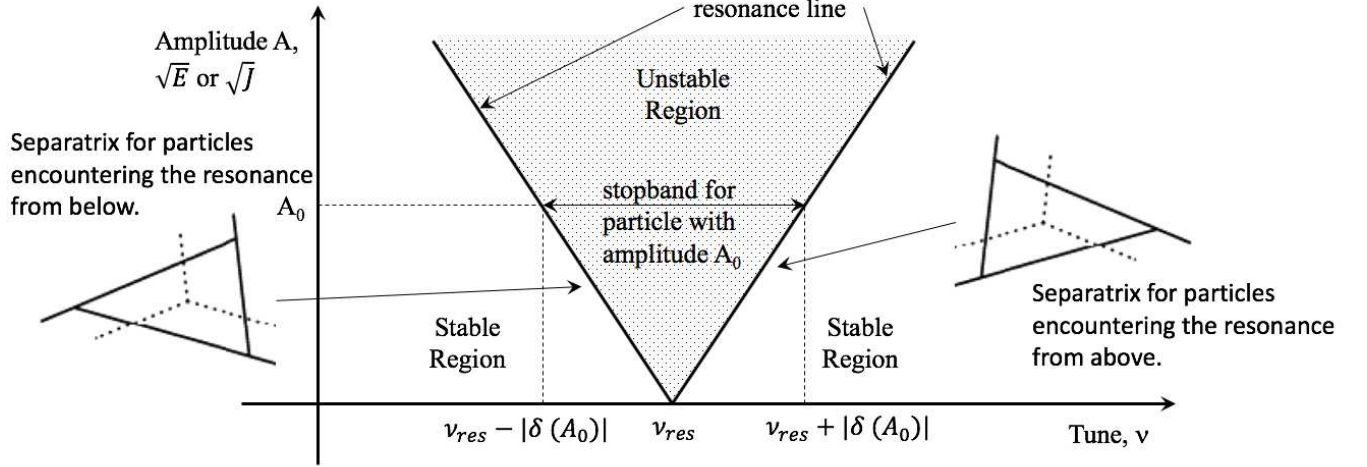

FIG. 6: Steinbach diagram for a third order resonance stopband. This diagram shows the particle amplitude dependance on the resonance as a function of the betatron tune for a given particle. In this case a particle with a tune of $\nu_{res} + |\delta(A_0)|$, with a corresponding amplitude of A_0, will encounter the third order resonance and enter into nonlinear/unstable motion.

For particles with different amplitudes, $A(\delta)$, there are corresponding tune distances between the point the particle enters the unstable region and the central resonant tune. So, ν_{res} really is the resonant tune for zero amplitude particles. All non-zero amplitude particles enter into resonance at a slightly different tune. Rewriting, to define the resonance line, we see,

$$A = \sqrt{48\sqrt{3}\pi}\left|\frac{\delta}{S}\right| \tag{33}$$

From this diagram it is easy to see that there are three basic ways to drive particles into the resonance.

1. Widening the stopband by increasing S

2. Driving particles into the resonance by changing their tunes

(a) dedicated quadrupole(s) that ramp, matching $dN/d\nu$ distribution to deliver uniform time distribution of beam

(b) for a machine with non-zero chromaticity, ramping the main tune bus and the main dipole bus together, for 'chromatic' extraction (most common method)

(c) for a machine with non-zero chromaticity, ramping the beam energy using either a betatron core (as at some medical accelerators) or shifting RF frequency in open loop mode, for bunched beam slow extraction

(d) tune can be shifted by driving dipoles alone, with fixed quadrupoles, while keeping the RF on in closed loop mode (another way to change energy - don't know of anyone who does this)

3. increasing the particle amplitudes, to drive them to the critical $A(\delta)$ point.

Extracting the particles by increasing the stopband width, consider two particles with the same tune, but different amplitudes. As S is increased, the resonance line first encounters the higher amplitude particle. At this moment the particle is moving on a stable triangle and will soon go outwards on one of the separtrix arms to be extracted. As we continue to increase S, the second particle encounters the resonance line and it gets extracted (later in time). The first particle will have been extracted from a larger stable triangle than the second particle. So if the particle tunes are fixed, this method will extract larger amplitude particles first. Since S can only go to finite values, low amplitude particles can never be extracted. One can imagine combining this meathod with an AC-dipole, where the dipole drives the particle amplitude up while the sextupoles extract higher amplitude particles. In this case, all particles can be extracted, at a fixed tune and within a small range of amplitudes.

Extracting particles by shifting tune is the most common method employed. Again, considering the two particles, the large amplitude particle will encounter the resonance line first, being extracted first. The smaller amplitude particle will encounter the resonance later and also be extracted. The stable triangles for the two particles will be the same size as in the previous example, since they are determined only from the initial particle amplitude. However, all particles will encounter the resonance, in principle, as the betatron tune only needs to be shifted over some finite range. It is worth noting that if there is power supply ripple, the particles can slip past the resonance and pass into the other stable region, never to be extracted. This is typically referred to as the 'remaining particle' problem.

When extracting particles using the third method, particles with the same tunes will be driven in amplitude until they encounter the resonance. So, all particles of the same tune have the same stable triangle size when they are extracted. This would be a way of creating a zero emittance extracted beam (assuming ripple compensation), given zero chromaticity. Naturally, having a smaller emittance beam prior to extraction makes this easier, technically, since the strength of the AC dipole or stripline would have to be proportional to the internal emittance.

A. Transit Time: static case

While particles are in stable motion, inside a stable triangle in phase space, their trajectories slowly get distorted as they get closer to the resonance condition given their amplitude, as the particles tune is shifted. As they move closer to the resonant tune, they spend more time near the unstable fixed points but can still fall anywhere along the edge of the triangle. At some point the particle falls outside the stable triangle and moves along the edges and soon lands on the separatrix arms. Keep in mind that each turn the particle lands on the consecutive edge or arm. Figure 4 describes the process.

Figure 8 shows the definitions of some of the parameters used to describe the transit time. Between points P_2 and P_3 a particle that has fallen out of the stable triangle will move along that edge (every third turn, spending the other two turns along the other two edges) until it arrives very near the unstable fixed point at P_3. Here we are looking at the triangle in normalized phase space, where $x = \sqrt{\beta_x}X$ and $dX' = dx'/\sqrt{\beta_x}$. In addition, the coordinate system has been shifted to place the point P_3 at $(0,0)$. In this coordinate system, and given the pararameters λ, Λ, and n, the point P_2 is located at $(0, -2\sqrt{3}h)$ and P_1 is located at $(3h, -\sqrt{3}h)$. The definitions are,

$$h = \frac{2}{3}\frac{\epsilon}{S} \tag{34}$$

where S is as defined in Eq. 8 and $\epsilon = 6\pi\delta$. The area inside the triangle is $Area = 3\sqrt{3}h^2$.

The Hamiltonian, for this shifted separatrix, is,

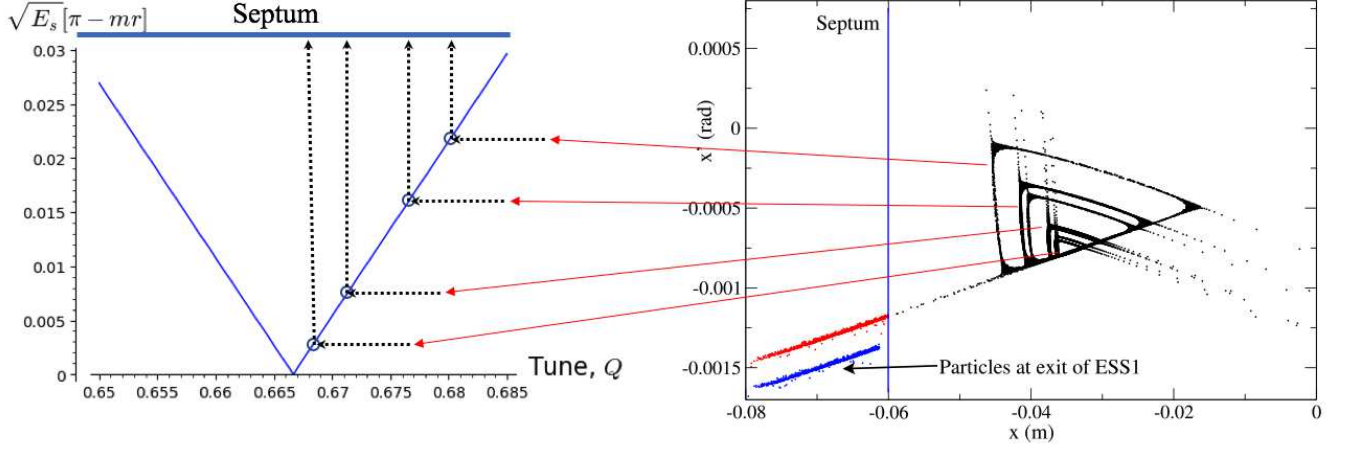$$H_{shifted} = \frac{S}{4}\left(4h^3 + 6hX^2 + 6\sqrt{3}hXX' + 3XX'^2 - X^3\right) \tag{35}$$

FIG. 7: This shows slow extraction as done at JPARC on the right, where a dynamic bump has aligned all the separatices at the lower edge, to create a very narrow angular distribution at the exit of the electrostatic septum (ESS1). On the left we illustrate how each particle, from four of the separatrices, is driven down in tune into the resonance. JPARC actually extracts on the 67/3 resonance, so the x-axis is not quite correct, but the diagram is just meant to show the principle and is not to scale. Once particles encounter the resonance they 'hover' near the border of the stable and unstable regions. This time spent locally at a roughly fixed amplitude and tune corresponds to the particles motion along the edge of the separatrix and time spent slowly moving past the three unstable fixed points, as seen in the right figure. Once the particle is spiralling outward, on the separatrix arms, it quickly moves up in amplitude, at mostly a fixed tune, until it encounters the septum. There are two relevent times, the time spent traveling along the edge of the separatrix and time spent spiralling away from the unstable fixed point. Clearly smaller amplitude particles take longer to reach the septum. But that also means they have a larger step size when they arrive at the septum and so have a lower probability of hitting it.

We define a time interval, $d\tau = 3T_{rev}$, in unitless terms of number of turns, T_{rev}. Then,

$$\frac{dX}{d\tau} = \frac{\partial H_{shifted}}{\partial X'} = \frac{S}{4}\left(6\sqrt{3}hX + 6XX'\right) \tag{36}$$

and,

$$\frac{dX'}{d\tau} = -\frac{\partial H_{shifted}}{\partial X} = -\frac{S}{4}\left(12hX + 6\sqrt{3}hX' + 3X'^2 - 3X^2\right) \tag{37}$$

Then trajectories can be calculated from initial conditions, (X_0, X'_0), and substituted into the shifted hamiltonian.

$$\frac{S}{4}\left(4h^3 + 6hX_0^2 + 6\sqrt{3}hX_0X'_0 + 3X_0X_0'^2 - X_0^3\right) = \frac{S}{4}\left(4h^3 + 6hX^2 + 6\sqrt{3}hXX' + 3XX'^2 - X^3\right) \tag{38}$$

Since very near P_3, $(|X|, |X'| \ll h)$ (the third order terms in X and X' can be neglected), we can drop all terms that don't have h, so that this can be approximated (simplified) to,

$$6hX_0^2 + 6\sqrt{3}hX_0X'_0 = 6hX^2 + 6\sqrt{3}hXX', \tag{39}$$

$$X' = \frac{X_0^2 + \sqrt{3}X_0X'_0 - X^2}{\sqrt{3}X} \tag{40}$$

Substituting back into Eq. 36, then

$$\frac{dX}{d\tau} = \frac{S}{4}\left(6\sqrt{3}hX + 6X\left[\frac{X_0^2 + \sqrt{3}X_0X'_0 - X^2}{\sqrt{3}X}\right]\right) \tag{41}$$

$$\frac{dX}{d\tau} = \frac{S}{4}\left(6\sqrt{3}hX + \frac{6}{\sqrt{3}}X_0^2 + 6X_0X'_0 - \frac{6}{\sqrt{3}}X^2\right) \tag{42}$$
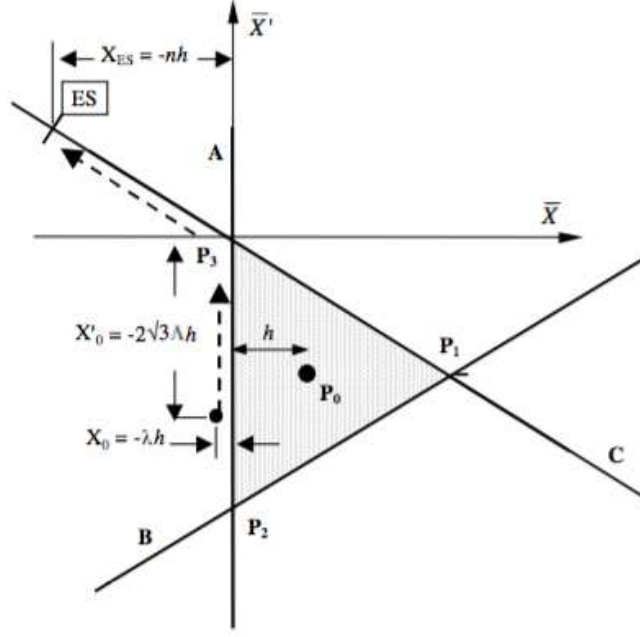
FIG. 8: Definition of λ, Λ, and n (from reference [8].)

We define the coordinates (X_0, X_0') that correspond to the point where a particle steps from the stable region to the unstable region. These are the initial conditions for Eq. 38. Equation 42 is valid as long as the particle is very near P_3, but it is also valid for particles traveling along the edge of the separatrix. Since a particle asymptotically approaches the separatix from within the stable region of the triangle, at the separatrix $X' \Rightarrow -\frac{1}{\sqrt{3}}X$, then from Eq. 36, we see that the third order terms all cancel out. So along the separatirx lines and near P_3, Eq. 42 remains valid.

The transit time is the sum of the time a particle spends along the edge of the separatrix (starting at (X_0, X_0')) to P_3 and then from P_3 to the septum (labeled X_{ES}).

$$T_{TT} = \frac{2}{\sqrt{3}S} \int\limits_{X_0}^{X_{ES}} \frac{1}{-X^2 + 3hX + X_0^2 + \sqrt{3}X_0 X_0'} dX \tag{43}$$

This can be evaulated using the standard rational integral in the form [5], as long as $4ac - b^2 < 0$,

$$\int \frac{1}{ax^2 + bx + c} dx = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + C \tag{44}$$

This then leads to the very messy looking total transit time formula.

$$T_{TT} = \frac{2}{\sqrt{3}S} \frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0 X_0')}} \left[\ln \left| \frac{-2X + 3h - \sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0 X_0')}}{-2X + 3h + \sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0 X_0')}} \right| \right]_{X_0}^{X_{ES}} \tag{45}$$

As seen in figure 8, we define the three parameters $\{n, \lambda, \Lambda\}$ as $X_{ES} = -nh$, $X_0 = -\lambda h$, and $X_0' = -2\sqrt{3}\Lambda h$. When a particle lands near P_3, Λ is small, so,

$$T_{TT} \approx \frac{1}{\sqrt{3}\epsilon} \ln \left| \frac{n}{n+3} \frac{3}{\lambda} \right| \tag{46}$$

A particle that lands along the edge of the separatrix,

$$T_{TT} \approx \frac{1}{\sqrt{3}\epsilon} \ln \left| \frac{1}{(1-\Lambda)^2} \frac{n}{n+3} \frac{3}{\lambda} \right| \tag{47}$$

B.   Transit Time: dynamic case

The previous analysis considered a slow extraction process given fixed parameters. However, to actually extract beam various parameters need to change continuously.

There are a few concepts that need to be explained, since this analysis can be complex and a little confusing. First, most of the time we are speaking about the behavior of single particles. A single particle has a fixed amplitude in phase space, so from that particles point of view, when it is far from resonance, the separatrix is large and far away. As the focusing of the accelerator changes, as seen by that particle, and its tune moves closer to resonance, the separatrix triangle shrinks and the particles motion in phase space becomes distorted, with more time being spent near the unstable fixed points. Eventually, the particles tune will shift to where it will now land outside of the stable triangle and begins its journey in the unstable region. When it steps from the stable region out to the unstable region, its position can be anywhere around the separatrix triangle.

It can take a very long time for a particle to get to the point where it crosses into the unstable region. If the particle tune spread is on the order of 0.1 and the beam spill is on the order of 1 second, then given a revolution frequency on the order of $2.7\mu sec$, some particles will circulate in the machine for as many as $370,000$ turns before being extracted. The transit times, as we will see, are on the order of hundreds of turns (a couple of thousand at the most). So, given the amount that a single particles tune shifts is on the order of $2.7 \times 10^{-7}\ dQ/turn$, which is how much the particle tune changes when it steps from the unstable to the stable region, the particles tune will only change another 1.4×10^{-4} before getting to the septum. The spread in tunes of the particles that have just stepped out of the separatrix is then very small. They will be confined to within a narrow strip along the separatrix edge, as shown in Figures 9 and 7.

We should also explain that from the particles point of view, its tune is being changed in some manner, but for these dyanmics it doesn't matter how. The tune can be shifted by changing a quadrupole gradient, by changing the normalized quadrupole gradient (K_1), of by changing the energy with a non-zero chromaticity lattice ($dQ = Q_0\xi dp/p$). The formula's remain the same in any case, although the parameters used may differ.

We assume that the tune is changed linearly in time ($dQ/dt = Constant$). As explained, above, we can also assume that the change in the separatrix size per turn is small, $\Delta h \ll h$. We assume a particle is moving along the separatrix $X' = -1/\sqrt{3}X$. The velocity of the separatrix as it recedes from a particle is,

$$\frac{dh}{dt} = \frac{4\pi}{S}\frac{dQ}{dt} \tag{48}$$

where we have taken $X = -h$. As the separatrix and particle move apart, the relative velocity of the particle is given by,

$$\frac{dX}{dt} = \frac{\sqrt{3}S}{2}(3hX - X^2) + \frac{4\pi}{S}\dot{Q} \tag{49}$$

Integrating Eq. 49, we get the transit time for a particle to travel from the nearest fixed point (e.g., P_3), to the septum.

$$T_{TD} \approx \frac{1}{\sqrt{3}\epsilon}\ln\left|\frac{n}{n+3}\frac{3}{\lambda - \frac{1}{\sqrt{3}}\dot{\epsilon}/\epsilon^2}\right| \tag{50}$$

where $\dot{\epsilon} = 6\pi\Delta\delta Q/dt$ ($\Delta\delta Q$ in 3 turns $= \dot{Q}$) is on the scale of 10^{-8}).

To determine the time spent traveling along the side of the triangle, the velocity of the separatrix is now multiplied by $2/\sqrt{3}$. However, the motion in X' is of most relevance.

$$\frac{dX'}{dt} = \frac{S}{4}(6\sqrt{3}hX' + 3X'^2) - \frac{8\pi}{\sqrt{3}S}\dot{Q} \tag{51}$$

Note that here $A = \sqrt{X_0^2 + X_0'^2} = \sqrt{3\sqrt{3}}h = \sqrt{48\sqrt{3}}\pi\delta Q/S$, the transit time is the same as in the static case,

$$T_{ED} \approx \frac{1}{\sqrt{3}\epsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}-A}{A}\right| \tag{52}$$

where the integration is from $X_0' = 2\sqrt{3}\Lambda h$ to $X_F' = -Ah$, which is the point near the corner where the curve will be joined to the trajectory from P_3 to the septum.
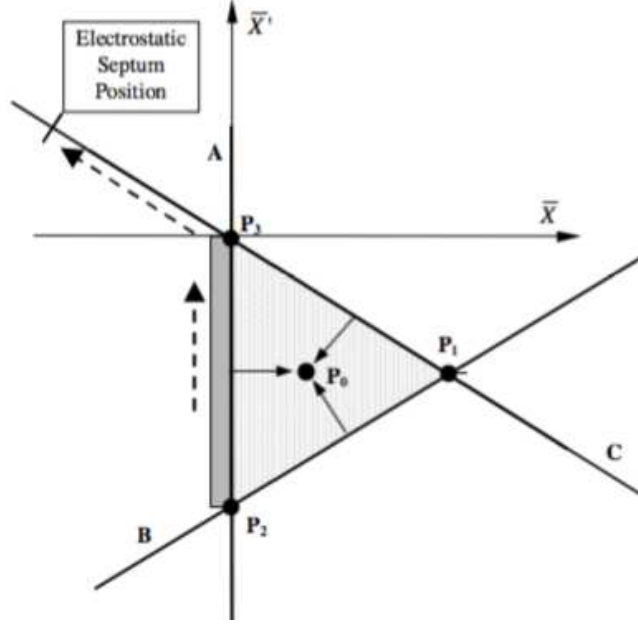
FIG. 9: Shrinking the stable region leaves a narrow strip of particles unstable. In the figure only one of the three sides of the triangle is considered and the motion of the particles is shown by the arrows (from reference [8].)

The total time from the particle starting point to move along the separatrix edge and the time to reach the septum is,

$$T_D \approx \frac{1}{\sqrt{3}\epsilon} \ln \left| \frac{n}{n+3} \frac{\Lambda}{1-\Lambda} \frac{2\sqrt{3}-A}{A} \frac{3}{\lambda_{F,d} - \dot{\epsilon}/(\sqrt{3}\epsilon^2)} \right|. \tag{53}$$

All particles starting with $0 < \Lambda < 0.1$ reach the septum at essentially the same time. This implies the spill will start with a spike at $t = T_{TD}$. If the distribution, $\rho(\Lambda)$, is the linear probability density of particles in the strip (seen in Figure 9), the spike will contain N_{spike} particles where,

$$N_{spike} = N_{strip} \int_0^{0.1} \rho(\Lambda) d\Lambda + N_{strip} \int_{\Lambda_F}^1 \rho(\Lambda) d\Lambda \tag{54}$$

The first term represents the particles in the vicinity of P_3 and the second term is all the particles overtaken by the separatrix. This requires some explanation. Looking at Figure 9, particles in the strip between points P_2 and P_3 are moving towards the point P_3. But the tune is shifting such that the three separatrix lines, A, B, C are moving towards the point, P_0. So particles in the strip are being overtaken by the separatrix line, B, and the point P_3 is also moving as C and A move toward P_0. Note that $\int_0^1 \rho(\Lambda) d\Lambda = 1$. Also note, the particles in these distributions are monoenergetic. A beam with momentum spread will be described later.

Given that the density of particles arriving at the septum over time is $P(t)$ where $P(t)dt$ is the number of particles reaching the septum between t and $t + dt$, the initial spike can be approximated by,

$$N_{strip} P_{spike}(t)dt = N_{spike}\delta(t + T_{TD})dt \tag{55}$$

After the inital spike, particles coming out between $T(\Lambda)$ and $T(\Lambda) + dt$ started between Λ and $\Lambda + d\Lambda$.

$$N_{strip} P_{tail}(t)dt = N_{strip} P_{tail}(T(\Lambda))dt = N_{strip}\rho(\Lambda)d\Lambda = N_{strip}\rho(\Lambda)\frac{d\Lambda}{dt}dt \tag{56}$$

and for $t_0 < t < t_F$ corresponding to $0.1 < \Lambda < \Lambda_F$, the time profile for the strip is then the sum of P_{spike} and P_{tail},

$$P_{strip}(t)dt = P_{spike}(t)dt + P_{tail}(t)dt. \tag{57}$$

Building up a model from this approach is now a matter of determining the values for these strip parameters, Λ_F, $\rho(\Lambda)$, and $\Lambda(t)$. The model is imprecise and what matters up to this point is the principles behind the timing of

the particle trajectories as they move along the separatrix lines. Typically the number of revolutions it takes for a particle to be extracted once stepping into the unstable region is on the scale of a few hundred to over one thousand, depending on the initial state of the particle. Most particles will be extracted fairly quickly, though.

C. Particle Tracking

To simulate slow extraction, we want to follow the trajectory of particles launched from some given distribution and observe their motion as they pass through the resonance condition and continue until they encounter the septum device. There are a number of ways to do this. Most simulation codes are built to allow particle tracking although they can be very slow or very fast depending on the approach. Using MadX, for example, is possible, but an element by element tracking in MadX can be slow. It can be made faster by learning how to run in a turn by turn mode. The classic method is to just use a Hamiltonian, keeping a close eye on understanding the limitations of the Hamiltonian. Such an approach for third integer extraction was developed well before powerful enough computerrs could be employed to do element by element simulations. One approach was developed by Y. Kobayashi [7], and the Hamiltonian has been named after him.

The method for using a third integer (Kobayashi) Hamiltonian is well described in the CERN PIMMS report [8]. The method uses a sextupole as a thin kick,

$$\Delta X = 0 \quad and \quad \Delta X' = S\left(X^2 - \frac{\beta_z}{\beta_x}Z^2\right) \tag{58}$$

and,

$$\Delta Z = 0 \quad and \quad \Delta Z' = -2S\frac{\beta_z}{\beta_x}XZ \tag{59}$$

where X is the transverse coordinate in the horizontal plane and Z is the transverse coordinate in the vertical plane. The beta-functions β_z and β_x are defined at the sextupole (where it is assume there is a single sextupole), which has a strength S, as defined before. We consider a particle with horizontal betatron tune close to the third integer, $Q_x = m \pm 1/3 + \delta Q$, where m is an integer and $|\delta Q| \ll 1/3$. This is the key assumption, since tracking particles far from $1/3$ will lead to incorrect/unrealistic results. The resulting Hamiltonian is then,

$$H = \frac{\epsilon}{2}(X^2 + X'^2) + \frac{S}{4}(3XX'^2 - X^3) \tag{60}$$

where we are only looking at motion in the horizontal plane. In this form, time is dimensionless. Then,

$$\Delta X_3 = \left(\frac{\Delta X}{\Delta t}\right)_{\Delta t=1(3turn)} = \frac{\partial H}{\partial X'} = \epsilon X' + \frac{3}{2}SXX' \tag{61}$$

$$\Delta X_3' = \left(-\frac{\Delta X'}{\Delta t}\right)_{\Delta t=1(3turn)} = -\frac{\partial H}{\partial X} = \epsilon X + \frac{3}{4}S(X^2 - X'^2) \tag{62}$$

Using this approach, as long as we only launch particles near the resonance condition and where their tunes are very close to $1/3$, we can realistically explore particle trajectories for slow extraction. Note that using this approach, we are only watching where the particles go every three turns. Figure 10 shows six particles launched using this approach, where five of the particles were launched very close to the three unstable fixed points, labeled A, B, and C. The sixth particle was launched near $(X, X') = (0, 0)$ and is far from the resonance, even if near the $1/3$ tune. Clearly this sixth particle is not showing realistic trajectory, as it is slowly spiralling outward when its motion should remain stable on a circlular trajectory in phase space. Although it is not so evident from figure 10, the particle spacings are increasing as they spiral away from the unstable fixed points. This is seen more clearly in figure 11, where we show the steps in position versus the steps in angle for three of the particles, one launched near the unstable fixed point, C, one launched near the unstable fixed point, A, and one launched just outside and above the fixed point at B. This last particle, as seen in figure 10 moves up from B to A and then its trajectory changes and it follows the separtrix line from A to the Septum. This can be seen more clearly in figure 12, where we have shown just that one particle along with a particle that is still in the stable region but very close to the resonance.

Given this simulation approach, we can see what happens when we launch four different particles from different locations, all at exactly the same time. The simulation can tell us how many turns (every three turns) it takes to
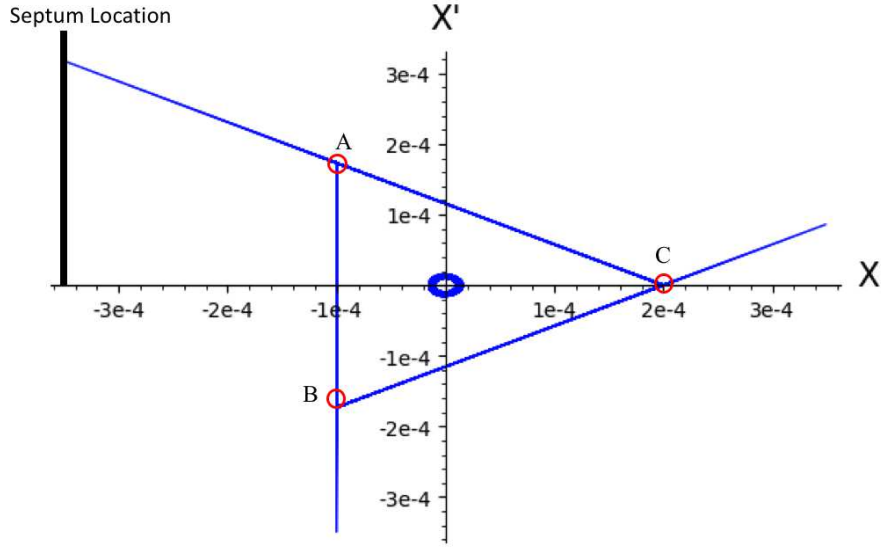
FIG. 10: Six particles, starting at different locations. Two of the particles stay in stable motion, staying within the region defined by the unstable fixed points located near points A, B, and C. Of these two, one is spiralling outward in phase space very near the center of the triangle. This particle does not fit the assumptions of the simulation and is not a valid trajectory. The other particle makes a triangle, traveling just barely inside the stable region, but right next to the boundary between the stable and unstable regions. The other four particles travel along the separatrix lines until reaching the extraction point on the left. (note: not an ideal example, just to illustrate).
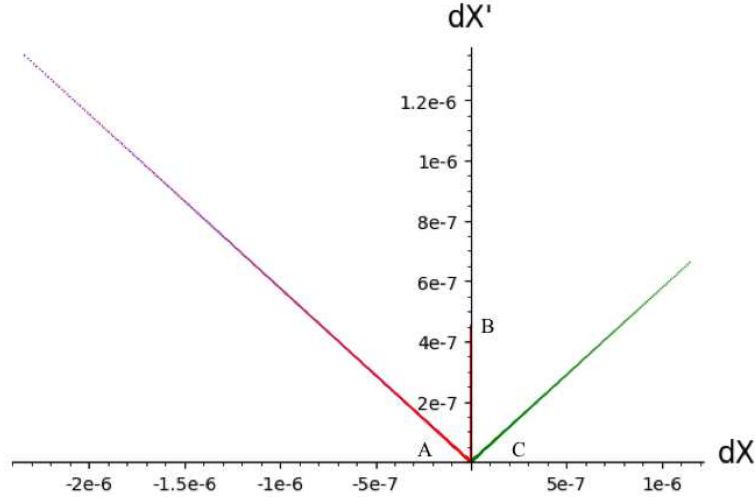

FIG. 11: Three particles, showing how the step size increases as their amplitudes increase. One particle (B) started out just to the left and above the unstable fixed point by B, traveling up to A, so its angular step size got smaller as it got closer to A and then grew again after passing by A and travel out to the septum.

get to the septum. This is shown in figure 13. Two particles, $P0$ and $P1$ are launched from the same amplitude but two different locations, and two other particles are launched from smaller amplitudes. Since they are all launched at exactly the same time, we look at the transit time differences. Particle $P0$ takes 3×1394 turns to arrive at the septum. Particle $P1$ takes 3×4756 turns. At 2.7 μsec per turn, the time difference for the two particles is about $10,000$ turns, or about $27 msec$. This example is just to illustrate the concept. The time it takes for $P2$ and $P3$ to arrive at the septum is much longer, 3×8716 turns and 3×61730 turns respectively. Particles can spend a large amount of time in the unstable region before arriving at the septum.

While we learn that the time difference for the particles is significant, this doesn't help us understand the time
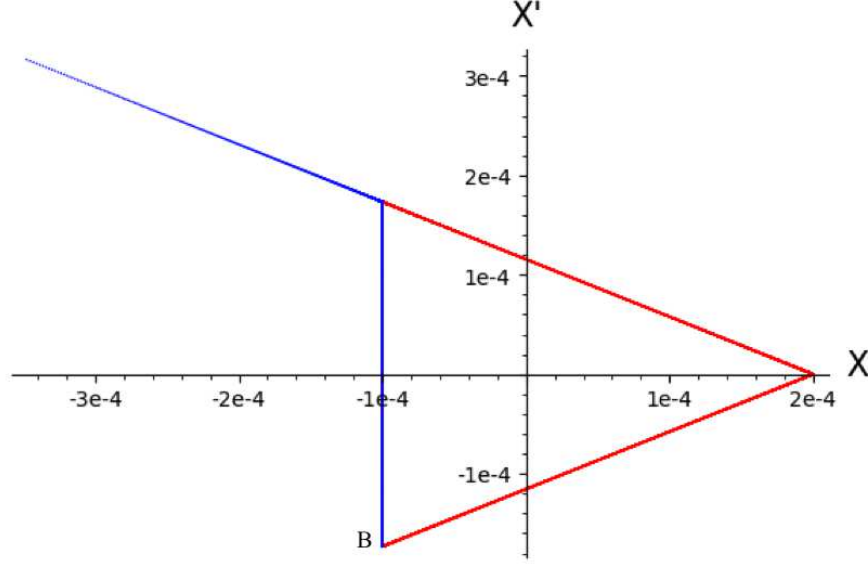
FIG. 12: This shows how the simulation (based on the Hamiltonian) only shows each particle every third turn. Depending on each particles starting point, we may see it appear on one of the three separatrix arms. In this case we show the particle that launched just to the left and above the unstable fixed point at B. On the alternative two turns not printed out in the simulation, this particle would appear on the other two sides of the triangle and on the other two arms, once passing by the unstable fixed points.

structure of the resulting stream of particles that get past the septum. To look at this requires a more detailed simulation that includes modulating the tunes to simulate ripple on power supplies. Also, we have not included any energy modulation in the particles. If the four particles in figure 13 have the same momentum, their relative arrival time to some longitudinal phase will lag by many turns but may stay very close to the same longitudinal phase coordinate. So this transit time doesn't tell us much about the bunch structure in the beam, when we perform bunched beam extraction.

Finally, we should point out that the Hamiltonians, Eq. 60 and Eq. 6 are not that different. The main difference is in how the kick from the sextupoles is treated. However, 60 is only valid near the resonance and only shows what happens every third turn. Equations 2 and 6 can be used to do turn by turn simulations over a larger tune range. However, taking this approach is not as fruitful as doing a more realistic turn by turn simulation, described later.

D. Reconstructing a Spill

Tracking particles and keeping account of the transit time for each particle is the first step for reconstructing the stream of particles being extracted over time. To keep the simulation simple, we build on using the Hamiltonian, generate random particles, but just for one edge of the separatrix. Since the three edges of the triangle all move towards zero, as particles are driven in tune into the resonance, we can build a distribution that is one-half Gaussian in X and Gaussian in X'. We adjust the particle coordinates so that all particles stream along the left separatrix edge and move up to the unstable fixed point above zero and stream out to the septum.

The first approach we take is using static slow extraction (betatron tune is not being shifted). So, we are only launching particles just into the unstable region and watching them as they spiral out to the septum. We can think of this as a collection of particles with various amplituds but with tunes that place them on the boundary between the stable and unstable regions. This is not quite realistic, since it suggests the separatrix just appears out of nowhere and the particles that happen to fall just outside of the stable region will begin to travel around the triangle and out along the separtrix arm. In reality the particles slowly see the separatrix and adiabatically their tunes are shifted until they encounter the resonance and jump into the unstable region.

By launching particles with different times, we can build a collection of transit times for groups of particles launched together at discrete times. Sorting and building a histogram of time of arrivals of particles (final coordinates assume arrival at some septum), we can simulate a simple spill. This simple spill uses randomly generated particles that
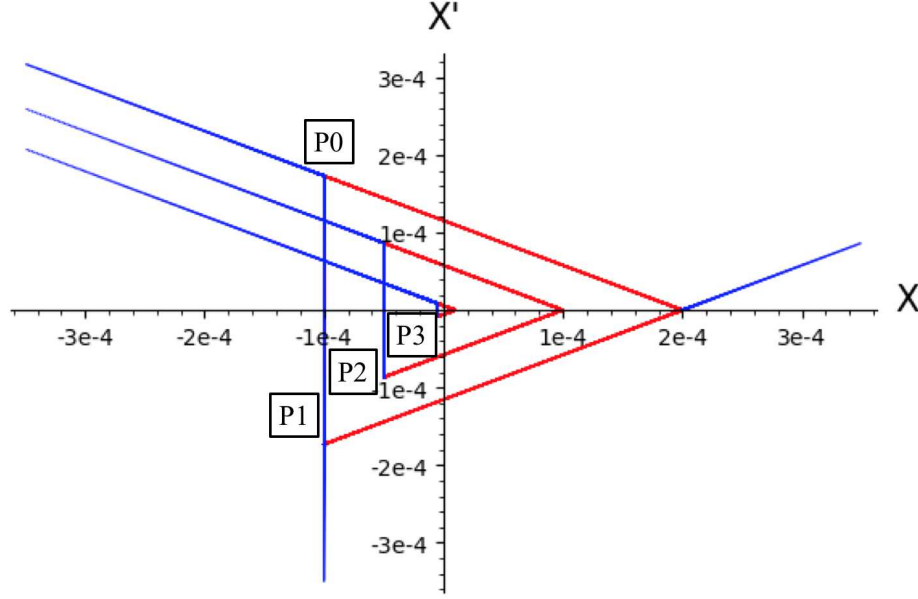
FIG. 13: Four different particles launching from different locations and at different amplitudes. If we assume all four cross into the unstable region at the same time, they will arrive at the septum at very different times, simply because their transit times are much different.
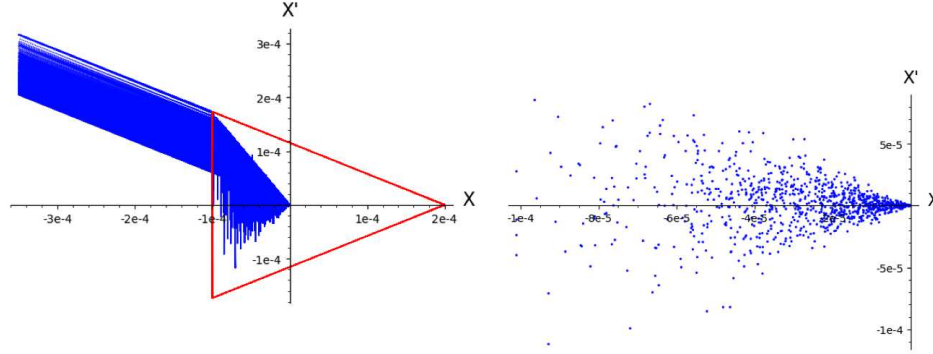

FIG. 14: A thousand particles launched just on the edge of the resonance from one sector of a 2D Gaussian distribution of particles.

are launched from just within the unstable region. To properly simulate the process we would need to launch them from within the stable region and observe their passage from the stable region to the unstable region. However, the simpler approach still simulates a spill quite well. Figure 16 shows two spills using particles launched directly into the unstable region. The one on the left is with no tune ripple and the variations in intensity are purely from the random nature of the simulation and the discrete time intervals used to build this distribution. The spill on the right has 100% tune ripple at 7.5 Hz applied, per Eq. 63.

$$\Delta Q_r = dQ_r \cos(\omega t)/S \qquad (63)$$

where $dQ_r = \delta_r hS$, with δ_r is some value close to 1. For figure 16, $\delta_r = 1.0$.

Simulating the launching of particles from inside the stable region requires more cpu time and the time spent inside the stable region needs to be ignored in the calculation of the transite times. However, particles will now stream out from any one of the unstable fixed points, since they will circulate inside the stable region for some period of time and may cross into the unstable region on any edge of the separatrix. So, a technique is needed to determine when particles streaming out on the other arms will arrive at the septum, seen only by particles on the one arm to
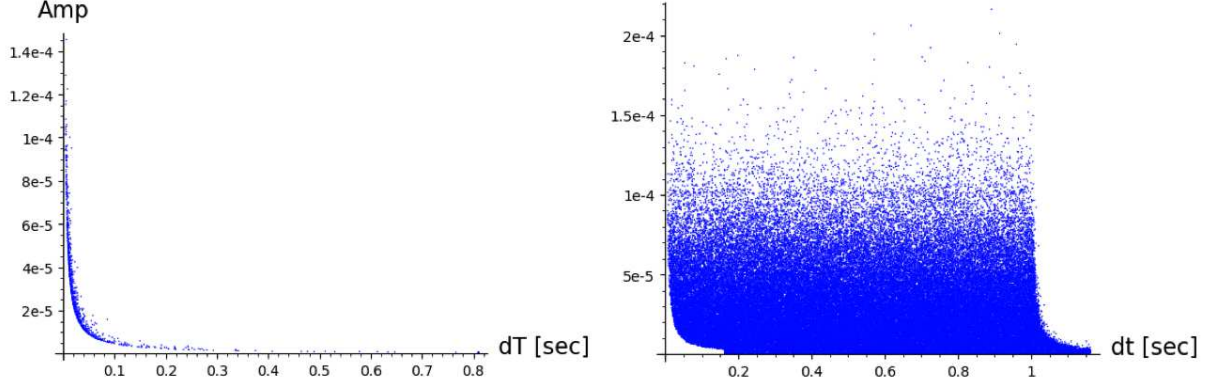
FIG. 15: Left figure: Distribution of transit times for one thousand particles. The majority of particles have transit times below 100 msec. So, particles, in principle, can spend quite a long time in the unstable region, particulary near the unstable fixed points. In the right figure, 100,000 particles are shown, distributed into 1000 independent transit time profiles. Each profile is 100 particles launched at exactly the same time. The sum of these distributions is what builds a spill profile, as shown in figure 16
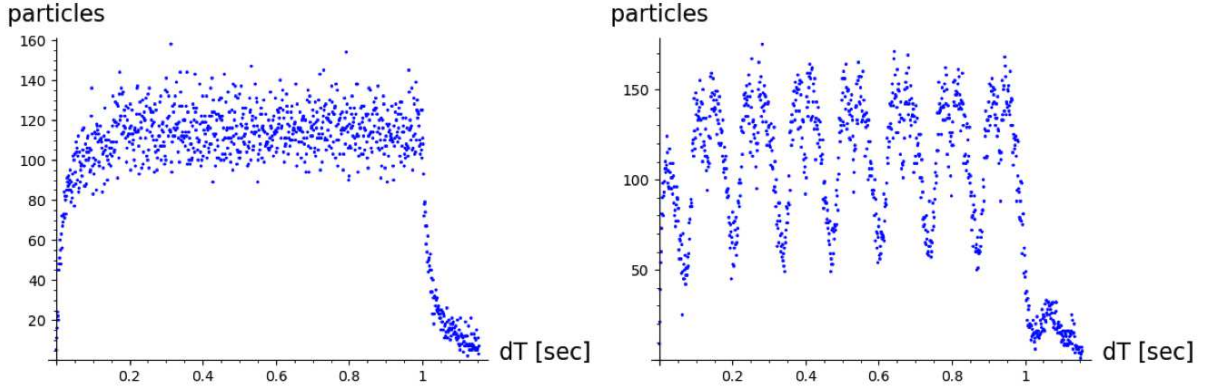

FIG. 16: Two simulated spills using 100,000 particles. For this simulation, 100 randomly generated particles are launched simultaneously every 1 msec, over 1000 msec's. This simulates the tune linearly driving particles into resonance, although assumes the tune distribution is uniform. On the left, no ripple in tune is applied. On the right the tune is modulated at 7.5 Hz.

the left (see figure 10). Another approach is to perform pseudo-element-by-element simulations based on linear beam dynamics, as described in the Hamiltonians, either Eq. 2 or Eq. 6. An approach for this method is described next.

E.   JPARC Example

To illustrate what has been discussed, so far, consider an example using parameters from JPARC slow extraction of 30 GeV/c protons. The parameters for JPARC are as follows:

1. Eight sextupoles operate at $S = 15.9$, where $L_S = 0.7m$, $B\rho \approx 100 \ Tm$, bore radius $= 68mm$, $\beta_x = 11m$, and the pole tip field in each sextupole is about $0.57T$. This configuration gives a virtual sextupole strength of about $S_v \approx 120$.

2. The normalized emittance in JPARC is $\approx 7\pi mm - mrad$.

3. The revolution period is $5.23\mu sec$

4. The electrostatic septum (ESS1) is located at 0.06 m to the inside of the equilibrium orbit at a $\beta_x = 22m$.

In figure 17 a more realistic version of what was shown in figure 7 is shown. In this figure the left figure is built using the above parameters. The black horizontal line at the top of the figure corresponds to the 99 % emittance of

$2.1e^{-6}\pi$ $mm - mrad$. The red lines and dashed black lines with arrows are not exact, but just meant to show the principle. The figure on the right was generated using a different particle tracking code (described in the next section) and so parameters will not correspond exactly.
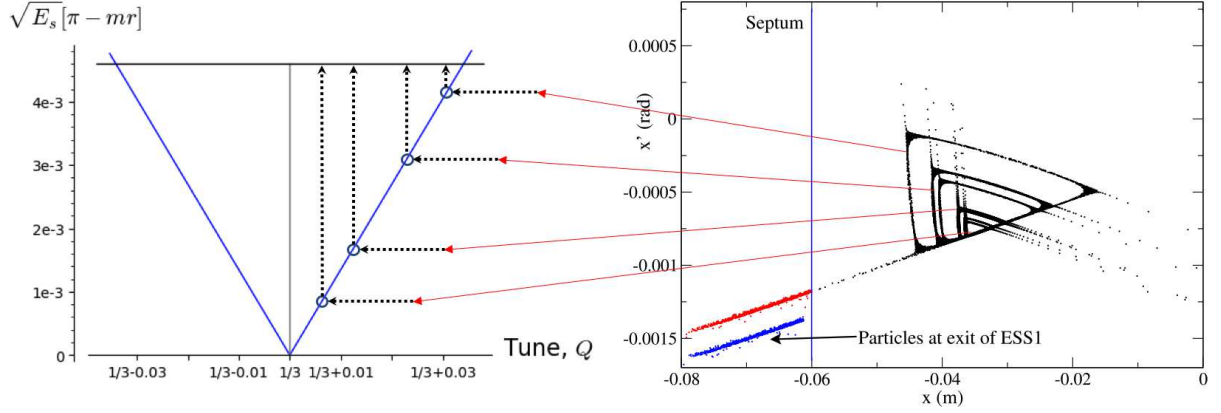

FIG. 17: JPARC Slow extraction using the above parameters.

Since JPARC uses a dynamic bump with zero horizontal chromaticity, large amplitude particles are extracted first and small amplitude particles are extrated last. Only particles of a single amplitude/tune are being extracted at any give moment in time.
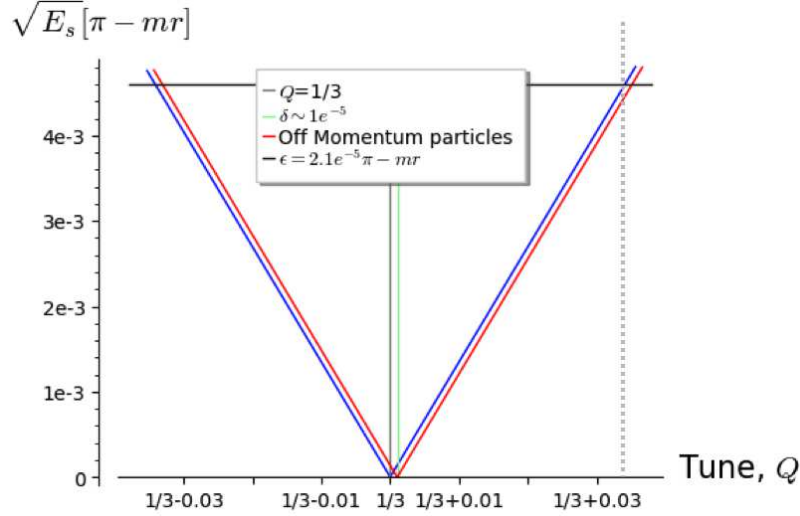

FIG. 18: Resonance lines for two different momenta in Steinbach diagram.

In figure 18, the Steinbach diagram for two different momenta is show, given the above parameters. The blue lines correspond to on momentum particles while the red lines correspond to particles with momenta that deviate by $\delta \approx 1e^{-5}$. Since JPARC extracts using zero chromaticity, particle momenta are not correlated with tune, so particles with different momenta may enter into resonance earlier or later from the on momenta particles at a given amplitude. Keeping the momentum spread small is still important since this can limit the effectiveness of the dynamic bump.

In figure 19, two particles are shown going into resonance, one very small amplitude particle and one large amplitude particle. The smaller amplitude particle takes longer to get to the septum, represented by the red line to the left. This figure does not show the effect of the dynamic bump, but does clearly show how many more turns are needed for the small amplitude particle to launch away from the resonance and arrive at the septum.

Figure 20 is similar to figure 19, except now 2000 particles are shown launching from many different initial coordinates. Since the initial phase space has a Gaussian distribution, there is a denser distribution of particles launching from small amplitudes than large amplitudes. One can imagine that to transform this Gaussian distribution into a uniform distribution in time, the large amplitude particles are extracted very quickly but smaller amplitude particles
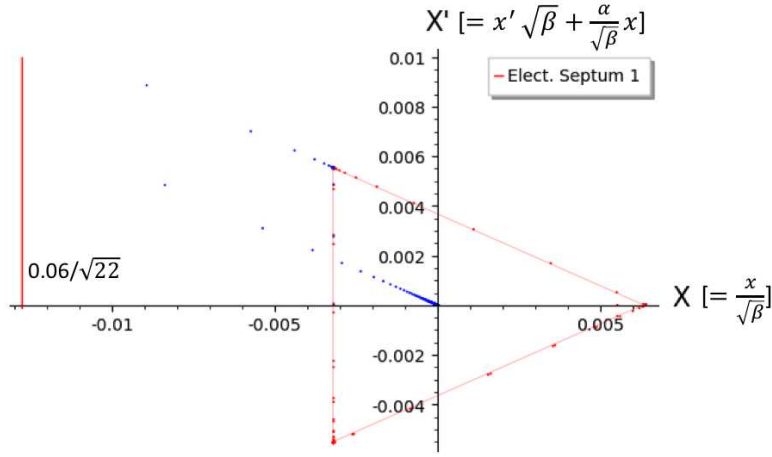
FIG. 19: Two particles, one small amplitude and one large amplitude being tracked using the Kobayashi Hamiltonian approach.

get extracted more slowly, until the beam is depleted to the core of the original distrubiton which gets extracted quickly again (being there are fewer particles left to extract by that time.)
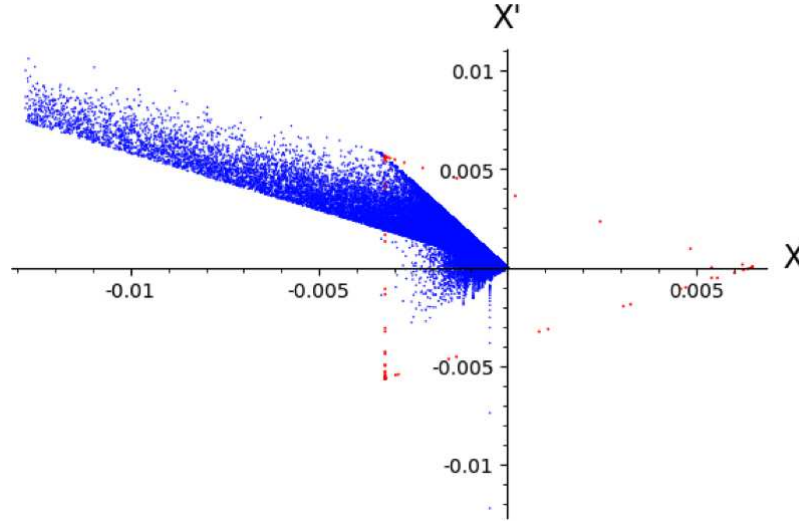
FIG. 20: 2000 randomly generated particles tracked using the Kobayashi Hamiltonian approach.

Finally, in figure 21, the transit time distribution is shown for the particles in figure 20. This clearly shows there is a direct correspondance between a particles initial amplitude and the time it takes to be extracted. Large amplitude particles can be extrated within 50 or 60 turns while small amplitude particles may take $> 10,000$ turns. Results for very small amplitude particles should be viewed with a skeptical eye, since this is tracking using an approximate Hamiltonian.

FIG. 21: Transit times for 2000 particles.

F. Pseudo-Element-by-Element Tracking

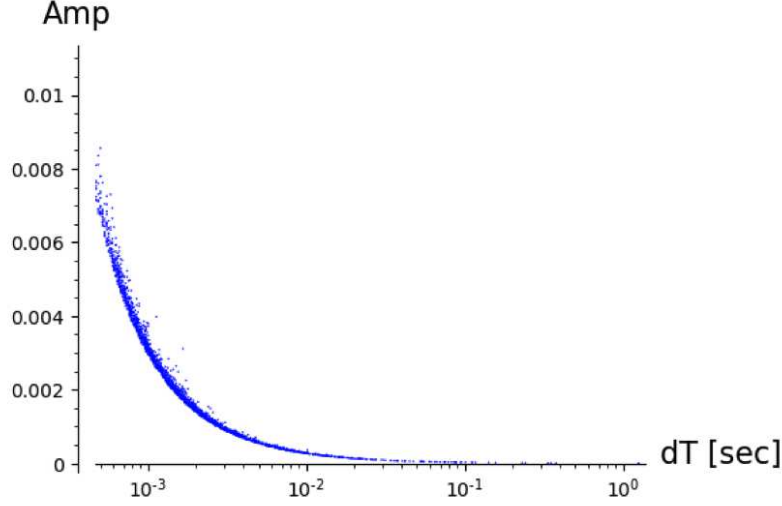The motion of charged particles in linear systems of magnets can be described as a simple harmonic oscillator with the form: (Most of what appears in this section can be found described in much greater detail in [9].)

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = 0 \tag{64}$$

where ν is the betatron tune and ϕ is the phase at a given point in the system. The general solution to this equation is,

$$\eta(\phi) = A\cos(\nu\phi + \delta) \tag{65}$$

where A and δ are constants of integration which depend on the initial conditions. The motion of this simple harmonic oscillator can be visualized by thinking of the system as a periodic system in equilibrium. The motion can then be described as a circle in (η, η') space, where $\eta' \equiv d\eta/d\phi$. In this case,

$$\eta'(\phi) = -A\sin(\nu\phi + \delta) \tag{66}$$

To do tracking simulations of single particles through arbitrary linear systems we generate a distribution of coordinates using random numbers,

$$\eta = \sigma_x\sqrt{-2\ln(i_1)}\cos(2\pi i_2) \tag{67}$$

and,

$$\eta' = -\sigma_x\sqrt{-2\ln(i_1)}\sin(2\pi i_2) \tag{68}$$

where i_1 and i_2 are two independent random numbers between 0 and 1.

The distribution of coordinates will then describe a Gaussian distribution of particles with a one sigma width of σ_x. We now need to make a transformation from (η, η') to coordinates useful for tracking the generated particles through a secondary system. So far the motion described is only for that of a particle in a single plane. To describe a distribution of particles in the two transverse planes (vertical and horizontal) we assume no coupling and use an independent set of random numbers to generate an independent distribution of coordinates.

The nominal coordinate system used in accelerators is a curve-linear coordinate system. In this case the trajectory is along a planar closed curve. The path length along the curve of motion is the independent variable, s. At any point

along this curve we can define three unit vectors: $\hat{s}, \hat{x}, \hat{x}'$. A more detailed description of this coordinate system can be found in the MAD manual [10].

We generally are interested in deviations of x and y from the reference orbit, so in our discussion we speak of x and y, meaning the deviations in horizontal and vertical planes from the reference path defined by the elements of the system.

In the curve-linear coordinate system, then, we write the equation of motion in the form of Hill's equation (note that $x' \equiv dx/ds$),

$$x'' + K(s)x = 0 \tag{69}$$

K(s) is a periodic function of the independent variable, s. The general solution to this equation is

$$x = A\omega(s)\cos(\psi(s) + \delta) \tag{70}$$

Again, A and δ are constants of integration reflecting the initial conditions. By substituting $\omega(s)$ and $\psi(s)$ into eq. (69) we find

$$2\omega\omega'\psi' + \omega^2\psi'' = (\omega^2\psi')' = 0 \tag{71}$$

Since what we want is a way of describing the motion of particles as they propagate through the system from some point s_0 to $s_0 + C$, we re-normalize the constants to derive the Courant-Snyder parameters,

$$\beta(s) = \frac{\omega(s)^2}{k} \tag{72}$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} \tag{73}$$

$$\gamma = \frac{1 + \alpha^2}{\beta} \tag{74}$$

Now we rewrite the equation of motion to describe a mapping from one point in the reference trajectory to another point in the reference trajectory:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \tag{75}$$

where,

$$M = I\cos\Delta\psi_c + J\sin\Delta\psi_c \tag{76}$$

where we have defined

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I \tag{77}$$

I and J satisfy the symplectic condition, $\mathcal{M}^T S \mathcal{M} = S$. One consequence of this condition is the transformations are very close to exact. If a matrix $\mathcal{M}$ is symplectic, then its inverse is symplectic and the determinant of that matrix is symplectic. If two matrices $\mathcal{M}$ and $\mathcal{N}$ are symplectic, then the product $\mathcal{MN}$ is also symplectic. The only loss in precision will be due to round off errors resulting from finite bit sizes in numeric representations.

The constant A in eq. (70) can be expressed in terms of x and x'.

$$\alpha(s)x(s) + \beta(s)x'(s) = -A\sqrt{\beta(s)}\sin(\psi(s) + \delta) \tag{78}$$

or,

$$A^2 = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2 \tag{79}$$

This invariant form describes an ellipse in (x, x') phase space. The area of the ellipse is a constant of the motion and is

$$\text{Area} = \frac{\pi A^2}{\sqrt{\beta\gamma - \alpha^2}} = \pi A^2 \tag{80}$$

The constant A^2 is called the unnormalized emittance, ϵ, in which case,

$$\frac{\epsilon}{\pi} = \gamma x^2 + 2\alpha x x' + \beta x'^2 \tag{81}$$

The canonical transformations between (η, η') coordinates and (x, x') coordinates is then a Floquet transformation,

$$\eta = \frac{x}{\sqrt{\beta}} \tag{82}$$

$$\eta' = \frac{x\alpha}{\sqrt{\beta}} + x'\sqrt{\beta} \tag{83}$$

$$x = \eta\sqrt{\beta} \tag{84}$$

$$x' = \frac{\eta' - \eta\alpha}{\sqrt{\beta}} \tag{85}$$

The computer code takes eqs.[67,68] and makes the canonical transformation to (x,x'), generating a single set of coordinates. The distribution of these coordinates depends on the value of σ, which is calculated from,

$$\sigma^2 = \frac{\epsilon\beta}{-2\pi\ln(1 - F)} \tag{86}$$

Where F is the fraction of emittance (e.g., 95 %). The momentum part of the particle coordinate is calculated using:

$$\sigma_\delta^2 = \frac{1}{-2\ln(1 - F)}\left(D\frac{\Delta p}{p_0}\right)^2 \tag{87}$$

Where D is the momentum dispersion of the lattice and $\Delta p/p_0$ corresponds to the fraction F of the total momentum distribution.

$$\delta = \sigma_\delta\sqrt{-2\ln(i_1)}\cos(2\pi i_2) \tag{88}$$

The total beam size depends on both components as:

$$\sigma_T^2 = \sigma^2 + \sigma_\delta^2 \tag{89}$$

If one knows the Courant-Snyder parameters at two points in a lattice, a single particle can be tracked from the first point to the next using the following transformation.

$$\begin{pmatrix} x_f \\ x'_f \\ \delta_f \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} x_i \\ x'_i \\ \delta_i \end{pmatrix} \tag{90}$$

where

$$
\begin{aligned}
M_{11} =\ & \left(\tfrac{\beta_f}{\beta_i}\right)^{\frac{1}{2}} \left(\cos \Delta\psi + \alpha_i \sin \Delta\psi\right) \\
M_{12} =\ & \left(\beta_f \beta_i\right)^{\frac{1}{2}} \sin \Delta\psi \\
M_{13} =\ & D_f - \left(\tfrac{\beta_f}{\beta_i}\right)^{\frac{1}{2}} \left(\cos \Delta\psi + \alpha_i \sin \Delta\psi\right) D_i - \left(\beta_f \beta_i\right)^{\frac{1}{2}} \sin \Delta\psi D_i' \\
M_{21} =\ & -\frac{1+\alpha_i \alpha_f}{(\beta_i \beta_f)^{\frac{1}{2}}} \sin \Delta\psi + \frac{\alpha_i - \alpha_f}{(\beta_i \beta_f)^{\frac{1}{2}}} \cos \Delta\psi \\
M_{22} =\ & \left(\tfrac{\beta_i}{\beta_f}\right)^{\frac{1}{2}} \left(\cos \Delta\psi - \alpha_f \sin \Delta\psi\right) \\
M_{23} =\ & D_f' - \left(-\frac{1+\alpha_i \alpha_f}{(\beta_i \beta_f)^{\frac{1}{2}}} \sin \Delta\psi + \frac{\alpha_i - \alpha_f}{(\beta_i \beta_f)^{\frac{1}{2}}} \cos \Delta\psi\right) D_i - \left(\tfrac{\beta_i}{\beta_f}\right)^{\frac{1}{2}} \left(\cos \Delta\psi - \alpha_f \sin \Delta\psi\right) D_i' \\
M_{31} =\ & 0 \\
M_{32} =\ & 0 \\
M_{33} =\ & 1
\end{aligned}
\tag{91}
$$

where δ is the momentum deviation for the particle $(\Delta p/p)$, $D_{i,f}$ and $D_{i,f}'$ are the dispersion and angular dispersion at the two points. $\alpha_{i,f}$ and $\beta_{i,f}$ are the Courant-Snyder parameters for the two points. The phase advance from i to f is denoted $\Delta\psi$. The vertical plane is evaluated in the same way, although it is assumed there is no coupling terms in between kicks (only coupling produced by the sextupole or octupole kick is included).

As long as there is not a strong momentum dependence of the Courant-Snyder parameters (i.e., a strong radial dependence), then we can assume the values to be static over the range of particles that represent realistic cases. Nevertheless the code has the ability to linearly vary Courant-Snyder parameters if the dependence is specified.

In both the AGS and the Booster a sextupole resonance is created using 2 sets of sextupoles excited in opposite polarities. Therefore, in order to study sextupole resonances in the AGS or Booster we need to define at least 5 points in the lattice: 4 sextupoles and one septum. We can construct the phase space at the septum by putting in thin sextupole kicks (which keeps the transformations symplectic) and tracking particles starting at the septum and observing the particles at the septum every revolution. One then performs the mapping of eq. (90) from the septum location to the first sextupole, using Courant-Snyder parameters derived from MAD, gives a thin sextupole kick, and maps to the next sextupole. The sextupole kick is done by using,

$$
x_f' = x_i' + S \cdot (x_i^2 + y_i^2)
\tag{92}
$$

$$
y_f' = y_i' - S \cdot (2 x_i y_i)
\tag{93}
$$

where S is the sextupole kick strength. This strength can be given directly, but is related to the sextupole field gradient by,

$$
S = \frac{1}{2! \cdot B\rho} \int_{-\infty}^{\infty} \frac{\partial^2 B_r}{\partial r^2} dl
\tag{94}
$$

An octupole kick is done by using,

$$
x_f' = x_i' + O \cdot (x_i^3 + x_i y_i^2)
\tag{95}
$$

$$
y_f' = y_i' - O \cdot (3 x_i^2 y_i + y_i^3)
\tag{96}
$$

where O is the octupole kick strength. This strength can be given directly, but is related to the octupole field gradient by,

$$
O = \frac{1}{3! \cdot B\rho} \int_{-\infty}^{\infty} \frac{\partial^3 B_r}{\partial r^3} dl
\tag{97}
$$

In the case of chromatic slow extraction, to simulate traversal through a resonance there needs to be the ability to shift the tune (e.g., phase advances). To do this the phase advances, $\Delta\psi$, are all initially shifted up by a small amount, with a scaling sufficient to place all particles above the resonance, in tune, and then decreased a very small amount each revolution, until the particles pass into the resonance and are excited into large amplitude oscillations.

The scaling is done as,

$$\text{scale} = \frac{\sqrt{x^2 + (D \cdot \delta)^2}}{\sqrt{-\ln(1-F)(\sigma^2 + \sigma_\delta^2)}} \tag{98}$$

The phase advances are then shifted as,

$$\text{phaseshift} = \text{scale} \cdot \left| \xi \frac{\Delta p}{p_0} \right| \tag{99}$$

where ξ is the lattice chromaticity. For these purposes the method is arbitrary. It is computationally faster than other methods, but it is incompatible with defining a time structure to the extracted particles. To do this would slow down the simulations significantly. We need only get enough tune shift to get the particle away from the resonance (since the given Courant-Snyder parameters are for a lattice right at or very close to the resonant tune) and which can be parametrically driven back down into the resonance adiabatically. Figure 1 shows an example of 4 different particles being driven into a sextupole resonance using this approach.

In the case of zero chromaticity slow extraction, this approach doesn't work. Instead the tune is shifted by calculating the tune width of the resonance for a given particle amplitude, shifting the phase advances down by that amount, and then stepping the phase advances up (in the case of J-PARC) very slowly. Again this is done for computational efficiency. The tune width is:

$$\delta Q > \sqrt{x^2 + x'^2} \frac{|S|}{\sqrt{48\pi\sqrt{3}}} \tag{100}$$

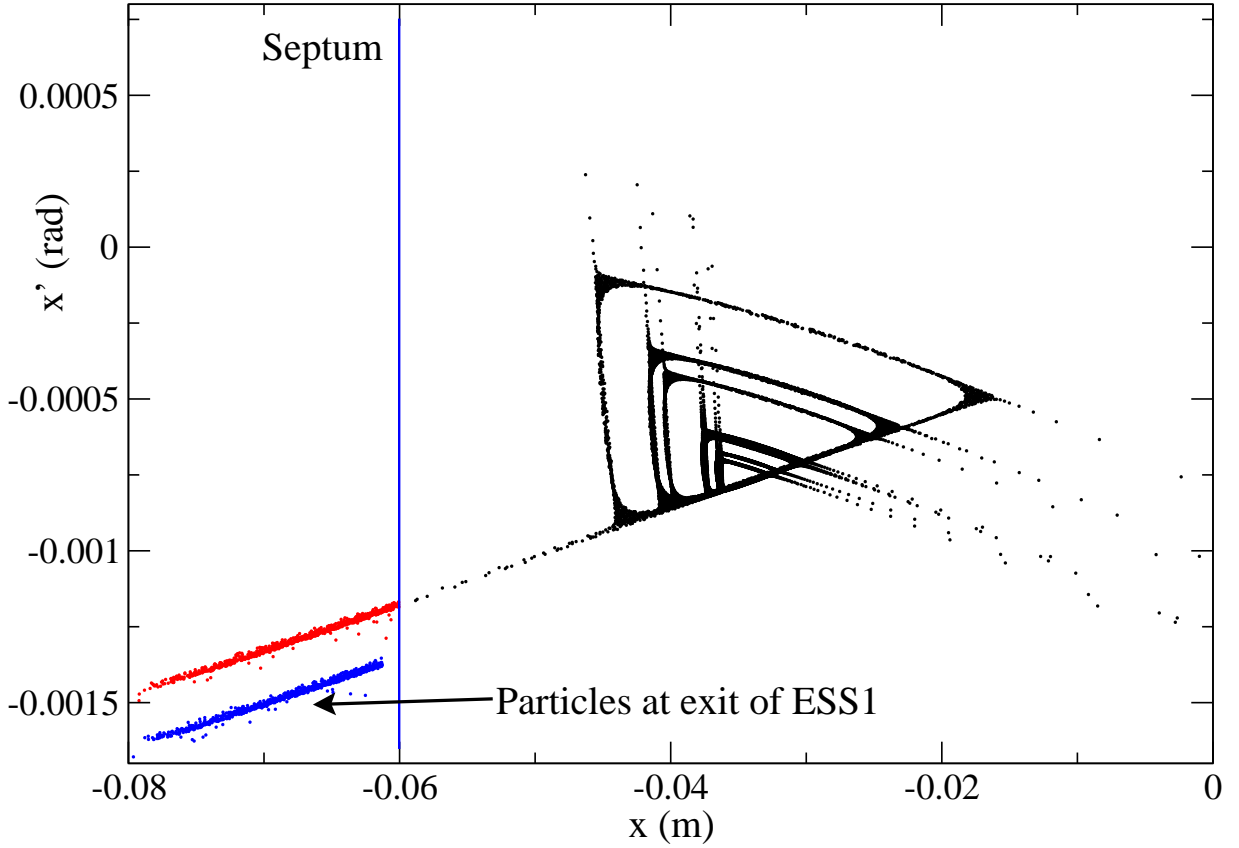Figure 22 shows an example for J-PARC slow extraction.

FIG. 22: Phase space at J-PARC ESS for 10 different particles (in black), including a dynamic bump. Step size in this example is 2cm for small amplitude particles. >990 Particles in red have stepped past the septum (blue line at -0.06m) out of 1000 initial particles. In this case the initial beam emittance is 2 π mm-mrad. The septum thickness is taken to be 0.051 mm (2/1000 of an inch). Extracted particle phase space trajectories are also shown at the exit of ESS1.

V. EXTRACTING SHORT BURSTS OF BUNCHED BEAM

Given that the betatron tune can be used to drive the beam into the resonance, it is possible to combine systems to do more sophisticated manipulations to achieve different extracted beam properties. For example, to slowly extract bunched beam, the main RF frequency is used to change the energy of the beam and through the non-zero chromaticity, drive the particles into resonance. The rate of the energy change will determine the number of bunches and the individual bunch intensities. To make bursts of bunched beam, the RF frequency can be adjusted to park the beam near the resonance and then a fast AC quadrupole can pulse to drive bunches into resonance and create short bursts of bunched beam. The intensity of the individual bunches will depend on the rate of the AC quadrupole pulse. There are two ways to do this. One is to pulse an AC quadrupole with a flat current (no ramp) and use the RF to create the ramp rate, or fix the RF to a constant frequency and put a ramp on the AC Quadrupole pulse.

A. Bunched Beam Slow Extraction

To extracted bunched beam is relatively straight forward. The RF is set to be open loop, the magnetic field is set to be constant, and then the radial steering function (which is a way of controlling the RF frequency) is ramped to drive beam into the resonance. The rate of the ramp will determine the length of the total spill, the intensity per mini-bunch, and the number of mini-bunches. The lengths of the mini-bunches will be comparable to the bunches in the ring, although slightly shorter.
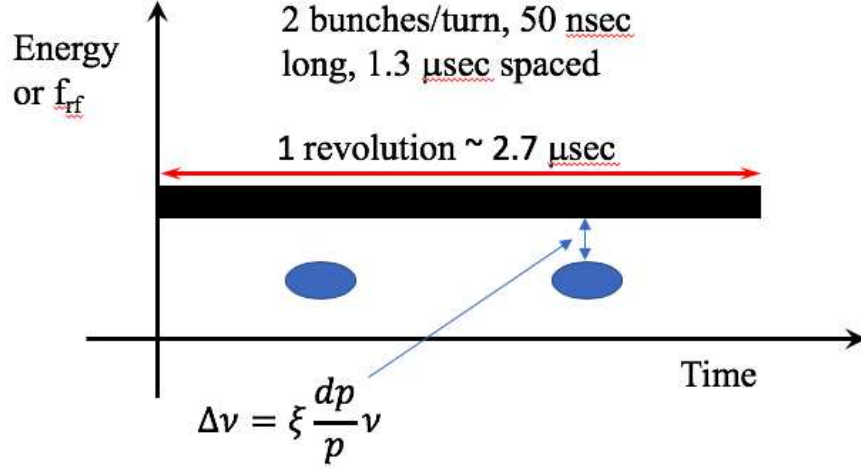
FIG. 23: This is an example of setting up to extract bunched beam. There are two bunches per turn in the AGS, which are some distance from the resonance (thick black band). The RF frequency will need to move the tune of the particles by $\Delta\nu$ pluse the tune width of the bunches.
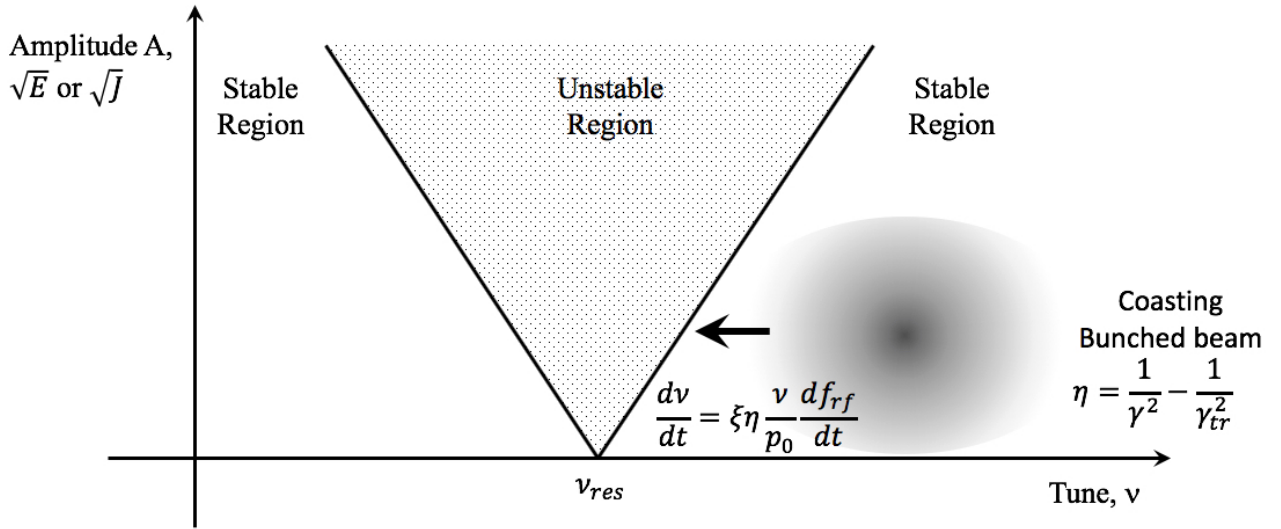

FIG. 24: The process as seen from the Steinbach diagram. The tune is shifted by changing the RF frequency.

B. Extracting short bursts

Returning to Eq. 9, we would like to determine what δ is for nominal AGS conditions. The definition of A is

$$A = \sqrt{X^2 + X'^2} \tag{101}$$

where $X = x/\sqrt{\beta}$ and $X' = x'\sqrt{\beta} + x\alpha/\sqrt{\beta}$, where β and α and the lattice parameters at the given location in the lattice. In this case, the amplitude function is as seen at the first septum magnet. In the AGS, given nominal conditions for 7 GeV proton beams, $A \approx 0.013$. Typically, for protons in the AGS, normalized beam emittances are on the order of 20π mm-mrad at 95%. A value of S at this energy is on the order of 0.5. Given these parameters we expect $\delta \approx 0.0003$. It is easy to see, now, why small variations in power supplies can cause significant spill structure. The way to intepret this value of δ is that this is the range of particle tunes currently entering resonant conditions. If the total beam has a tune spread of 0.1, we can see that at any given moment, only about 0.3% of the total beam
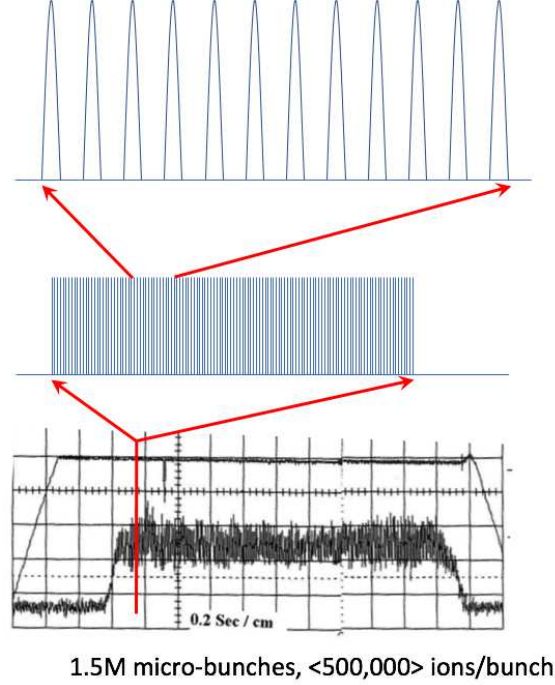
1.5M micro-bunches, <500,000> ions/bunch

FIG. 25: The result is what may look like a normal spill, but on close inspection will be composed of thousands of mini-bunches.

is being extracted.

In normal slow extractioin we tend to make the chromaticity large and the momentum spread large. This spreads out the tunes and reduces the spill ripple by increasing the rate that we drive the beam into the resonance. But, if we wanted to create a burst of extracted beam of some arbitrary, but extremely short length of time, we would want to make the momentum spread as small as possible and reduce the chromaticity, to get as small a tune spread as possible. Assuming we could reduce the momentum spread of the beam by a factor of 2 and the chromaticity could be reduced by a factor of 10, we would reduce the tune spread from 0.1 to 0.005. The value of δ remains the same, since it is the tune width of particles in resonance at any moment and depends only on the emittance and the sextupole strengths.

Using the RF frequency to ramp the beam into resonance we would need

$$\frac{d\nu}{dt} = \xi\eta\frac{\nu_0}{p_0}\frac{df_{rf}}{dt}. \tag{102}$$

where $\xi = (d\nu/\nu_0)/(dp/p_0)$ is the normalized chromaticity and $\eta = 1/\gamma^2 - 1/\gamma_{tr}^2$ is the frequency slip factor.

Figure 26 shows the amount of frequency shift needed to make a -0.005 change in betatron tune, for a chromaticity of -0.25 as a function of energy. The γ_{tr} point in the cycle (for protons) is shown as a red vertical line. This highlights some of the issues with using the RF frequency to control the slow extraction. Very low energies require very little frequency shift, but may be less stable, as a result. Near transition, the amount of shift is very large.

If we wanted to do bunched slow extraction with a pulse train of 100 mini-bunches, spaced 1.35 μsec apart, we would have a 135 μsec pulse of beam. The amount of frequency shift needed to move the tune by 0.005 units, for 7 GeV beams, is only 0.4 Hz. For normal slow extraction parameters (with normal chromaticity and momentum spread), to extract all the beam in 2 seconds would require a $df_{rf}/dt = 4.24\ Hz/sec$, where with a tune spread of 0.1 and we need a total frequency shift of $8.5Hz$. But to extract all the beam in $135\mu sec$, with these parameters, would require $df_{rf}/dt = 63,000\ Hz/sec$, since we still need to move the frequency by $8.5Hz$, but now in $135\mu sec$. If we reduce the tune spread to 0.005, with the smaller chromaticity, we still need to move the frequency by $4.2Hz$ (since we made the momentum spread smaller by 1/2). So to extract using the RF frequency we would require $df_{rf}/dt = 31,400\ Hz/sec$. Clearly, the only way to get this method to work would be to reduce the momentum spread by a large factor. We conclude that using the RF to ramp the beam into the resonance in 135 μsec is probably not practical. However, the RF can be used to place the beam near the resonance and a fast quadrupole could drive the beam through the
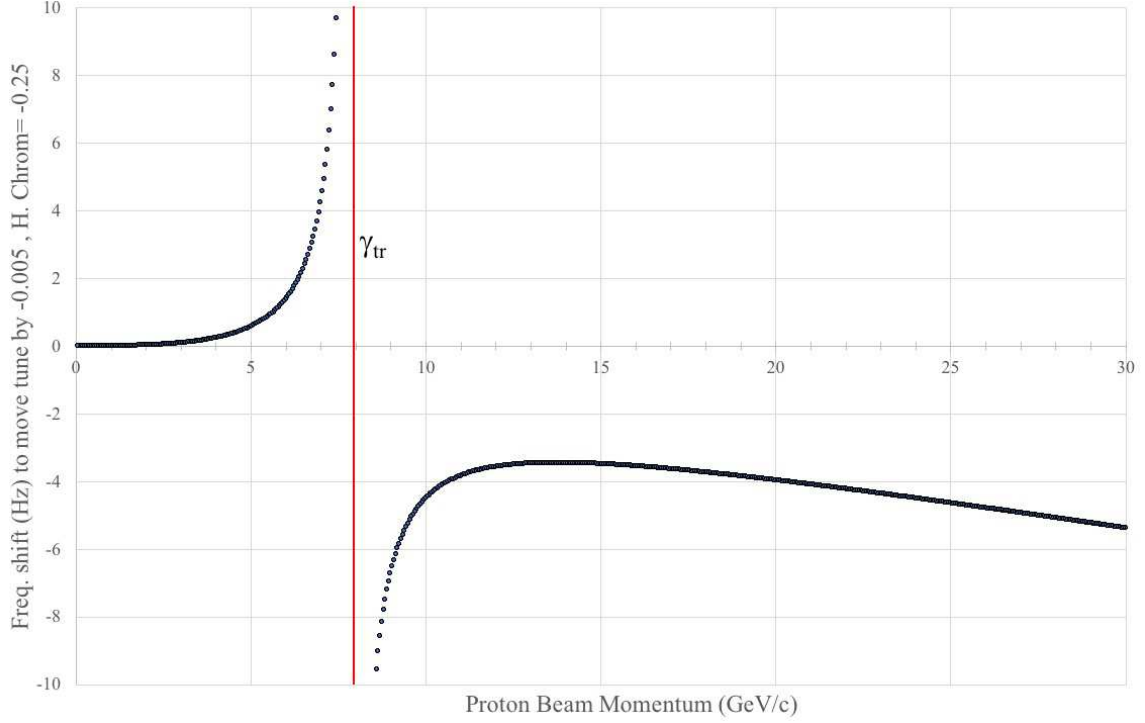
FIG. 26: Required frequency shift to change betatron tune by -0.005 units for a chromaticity of -0.25.

resonance, with a sloped current driving it from a maximum current to zero. The quadrupole would have to be able to change the tune by about 0.005 units. Is this possible? The most difficult problem will be the rise time for this quadrupole, which will need to be $< 1.3 \mu sec$.

FIG. 27: The J-PARC ripple correction quadrupole.

RQ	
Core Material	0.1mm thick lamination steel
Bore Radius	80mm
Magnet Length	0.62m
Coil Turn Number	6
Field Gradient	0.94T/m@400A
Inductance	0.65mH
Resistance	11.25mΩ
Current	75Arms
Voltage	400Vrms
Band Frequency	40Hz~10kHz
ΔH tune	0.001439/100A

FIG. 28: RQ Parameters.

One example of a fast quadrupole is the AC quadrupole used for ripple control at J-PARC. Figure 27 shows a photograph of the magnet and Fig. 28 shows the parameters for this magnet. Note, that the tune shift values corresponds to $30 GeV/c$ proton operation at J-PARC, which is $B\rho \approx 100 \ Tm$.

In our case, if we wanted to extract a tune spread of 0.005, using this magnet we would need $\approx 50 amp$. But to get a rise time of $1.3 \mu sec$ would require very high voltage ($\approx 25 kV$). However, if we used 6 such magnets, but with 1 turn instead of 6 turns, the voltage required would drop to $\approx 5000V$. Figure 29, shows the amount of voltage needed

to ramp from 0 to $50 amp$ as a function of rise time, for the J-PARC RQ magnet.
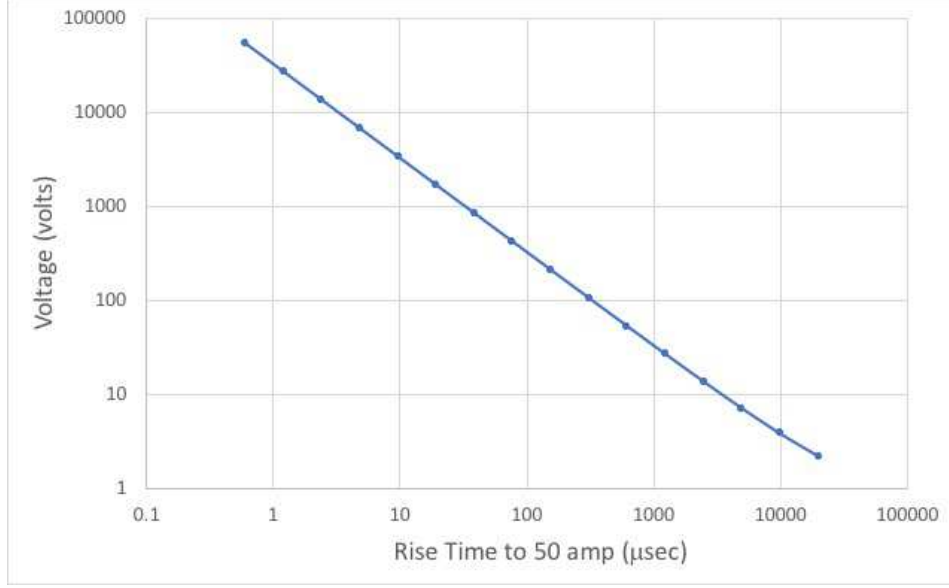

FIG. 29: Required voltages to get a given rise time for the J-PARC RQ quadrupole, to go from 0 to 50 amp.

Additionally, to extract the bunch train with uniform intensity per bunch, the shape of the down ramp for the tune quadrupole will need to follow an error function, not a simple linear ramp. This assumes the distribution in tune space reflects the distribution of the momentum spread and is roughly Gaussian. For this to work properly, the timing of the pulse and the height of the pulse should reflect the beam distribution as best as possible. The height of the pulse should be close to the amount of current to fully extract a given tune spread. The center of the pulse would correspond to the center of the tune distribution. So a measure of the tune of the non-resonant beam just before pulsing the quadrupole will give the central tune. A measure of the momentum spread will give the tune spread. As long as these parameters do not change, the pulse can be programmed, as long as it is locked to the RF, to come on at the correct energy and the rise of the pulse falls between two bunches.

VI. SUMMARY

Bunched beam slow extraction is essentially no different from normal slow extraction, except to keep the bunch structure the RF must remain on and active during extraction. To best control the extraction process the radial loop during extraction must be set to open, or else the beam position feedback will fight the radial control as beam is extracted. To extract short bursts of bunched beam using slow extraction is a challenging, but feasible method.

VII. APPENDIX

Sage Math 9.1 (Python) code is listed here.
From the Kobayashi Hamiltonian, the position of a particle, near the resonant tune, every three turns, is given by,

```
def CalcX(j, S, h, X, Xp):
    if j==0:
        Xn=X0
    else:
        Xn = X+(3.*S/2.)*(h*Xp + X*Xp)
    return Xn
```

The angle is given by,

```
def CalcXp(j, S, h, X, Xp):
    if j==0:
```

```
        Xpn = Xp0
    else:
        Xpn = Xp-(3.*S/2.)*(h*X-(-Xp*Xp + X*X)/2.)
    return Xpn
```

Given these two routines, we can perform resonant extraction calcuations. In this routine we launch a single particle with coordinates starting at X_0 and X_0', calculating new coordinates, $x[i]$ and $xp[i]$, and then repeatedly tracking, updating the coordinates every loop. The coordinates are placed in independent arrays (lists) and a stop condition allows the loop to terminate as soon as the particles arrive at a septum location, defined arbitrarily to be at $|x| = 0.00035\ m$. In a Sage Math environment, for simplicity, statments are commented in or out, depending on what is desired. In this case the code has a modulation of the tune turned on. This example fits to the AGS, where the revolution frequency is $2.7\ \mu sec$.

```
def runsx(S,h,Tz):
    debug = False
    x.clear()
    xp.clear()
    dx.clear()
    dxp.clear()
    #S=20.
    #h=0.0001
    Tref=Tz
    Frip=7.5
    AQ=1.00005*h*S  # value small compared to dQ,e.g. 0.0001*dQ
    for i in range(0,20000):
        #h-=h/3000000
        h = AQ*cos(2.*3.14159265358979*Tref*Frip)/S
        Tref += 0.0000027   # 2.7 usec/turn
        if i == 0:
            x.append(CalcX(i,S,h,X0,Xp0))
            xp.append(CalcXp(i,S,h,X0,Xp0))
            dx.append(0)
            dxp.append(0)
            if debug:
                print("i==0")
                print(x,xp)
        else:
            x.append(CalcX(i,S,h,x[i-1],xp[i-1]))
            xp.append(CalcXp(i,S,h,x[i-1],xp[i-1]))
            dx.append(x[i]-x[i-1])
            dxp.append(xp[i]-xp[i-1])
            if abs(x[i])>0.00035 or abs(xp[i])>0.00035:
                return i
            if debug:
                print("i>0")
                print(x,xp)
    return i
```

Here is an example off using the above routines. To plot the data, the two arrays, x and xp, need to be placed into a points array, where graphics parameters can be defined, and then that can be displayed using the show() function.

```
x = []
xp= []
dx= []
dxp= []
Nm=100000
S=20.
h=0.0001
X0 =-0.0000001-h
Xp0=-0.00000+h*1.732050807569
```

```
Nm=runsx(S,h,0)
print("A", Nm)
v1=points([(x[j],xp[j]) for j in range(Nm)], size=1)
dv1=points([(dx[j],dxp[j]) for j in range(Nm)], size=1, axes_labels=['dX','dX\''])
show(v1)
```

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