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Flattened multi-harmonic RF Buckets for acceleration of polarized protons in AGS

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1 Summary

The principle results of this report are summarized in this and the following three sections. The remaining sections provide details as needed and may be treated as appendices.

The RF voltages and phases that produce flattened multi-harmonic accelerating buckets are derived in Sections 5–8, 34–36, 52–54, and are shown to be calculable in terms of voltages V_1 and phases ϕ_1 that satisfy

$$V_1 \sin\phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c} \frac{dB}{dt} \right). \tag{1}$$

Here R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. For given programmed voltages V_1 and programmed guide field, the phase ϕ_1 can be calculated throughout the field cycle just as it is for the usual single-harmonic setup where one has voltages V_1' and phases ϕ_1' that satisfy

$$V_1' \sin\phi_1' = 2\pi R_s \rho_s \left(\frac{1}{c} \frac{dB}{dt} \right). \tag{2}$$

The phases ϕ_1 obtained from (1) can be used along with lookup tables to obtain the required multi-harmonic voltages and phases throughout the guide field cycle. With sufficiently short lookup times this could be done in real time during the cycle. The formulae needed to produce the lookup tables are summarized in Sections 18–19, 41, 59–60 for triple, double, and quad-harmonic buckets respectively.

Having obtained the voltages and phases that produce flattened multi-harmonic buckets, the widths and heights of matched bunches held in the buckets can be calculated and compared with those of bunches matched to a single-harmonic bucket. For the acceleration of polarized protons in AGS, the matched bunches are assumed to have a uniform distribution of particles with a longitudinal emittance of 1 eV-s. The parameters of the single and multi-harmonic buckets are adjusted so that all buckets have an area of 2 eV-s, twice that of the matched bunches. This is done for various values of $G\gamma$ and dB/dt.

The matched bunches in the flattened multi-harmonic buckets are longer (wider) than those in the single-harmonic bucket, thereby reducing the strength of the electromagnetic field seen by a given particle due to the other particles in the bunch. One calculable effect of this field is the incoherent tune shift. In Sections 29–32, 50, and 69, this is shown to be inversely proportional to the bunch width. We then have

$$\frac{\delta Q_n}{\delta Q_1} = \frac{W_1}{W_n} \tag{3}$$

where δQ_n and δQ_1 are the tune shifts in bunches matched to the n and single-harmonic buckets, respectively, and W_n and W_1 are the corresponding bunch widths. As shown in Sections 32, 50, and 69 we may also take

$$\frac{\delta Q_n}{\delta Q_1} = \frac{H_n}{H_1} \tag{4}$$

where H_n and H_1 are the bunch heights.

The ratios W_1/W_n and H_n/H_1 for double, triple, and quad-harmonic buckets are tabulated in Section 2 for polarized protons in AGS. These quantify the extent to which the incoherent tune shift is reduced in the flattened multi-harmonic buckets.

Using subscripts I and F to denote incoherent tune shifts, bunch widths, and $\beta\gamma^2$ at $G\gamma = 4.5$ and 14.0, respectively, we have

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{W_I\,(\beta\gamma^2)_I}{W_F\,(\beta\gamma^2)_F} = 0.591,\ 0.357,\ 0.273,\ 0.232 \tag{5}$$

for matched bunches sitting in single, double, triple, and quad-harmonic buckets, respectively, as shown in Sections 33, 51, and 70. This shows the extent to which the incoherent tune shift is reduced as polarized protons are accelerated in AGS.

The RF voltages and phases needed to produce the double, triple, and quad-harmonic buckets in AGS are summarized in Sections 3 and 4.

2 Matched bunch data summary

The following tables summarize the matched bunch data of Sections 22–27, 43–48, and 62–67. The bunch widths are given in degrees.

Table 1: Double-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_2	W_1/W_2	H_2/H_1
0.01	4.5	5.7784	195.4	222.9	0.877	0.7885
9.0	6.0	10.6883	86.79	122.8	0.707	0.634
18.0	7.5	16.9926	66.02	100.92	0.654	0.586
22.0	10.0	30.6069	50.94	83.89	0.607	0.544
25.0	12.5	48.1083	39.12	69.51	0.563	0.503
25.0	14.0	60.4754	31.61	59.73	0.529	0.473

Table 2: Triple-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_3	W_1/W_3	H_3/H_1
0.01	4.5	5.7784	195.4	239.6	0.816	0.703
9.0	6.0	10.6883	86.8	147.8	0.587	0.507
18.0	7.5	16.9926	66.0	126.5	0.522	0.448
22.0	10.0	30.6069	50.9	109.4	0.466	0.400
25.0	12.5	48.1083	39.1	94.3	0.415	0.355
25.0	14.0	60.4754	31.6	83.8	0.377	0.323

Table 3: Quad-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_4	W_1/W_4	H_4/H_1
0.01	4.5	5.7784	195.4	251.2	0.778	0.655
9.0	6.0	10.6883	86.79	166.3	0.522	0.439
18.0	7.5	16.9926	66.02	145.9	0.452	0.380
22.0	10.0	30.6069	50.94	129.2	0.394	0.331
25.0	12.5	48.1083	39.12	114.3	0.342	0.287
25.0	14.0	60.4754	31.61	103.6	0.305	0.256

3 Multi-harmonic bucket voltage summary

The following tables summarize the multi-harmonic voltage data of Sections 22–27, 43–48, and 62–67. Formulae for the voltages are derived in Sections 6–8, 34–36, and 52–54. The voltage V_1' needed to make a 2 eV-s single-harmonic bucket is given in kV. The remaining voltages follow from the tabulated ratios.

Table 4: Double-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1
843.9	0.01	4.5	7.498	0.7566	0.49995
1170.7	9.0	6.0	89.942	1.1042	0.3464
1489.0	18.0	7.5	152.715	1.2006	0.3122
2011.5	22.0	10.0	171.219	1.2583	0.2915
2529.4	25.0	12.5	184.954	1.2934	0.27775
2838.9	25.0	14.0	180.410	1.3104	0.2702

Table 5: Triple-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1	V_3/V_1
843.9	0.01	4.5	7.498	0.6270	0.7999	0.2000
1170.7	9.0	6.0	89.942	1.1656	0.5119	0.1096
1489.0	18.0	7.5	152.715	1.3049	0.4708	0.0949
2011.5	22.0	10.0	171.219	1.3880	0.4478	0.0863
2529.4	25.0	12.5	184.954	1.4390	0.4330	0.0805
2838.9	25.0	14.0	180.410	1.4640	0.4249	0.0772

Table 6: Quad-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1	V_3/V_1	V_4/V_1
843.9	0.01	4.5	7.498	0.5440	0.9997	0.4284	0.07140
1170.7	9.0	6.0	89.942	1.2048	0.6092	0.2163	0.03304
1489.0	18.0	7.5	152.715	1.3681	0.5686	0.1907	0.02804
2011.5	22.0	10.0	171.219	1.4654	0.5469	0.1763	0.02513
2529.4	25.0	12.5	184.954	1.5254	0.5331	0.1669	0.02317
2838.9	25.0	14.0	180.410	1.5552	0.5255	0.1616	0.02204

4 Multi-harmonic bucket phase summary

The following tables summarize the multi-harmonic phase data of Sections 22–27, 43–48, and 62–67. Formulae for the phases are derived in Sections 6–8, 34–36, and 52–54. The phases are tabulated in degrees.

Table 7: Double-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2
0.01	4.5	0.52657	0.69601	0.9280	0.6960
9.0	6.0	43.59258	38.6424	56.3678	37.90275
18.0	7.5	54.3111	42.5683	64.4165	41.2951
22.0	10.0	62.30122	44.7193	69.7478	42.9610
25.0	12.5	68.6577	46.0637	73.7703	43.8738
25.0	14.0	72.72346	46.7777	76.3130	44.2976

Table 8: Triple-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2	ψ_3
0.01	4.5	0.52657	0.8399	1.2599	0.9449	1.1199
9.0	6.0	43.59258	36.2684	62.5425	40.5938	51.6492
18.0	7.5	54.3111	38.4927	69.0067	42.7591	55.3468
22.0	10.0	62.30122	39.63665	73.1121	43.7449	57.2210
25.0	12.5	68.6580	40.3377	76.1527	44.2743	58.3137
25.0	14.0	72.7235	40.7118	78.0647	44.5229	58.8580

Table 9: Quad-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2	ψ_3	ψ_4
0.01	4.5	0.52657	0.9679	1.5488	1.1615	1.3766	1.4519
9.0	6.0	43.59258	34.9115	66.3048	41.9421	53.8989	58.8879
18.0	7.5	54.3111	36.4174	71.7797	43.4592	56.6599	62.4758
22.0	10.0	62.3012	37.1726	75.1831	44.1237	57.9947	64.3374
25.0	12.5	68.6577	37.6330	77.6775	44.4776	58.7570	65.4641
25.0	14.0	72.7235	37.8801	79.2400	44.6452	59.1360	66.0500

5 Hamiltonian for multi-harmonic RF bucket

The hamiltonian for particle motion in an RF bucket is [1]

$$H(\phi, W) = \frac{1}{2} aW^2 + U(\phi) \tag{6}$$

where

$$a = \frac{h^2 c^2 \eta_s}{R_s^2 E_s}, \quad \eta_s = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}, \quad E_s = mc^2 \gamma_s \tag{7}$$

and

$$W = \frac{E - E_s}{h\omega_s}, \quad \omega_s = \frac{c\beta_s}{R_s}. \tag{8}$$

The "potential" U satisfies

$$\frac{\partial U}{\partial \phi} = -F(\phi + \phi_s) + F(\phi_s) \tag{9}$$

where the "force" function

$$F(x) = A_1 \sin x - A_2 \sin(2x - 2\psi_2) + A_3 \sin(3x - 3\psi_3) - A_4 \sin(4x - 4\psi_4) \tag{10}$$

for quad-harmonic buckets, and

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h}, \quad A_4 = \frac{eQV_2}{2\pi h}, \quad A_4 = \frac{eQV_3}{2\pi h}. \tag{11}$$

The subscript s denotes parameters of the synchronous particle.

ϕ_s is the synchronous phase and ϕ is the deviation from ϕ_s.

ψ_2, ψ_3, and ψ_4 are adjustable phase offsets.

eQ is the charge of the particle and e is the elementary electric charge.

h is the fundamental harmonic number.

V_1, V_2, V_3, and V_4 are the adjustable amplitudes of the first, second, third, and fourth harmonic voltages. The associated RF frequencies are hf, $2hf$, $3hf$, and $4hf$, respectively, where

$$f = \frac{c\beta_s}{2\pi R_s} \tag{12}$$

is the revolution frequency of the synchronous particle.

For single, double, triple, and quad-harmonic RF buckets we have respectively

$$V_1 \neq 0, \quad V_2 = 0, \quad V_3 = 0, \quad V_4 = 0 \quad \text{(single)} \tag{13}$$

$$V_1 \neq 0, \quad V_2 \neq 0, \quad V_3 = 0 \quad V_4 = 0 \quad \text{(double)} \tag{14}$$

$$V_1 \neq 0, \quad V_2 \neq 0, \quad V_3 \neq 0, \quad V_4 = 0 \quad \text{(triple)} \tag{15}$$

$$V_1 \neq 0, \quad V_2 \neq 0, \quad V_3 \neq 0, \quad V_4 \neq 0 \quad \text{(quad).} \tag{16}$$

6 Potential for a triple-harmonic bucket

Defining

$$\psi = \phi + \phi_s \tag{17}$$

we have

$$U = A_1 \cos\psi - \frac{1}{2}A_2\cos(2\psi - 2\psi_2) + \frac{1}{3}A_3\cos(3\psi - 3\psi_3) + C\psi \tag{18}$$

$$\frac{\partial U}{\partial \psi} = -A_1 \sin\psi + A_2 \sin(2\psi - 2\psi_2) - A_3 \sin(3\psi - 3\psi_3) + C \tag{19}$$

$$\frac{\partial^2 U}{\partial \psi^2} = -A_1 \cos\psi + 2A_2 \cos(2\psi - 2\psi_2) - 3A_3 \cos(3\psi - 3\psi_3) \tag{20}$$

$$\frac{\partial^3 U}{\partial \psi^3} = A_1 \sin\psi - 4A_2 \sin(2\psi - 2\psi_2) + 9A_3 \sin(3\psi - 3\psi_3) \tag{21}$$

$$\frac{\partial^4 U}{\partial \psi^4} = A_1 \cos\psi - 8A_2 \cos(2\psi - 2\psi_2) + 27A_3 \cos(3\psi - 3\psi_3) \tag{22}$$

$$\frac{\partial^5 U}{\partial \psi^5} = -A_1 \sin\psi + 16A_2 \sin(2\psi - 2\psi_2) - 81A_3 \sin(3\psi - 3\psi_3) \tag{23}$$

$$\frac{\partial^6 U}{\partial \psi^6} = -A_1 \cos\psi + 32A_2 \cos(2\psi - 2\psi_2) - 243A_3 \cos(3\psi - 3\psi_3) \tag{24}$$

and so on, where

$$C = A_1 \sin\phi_s - A_2 \sin(2\phi_s - 2\psi_2) + A_3 \sin(3\phi_s - 3\psi_3). \tag{25}$$

Using integer superscripts to denote the number of differentiations with respect to ψ, we have

$$U^1(\phi_s) = 0 \tag{26}$$

$$U^2(\phi_s) = -A_1 \cos\phi_s + 2A_2 \cos(2\phi_s - 2\psi_2) - 3A_3 \cos(3\phi_s - 3\psi_3) \tag{27}$$

$$U^3(\phi_s) = A_1 \sin\phi_s - 4A_2 \sin(2\phi_s - 2\psi_2) + 9A_3 \sin(3\phi_s - 3\psi_3) \tag{28}$$

$$U^4(\phi_s) = A_1 \cos\phi_s - 8A_2 \cos(2\phi_s - 2\psi_2) + 27A_3 \cos(3\phi_s - 3\psi_3) \tag{29}$$

$$U^5(\phi_s) = -A_1 \sin\phi_s + 16A_2 \sin(2\phi_s - 2\psi_2) - 81A_3 \sin(3\phi_s - 3\psi_3) \tag{30}$$

$$U^6(\phi_s) = -A_1 \cos\phi_s + 32A_2 \cos(2\phi_s - 2\psi_2) - 243A_3 \cos(3\phi_s - 3\psi_3). \tag{31}$$

7 Conditions for a flattened triple-harmonic bucket

For a flattened RF bucket we want derivatives

$$U^1(\phi_s) = U^2(\phi_s) = U^3(\phi_s) = U^4(\phi_s) = U^5(\phi_s) = 0 \tag{32}$$

which gives

$$A_1 C_1 = 2A_2 C_2 - 3A_3 C_3 \tag{33}$$

$$-A_1 S_1 = -4A_2 S_2 + 9A_3 S_3 \tag{34}$$

$$-A_1 C_1 = -8A_2 C_2 + 27A_3 C_3 \tag{35}$$

$$A_1 S_1 = 16A_2 S_2 - 81A_3 S_3 \tag{36}$$

where

$$C_1 = \cos\phi_s, \quad C_2 = \cos(2\phi_s - 2\psi_2), \quad C_3 = \cos(3\phi_s - 3\psi_3) \tag{37}$$

$$S_1 = \sin\phi_s, \quad S_2 = \sin(2\phi_s - 2\psi_2), \quad S_3 = \sin(3\phi_s - 3\psi_3). \tag{38}$$

It then follows from (33) and (35) that

$$-6A_2 C_2 + 24A_3 C_3 = 0 \tag{39}$$

$$4A_3 C_3 = A_2 C_2 \tag{40}$$

$$4A_1 C_1 = 5A_2 C_2 = 20A_3 C_3 \tag{41}$$

and

$$A_2 C_2 = \frac{4}{5} A_1 C_1, \quad A_3 C_3 = \frac{1}{5} A_1 C_1. \tag{42}$$

Similarly from (34) and (36) we have

$$12A_2 S_2 - 72A_3 S_3 = 0 \tag{43}$$

$$6A_3S_3 = A_2S_2 \tag{44}$$

$$2A_1S_1 = 5A_2S_2 = 30A_3S_3 \tag{45}$$

and

$$A_2S_2 = \frac{2}{5}A_1S_1, \quad A_3S_3 = \frac{1}{15}A_1S_1. \tag{46}$$

Finally, dividing the equations in (46) by those in (42) we have

$$\frac{S_2}{C_2} = \frac{1}{2}\frac{S_1}{C_1}, \quad \frac{S_3}{C_3} = \frac{1}{3}\frac{S_1}{C_1} \tag{47}$$

which give

$$\tan(2\phi_s - 2\psi_2) = \frac{1}{2}\tan\phi_s, \quad \tan(3\phi_s - 3\psi_3) = \frac{1}{3}\tan\phi_s. \tag{48}$$

Thus, if the value of ϕ_s is given, we can calculate values for the phases ψ_2 and ψ_3 and for the ratios A_2/A_1 and A_3/A_1. Specifically one has

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\}, \quad \psi_3 = \phi_s - \frac{1}{3}\arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{49}$$

and, according to (42) and (46),

$$\frac{A_2}{A_1} = \frac{4}{5}\left\{\frac{\cos\phi_s}{\cos(2\phi_s - 2\psi_2)}\right\} = \frac{2}{5}\left\{\frac{\sin\phi_s}{\sin(2\phi_s - 2\psi_2)}\right\} \tag{50}$$

$$\frac{A_3}{A_1} = \frac{1}{5}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{15}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\}. \tag{51}$$

8 Synchronous phase for flattened triple-harmonic bucket

As a function of phase, the RF voltage must satisfy

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right)U^1(\psi) \tag{52}$$

where

$$-U^1(\psi) = A_1\sin\psi - A_2\sin(2\psi - 2\psi_2) + A_3\sin(3\psi - 3\psi_3) - C \tag{53}$$

$$C = A_1\sin\phi_s - A_2\sin(2\phi_s - 2\psi_2) + A_3\sin(3\phi_s - 3\psi_3). \tag{54}$$

Thus, using

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h}, \quad A_3 = \frac{eQV_3}{2\pi h} \tag{55}$$

we have

$$V(\psi) = V_1 \sin\psi - V_2 \sin(2\psi - 2\psi_2) + V_3 \sin(3\psi - 3\psi_3). \tag{56}$$

The synchronous phase ϕ_s must satisfy

$$V(\phi_s) = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{57}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. Here it is useful to define phase ϕ_1 such that

$$V_1 \sin\phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right). \tag{58}$$

We then have

$$V_1 \sin\phi_s - V_2 \sin(2\phi_s - 2\psi_2) + V_3 \sin(3\phi_s - 3\psi_3) = V_1 \sin\phi_1 \tag{59}$$

which gives

$$\sin\phi_s = \sin\phi_1 + \frac{V_2}{V_1}\,\sin(2\phi_s - 2\psi_2) - \frac{V_3}{V_1}\,\sin(3\phi_s - 3\psi_3). \tag{60}$$

Using

$$\frac{V_2}{V_1} = \frac{A_2}{A_1}, \quad \frac{V_3}{V_1} = \frac{A_3}{A_1} \tag{61}$$

we then have

$$S_1 = \sin\phi_1 + \frac{A_2}{A_1}\,S_2 - \frac{A_3}{A_1}\,S_3 \tag{62}$$

where, according to the conditions (46) for a flattened bucket,

$$\frac{A_2}{A_1} = \frac{2}{5}\left\{\frac{S_1}{S_2}\right\}, \quad \frac{A_3}{A_1} = \frac{1}{15}\left\{\frac{S_1}{S_3}\right\}. \tag{63}$$

Thus

$$S_1 = \sin\phi_1 + \frac{2}{5}\,S_1 - \frac{1}{15}\,S_1 \tag{64}$$

$$S_1\left\{1 - \frac{2}{5} + \frac{1}{15}\right\} = \sin\phi_1 \tag{65}$$

$$S_1 = \frac{3}{2}\sin\phi_1 \tag{66}$$

and therefore

$$\phi_s = \arcsin\left(\frac{3}{2}\sin\phi_1\right). \tag{67}$$

Here we see that in order to have a real synchronous phase we must have

$$0 \le \sin\phi_1 \le \frac{2}{3}. \tag{68}$$

This constraint is satisfied if

$$0 \le \phi_1 \le 41.8103^\circ \tag{69}$$

or

$$138.1897^\circ \le \phi_1 < 180^\circ. \tag{70}$$

If a particular ϕ_1 is given, we can obtain ϕ_s from (67). This in turn can be used in (49) to obtain ψ_2 and ψ_3. Finally, ϕ_s, ψ_2, and ψ_3 can be used in (50) and (51) to obtain the ratios V_2/V_1 and V_3/V_1. The voltage V_1 is given by (58). We therefore have the result that the phases ϕ_s, ψ_2, and ψ_3, and the ratios V_2/V_1 and V_3/V_1 are completely determined by the phase ϕ_1.

9 Normalized triple-harmonic voltage and potential

It is convenient to normalize the voltage and potential so that they are dimensionless and completely determined by the phase ϕ_1. We define normalized voltage

$$\mathcal{V}(\psi) = \frac{1}{V_1}\, V(\psi) \tag{71}$$

and normalized potential

$$\mathcal{U}(\psi) = \frac{1}{A_1}\left\{U(\psi_u) - U(\psi)\right\} \tag{72}$$

with derivatives

$$\mathcal{U}^m(\psi) = -\frac{1}{A_1}\, U^m(\psi). \tag{73}$$

Here

$$V(\psi) = V_1\sin\psi - V_2\sin(2\psi - 2\psi_2) + V_3\sin(3\psi - 3\psi_3) \tag{74}$$

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right) U^1(\psi) \tag{75}$$

$$U(\psi) = A_1 \cos\psi - \frac{1}{2} A_2 \cos(2\psi - 2\psi_2) + \frac{1}{3} A_3 \cos(3\psi - 3\psi_3) + C\psi \tag{76}$$

$$U^1(\psi) = -A_1 \sin\psi + A_2 \sin(2\psi - 2\psi_2) - A_3 \sin(3\psi - 3\psi_3) + C \tag{77}$$

$$U^2(\psi) = -A_1 \cos\psi + 2A_2 \cos(2\psi - 2\psi_2) - 3A_3 \cos(3\psi - 3\psi_3) \tag{78}$$

$$C = A_1 \sin\phi_s - A_2 \sin(2\phi_s - 2\psi_2) + A_3 \sin(3\phi_s - 3\psi_3) \tag{79}$$

and therefore

$$\mathcal{V}(\psi) - \mathcal{V}(\phi_s) = \mathcal{U}^1(\psi) \tag{80}$$

$$\mathcal{U}(\psi) = \mathcal{D} - \cos\psi + \frac{1}{2} R_2 \cos(2\psi - 2\psi_2) - \frac{1}{3} R_3 \cos(3\psi - 3\psi_3) - \mathcal{C}\psi \tag{81}$$

$$\mathcal{U}^1(\psi) = \sin\psi - R_2 \sin(2\psi - 2\psi_2) + R_3 \sin(3\psi - 3\psi_3) - \mathcal{C} \tag{82}$$

$$\mathcal{U}^2(\psi) = -\cos\psi + 2R_2 \cos(2\psi - 2\psi_2) - 3R_3 \cos(3\psi - 3\psi_3) \tag{83}$$

where

$$\mathcal{D} = \cos\psi_u - \frac{1}{2} R_2 \cos(2\psi_u - 2\psi_2) + \frac{1}{3} R_3 \cos(3\psi_u - 3\psi_3) + \mathcal{C}\psi_u \tag{84}$$

$$\mathcal{C} = \sin\phi_s - R_2 \sin(2\phi_s - 2\psi_2) + R_3 \sin(3\phi_s - 3\psi_3) \tag{85}$$

and

$$R_2 = \frac{A_2}{A_1}, \quad R_3 = \frac{A_3}{A_1}. \tag{86}$$

The phase ψ_u in these equations is the unstable fixed point phase associated with oscillations about the synchronous phase ϕ_s. It satisfies

$$\mathcal{U}(\psi_u) = 0, \quad \mathcal{U}^1(\psi_u) = 0. \tag{87}$$

Below transition one has

$$0 < \phi_s < \psi_u \tag{88}$$

and

$$\mathcal{U}^2(\psi_u) < 0. \tag{89}$$

There is an additional phase

$$\psi_e < \phi_s < \psi_u \tag{90}$$

that satisfies

$$\mathcal{U}(\psi_e) = \mathcal{U}(\psi_u) = 0. \tag{91}$$

The equations of this and the previous section show that the normalized voltage and potential are completely determined by the phase ϕ_1. The phases ψ_u and ψ_e are also completely determined by ϕ_1.

10 Triple-harmonic bucket width and area

The RF bucket associated with the stable fixed point phase ϕ_s extends from ψ_e to ψ_u. It is defined by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_u) - U(\psi)\right\} \tag{92}$$

$$a = \frac{h^2c^2\eta_s}{R_s^2 E_s}, \quad \eta_s = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}, \quad E_s = mc^2\gamma_s. \tag{93}$$

The phase width of the bucket is

$$\Delta\psi = |\psi_u - \psi_e|\,. \tag{94}$$

In terms of the normalized potential (72) we have

$$W^2(\psi) = \frac{2}{a}\,A_1\,\mathcal{U}(\psi) \tag{95}$$

and

$$W(\psi) = \pm\left(\frac{2A_1}{|a|}\right)^{1/2} |\mathcal{U}(\psi)|^{1/2}\,. \tag{96}$$

Here

$$A_1 = \frac{eQV_1}{2\pi h}, \quad \frac{1}{|a|} = \frac{R_s^2 E_s}{h^2c^2|\eta_s|} \tag{97}$$

which gives

$$\left(\frac{2A_1}{|a|}\right)^{1/2} = \frac{R_s}{hc}\left\{\frac{eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} = \frac{B_1}{8\sqrt{2}} \tag{98}$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{99}$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1. Thus we have

$$W(\psi) = \pm\frac{B_1}{8\sqrt{2}}\,|\mathcal{U}(\psi)|^{1/2}\,. \tag{100}$$

The normalized triple-harmonic RF bucket is defined by the curves

$$\mathcal{W}(\psi) = \pm\,|\mathcal{U}(\psi)|^{1/2} \tag{101}$$

which are dimensionless and completely determined by the phase ϕ_1.

The area of the triple-harmonic bucket is

$$\mathcal{A}_3 = 2\int_{\psi_e}^{\psi_u} |W(\psi)|\ d\psi \tag{102}$$

where

$$|W(\psi)| = \frac{B_1}{8\sqrt{2}}\ |\mathcal{U}(\psi)|^{1/2}. \tag{103}$$

Defining normalized bucket area

$$\mathcal{B}_3 = 2\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\, d\psi \tag{104}$$

we then have

$$\mathcal{A}_3 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_3. \tag{105}$$

The normalized bucket area is dimensionless and is completely determined by the phase ϕ_1.

11 Bunch matched to triple-harmonic bucket

Consider a particle moving along the boundary of a bunch matched to the triple-harmonic bucket and let ψ_R be the right turning point phase of the boundary. Below transition one has

$$\psi_e < \phi_s < \psi_R < \psi_u. \tag{106}$$

The corresponding left turning point phase ψ_L satisfies

$$\psi_e < \psi_L < \phi_s < \psi_R < \psi_u \tag{107}$$

and

$$\mathcal{U}(\psi_L) = \mathcal{U}(\psi_R). \tag{108}$$

Here, as shown in Section 9,

$$\mathcal{U}(\psi) = \frac{1}{A_1}\left\{U(\psi_u) - U(\psi)\right\} \tag{109}$$

where

$$\frac{1}{A_1}U(\psi_u) = \cos\psi_u - \frac{1}{2}R_2\cos(2\psi_u - 2\psi_2) + \frac{1}{3}R_3\cos(3\psi_u - 3\psi_3) + \mathcal{C}\psi_u \tag{110}$$

$$\frac{1}{A_1}\,U(\psi) = \cos\psi - \frac{1}{2}R_2\cos(2\psi - 2\psi_2) + \frac{1}{3}R_3\cos(3\psi - 3\psi_3) + \mathcal{C}\psi \quad (111)$$

$$\mathcal{C} = \sin\phi_s - R_2\sin(2\phi_s - 2\psi_2) + R_3\sin(3\phi_s - 3\psi_3) \quad (112)$$

and

$$R_2 = \frac{A_2}{A_1}, \quad R_3 = \frac{A_3}{A_1}. \quad (113)$$

Given either turning point phase, one can solve (108) numerically to obtain the other. The phase width of the bunch is then

$$\Delta\psi = \psi_R - \psi_L. \quad (114)$$

Alternatively, if $\Delta\psi$ is given, one can solve

$$\mathcal{U}(\psi_R - \Delta\psi) = \mathcal{U}(\psi_R) \quad (115)$$

to obtain ψ_R and $\psi_L = \psi_R - \Delta\psi$.

Below transition the boundary of the matched bunch is given by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\,\{U(\psi_R) - U(\psi)\} \quad (116)$$

and

$$\psi_L \le \psi \le \psi_R. \quad (117)$$

In terms of the normalized potential (109) we have

$$W^2(\psi) = \frac{2}{a}\,A_1\,\{\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\} \quad (118)$$

and

$$W(\psi) = \pm\left(\frac{2A_1}{|a|}\right)^{1/2}|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}. \quad (119)$$

According to (98) and (99) we then have

$$W(\psi) = \pm\frac{B_1}{8\sqrt{2}}\,|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2} \quad (120)$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} \quad (121)$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1.

The normalized bunch boundary is defined by the curves

$$\mathcal{W}(\psi) = \pm\left|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right|^{1/2} \tag{122}$$

where

$$\psi_L \leq \psi \leq \psi_R. \tag{123}$$

The area of the bunch is

$$B = \frac{B_1}{8\sqrt{2}}\,\mathcal{B} \tag{124}$$

where

$$\mathcal{B} = 2\int_{\psi_L}^{\psi_R} \left|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right|^{1/2} d\psi \tag{125}$$

is defined to be the normalized area. The normalized bunch boundary and area are dimensionless and completely determined by the turning point phase ψ_R and the phase ϕ_1.

12 Potential for single-harmonic bucket

For a single-harmonic bucket we have

$$\frac{\partial U}{\partial \phi} = -F(\phi + \phi_1') + F(\phi_1') \tag{126}$$

where

$$F(x) = A_1' \sin x, \quad A_1' = \frac{eQV_1'}{2\pi h} \tag{127}$$

and ϕ_1' is the synchronous phase. Defining, as before,

$$\psi = \phi + \phi_1' \tag{128}$$

we have

$$U = A_1' \cos\psi + C\psi \tag{129}$$

$$\frac{\partial U}{\partial \psi} = -A_1' \sin\psi + C \tag{130}$$

$$\frac{\partial^2 U}{\partial \psi^2} = -A_1' \cos\psi \tag{131}$$

where

$$C = A_1' \sin\phi_1'. \tag{132}$$

Using integer superscripts to denote the number of differentiations with respect to ψ, we have

$$U^1(\phi_1') = 0 \tag{133}$$

$$U^2(\phi_1') = -A_1' \cos \phi_1'. \tag{134}$$

13 Synchronous phase for single-harmonic bucket

The RF voltage must satisfy

$$V(\psi) - V(\phi_1') = -\left(\frac{2\pi h}{eQ}\right) U^1(\psi) \tag{135}$$

where here

$$-U^1(\psi) = A_1' \sin \psi - A_1' \sin \phi_1'. \tag{136}$$

Thus, using

$$A_1' = \frac{eQV_1'}{2\pi h} \tag{137}$$

we have

$$V(\psi) = V_1' \sin \psi. \tag{138}$$

The synchronous phase ϕ_1' must satisfy

$$V_1' \sin \phi_1' = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{139}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. According to (58) we also have, for the triple-harmonic bucket,

$$V_1 \sin \phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{140}$$

and therefore

$$V_1' \sin \phi_1' = V_1 \sin \phi_1. \tag{141}$$

14 Normalized single-harmonic voltage and potential

It is convenient to normalize the voltage and potential so that they are dimensionless and completely determined by the phase ϕ_1'. We define normalized voltage

$$\mathcal{V}(\psi) = \frac{1}{V_1'} V(\psi) \tag{142}$$

and normalized potential

$$\mathcal{U}(\psi) = \frac{1}{A_1'} \{U(\psi_u) - U(\psi)\} \tag{143}$$

with derivatives

$$\mathcal{U}^m(\psi) = -\frac{1}{A_1'} U^m(\psi). \tag{144}$$

Here

$$V(\psi) = V_1' \sin\psi \tag{145}$$

$$U(\psi_u) = A_1' \cos\psi_u + C\psi_u \tag{146}$$

$$U(\psi) = A_1' \cos\psi + C\psi \tag{147}$$

$$U^1(\psi) = -A_1' \sin\psi + C \tag{148}$$

$$U^2(\psi) = -A_1' \cos\psi \tag{149}$$

$$C = A_1' \sin\phi_1' \tag{150}$$

and we have

$$\mathcal{V}(\psi) = \sin\psi \tag{151}$$

$$\mathcal{U}(\psi) = \cos\psi_u - \cos\psi + (\psi_u - \psi)\sin\phi_1' \tag{152}$$

$$\mathcal{U}^1(\psi) = \sin\psi - \sin\phi_1' \tag{153}$$

$$\mathcal{U}^2(\psi) = \cos\psi. \tag{154}$$

The phase ψ_u is the unstable fixed point phase associated with oscillations about the synchronous phase ϕ_1'. It satisfies

$$\mathcal{U}(\psi_u) = 0, \quad \mathcal{U}^1(\psi_u) = 0 \tag{155}$$

and therefore

$$\psi_u = \pi - \phi_1' \tag{156}$$

which gives

$$\mathcal{U}(\psi) = -\cos\phi_1' - \cos\psi + (\pi - \phi_1' - \psi)\sin\phi_1'. \tag{157}$$

Thus the normalized potential is completely determined by the phase ϕ_1'.

Below transition one has

$$0 < \phi_1' < \psi_u \tag{158}$$

and

$$\mathcal{U}^2(\psi_u) < 0. \tag{159}$$

There is an additional phase

$$\psi_e < \phi_1' < \psi_u \tag{160}$$

that satisfies

$$\mathcal{U}(\psi_e) = \mathcal{U}(\psi_u) = 0. \tag{161}$$

Thus phases ψ_e and ψ_u are completely determined by the phase ϕ_1'.

15 Single-harmonic bucket width and area

The RF bucket associated with the stable fixed point phase ϕ_1' extends from ψ_e to ψ_u. It is defined by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_u) - U(\psi)\right\} \tag{162}$$

$$a = \frac{h^2 c^2 \eta_s}{R_s^2 E_s}, \quad \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}, \quad E_s = mc^2\gamma_s. \tag{163}$$

The phase width of the bucket is

$$\Delta\psi = |\psi_u - \psi_e|. \tag{164}$$

In terms of the normalized potential (143) we have

$$W^2(\psi) = \frac{2}{a} A_1' \,\mathcal{U}(\psi) \tag{165}$$

and

$$W(\psi) = \pm\left(\frac{2A_1'}{|a|}\right)^{1/2} |\mathcal{U}(\psi)|^{1/2}. \tag{166}$$

Here

$$A_1' = \frac{eQV_1'}{2\pi h}, \quad \frac{1}{|a|} = \frac{R_s^2 E_s}{h^2 c^2 |\eta_s|} \tag{167}$$

which gives

$$\left(\frac{2A_1'}{|a|}\right)^{1/2} = \frac{R_s}{hc}\left\{\frac{eQV_1' E_s}{\pi h|\eta_s|}\right\}^{1/2} = \frac{B_1'}{8\sqrt{2}} \tag{168}$$

where

$$B_1' = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1'E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{169}$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1'. Thus we have

$$W(\psi) = \pm\frac{B_1'}{8\sqrt{2}}\;|\mathcal{U}(\psi)|^{1/2}\,. \tag{170}$$

For comparison with the normalized triple-harmonic bucket we define

$$\mathcal{W}(\psi) = \left(\frac{B_1}{8\sqrt{2}}\right)^{-1} W(\psi) \tag{171}$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2}. \tag{172}$$

We then have

$$\mathcal{W}(\psi) = \pm\left(\frac{V_1'}{V_1}\right)^{1/2}|\mathcal{U}(\psi)|^{1/2} \tag{173}$$

where

$$\mathcal{U}(\psi) = -\cos\phi_1' - \cos\psi + (\pi - \phi_1' - \psi)\sin\phi_1'. \tag{174}$$

Here, according to (141), we have

$$\frac{V_1'}{V_1} = \frac{\sin\phi_1}{\sin\phi_1'} \tag{175}$$

therefore

$$\mathcal{W}(\psi) = \pm\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}|\mathcal{U}(\psi)|^{1/2}\,. \tag{176}$$

These curves define the normalized single-harmonic bucket and show that it is completely determined by the phases ϕ_1 and ϕ_1'.

The area of the single-harmonic bucket is

$$\mathcal{A}_1 = 2\int_{\psi_e}^{\psi_u}|W(\psi)|\;d\psi \tag{177}$$

where

$$|W(\psi)| = \frac{B_1'}{8\sqrt{2}}\;|\mathcal{U}(\psi)|^{1/2}\,. \tag{178}$$

Thus we can write

$$\mathcal{A}_1 = \frac{B_1'}{8\sqrt{2}}\,\mathcal{B}_1' \tag{179}$$

where

$$\mathcal{B}_1' = 2\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\,d\psi. \tag{180}$$

For comparison with the normalized triple-harmonic area we define normalized single-harmonic bucket area

$$\mathcal{B}_1 = \left(\frac{B_1}{8\sqrt{2}}\right)^{-1}\mathcal{A}_1. \tag{181}$$

Using (175) we then have

$$\mathcal{B}_1 = \left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}\mathcal{B}_1' \tag{182}$$

which shows that the normalized single-harmonic bucket area is completely determined by the phases ϕ_1 and ϕ_1'.

16 Bunch matched to single-harmonic bucket

Consider a particle moving along the boundary of a bunch matched to the RF bucket and let ψ_R be the right turning point phase of the boundary. Below transition one has

$$\psi_e < \phi_1' < \psi_R < \psi_u. \tag{183}$$

The corresponding left turning point phase ψ_L satisfies

$$\psi_e < \psi_L < \phi_1' < \psi_R < \psi_u \tag{184}$$

and

$$\mathcal{U}(\psi_L) = \mathcal{U}(\psi_R) \tag{185}$$

where

$$\mathcal{U}(\psi_L) = -\cos\phi_1' - \cos\psi_L + (\pi - \phi_1' - \psi_L)\sin\phi_1' \tag{186}$$

$$\mathcal{U}(\psi_R) = -\cos\phi_1' - \cos\psi_R + (\pi - \phi_1' - \psi_R)\sin\phi_1'. \tag{187}$$

Thus we must have

$$\cos\psi_R - \cos\psi_L = -(\psi_R - \psi_L)\sin\phi_1'. \tag{188}$$

Given either turning point phase, one can solve (188) to obtain the other. The phase width of the bunch is then

$$\Delta\psi = \psi_R - \psi_L. \tag{189}$$

Alternatively, if $\Delta\psi$ is given, one can solve

$$\cos\psi_R - \cos(\psi_R - \Delta\psi) = -(\psi_R - \psi_L)\sin\phi_1' \tag{190}$$

to obtain ψ_R and $\psi_L = \psi_R - \Delta\psi$. The solution is [2]

$$\psi_R = \frac{\Delta\psi}{2} + \arcsin\left\{\frac{\Delta\psi\sin\phi_1'}{2\sin(\Delta\psi/2)}\right\} \tag{191}$$

$$\psi_L = -\frac{\Delta\psi}{2} + \arcsin\left\{\frac{\Delta\psi\sin\phi_1'}{2\sin(\Delta\psi/2)}\right\}. \tag{192}$$

Below transition the boundary of the matched bunch is given by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_R) - U(\psi)\right\} \tag{193}$$

and

$$\psi_L \le \psi \le \psi_R. \tag{194}$$

In terms of the normalized potential (143) we have

$$W^2(\psi) = \frac{2}{a}\,A_1'\left\{\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right\} \tag{195}$$

which gives

$$W(\psi) = \pm\left(\frac{2A_1'}{|a|}\right)^{1/2}\left|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right|^{1/2}. \tag{196}$$

According to (168) and (169) we then have

$$W(\psi) = \pm\frac{B_1'}{8\sqrt{2}}\left|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right|^{1/2} \tag{197}$$

where

$$B_1' = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1'E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{198}$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1'.

For comparison with the normalized bunch matched to the triple-harmonic bucket, the normalized boundary of the bunch matched to the single-harmonic bucket is defined by the curves

$$\mathcal{W}(\psi) = \pm\left(\frac{B_1}{8\sqrt{2}}\right)^{-1} W(\psi) \tag{199}$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2}. \tag{200}$$

We then have

$$\mathcal{W}(\psi) = \pm\left(\frac{V_1'}{V_1}\right)^{1/2}|\,\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2} \tag{201}$$

where

$$\mathcal{U}(\psi) = -\cos\phi_1' - \cos\psi + (\pi - \phi_1' - \psi)\sin\phi_1' \tag{202}$$

$$\mathcal{U}(\psi_R) = -\cos\phi_1' - \cos\psi_R + (\pi - \phi_1' - \psi_R)\sin\phi_1' \tag{203}$$

and

$$\mathcal{U}(\psi) - \mathcal{U}(\psi_R) = \cos\psi_R - \cos\psi + (\psi_R - \psi)\sin\phi_1'. \tag{204}$$

Here, according to (141), we have

$$\frac{V_1'}{V_1} = \frac{\sin\phi_1}{\sin\phi_1'} \tag{205}$$

and therefore

$$\mathcal{W}(\psi) = \pm\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}|\,\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}. \tag{206}$$

The area of the bunch is

$$B = \frac{B_1}{8\sqrt{2}}\,\mathcal{B} \tag{207}$$

where

$$\mathcal{B} = 2\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}\int_{\psi_L}^{\psi_R}|\,\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}\,d\psi \tag{208}$$

is defined to be the normalized area. The normalized bunch boundary and area are dimensionless and completely determined by the turning point phase ψ_R and the phases ϕ_1 and ϕ_1'.

17 Adjusting the ratio of triple to single-harmonic bucket area

For the triple-harmonic bucket we have area

$$\mathcal{A}_3 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_3 \tag{209}$$

and normalized area

$$\mathcal{B}_3 = 2\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\,d\psi \tag{210}$$

where, as shown in Sections 8 and 9, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1.

Similarly, for the single-harmonic bucket we have area

$$\mathcal{A}_1 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_1 \tag{211}$$

and normalized area

$$\mathcal{B}_1 = 2\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\,d\psi \tag{212}$$

where, as shown in Sections 14 and 15, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1'.

Thus, the ratio

$$\frac{\mathcal{A}_3}{\mathcal{A}_1} = \frac{\mathcal{B}_3}{\mathcal{B}_1} \tag{213}$$

is completely determined by ϕ_1 and ϕ_1'. If either of these phases is given, the other can be adjusted to give a desired value for the ratio.

If, for example, we start with a given single-harmonic bucket having synchronous phase ϕ_1' and want a triple-harmonic bucket with the same area, the phase ϕ_1 can be adjusted to give

$$\mathcal{B}_3 = \mathcal{B}_1 \tag{214}$$

and therefore

$$\mathcal{A}_3 = \mathcal{A}_1. \tag{215}$$

Since, according to (141),

$$V_1\sin\phi_1 = V_1'\sin\phi_1' \tag{216}$$

we also have voltage ratio

$$\frac{V_1}{V_1'} = \frac{\sin\phi_1'}{\sin\phi_1}. \tag{217}$$

If the single-harmonic voltage V_1' is given, we then have the required triple-harmonic voltage V_1.

18 Application of triple-harmonic bucket to acceleration of polarized protons in AGS

For protons [3]

$$mc^2 = 938.272\,088\,16(29) \text{ MeV} \tag{218}$$

$$g = 5.585\,694\,6893(16) \tag{219}$$

$$G = (g-2)/2 = 1.792\,8473\,4465 \tag{220}$$

and in AGS [4]

$$R_s = 128.4526 \text{ m}, \quad \rho_s = 85.378351 \text{ m} \tag{221}$$

$$\gamma_t = 8.5, \quad h = 6. \tag{222}$$

Suppose we have acceleration of polarized protons in AGS set up using just a single RF harmonic and wish to move to a triple-harmonic setup. Let V_1', and ϕ_1' be the voltage and synchronous phase for the single-harmonic setup. For a given single-harmonic voltage V_1' the synchronous phase ϕ_1' must satisfy

$$V_1'\sin\phi_1' = 2\pi R_s\rho_s\left(\frac{1}{c}\frac{dB}{dt}\right) \tag{223}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. According to (58) we also have, for the triple-harmonic bucket,

$$V_1\sin\phi_1 = 2\pi R_s\rho_s\left(\frac{1}{c}\frac{dB}{dt}\right) \tag{224}$$

and therefore

$$V_1\sin\phi_1 = V_1'\sin\phi_1'. \tag{225}$$

As a starting point for moving to a triple-harmonic setup, we would like the triple-harmonic bucket to have the same area as that of the single-harmonic bucket. As shown in the previous section, we can adjust the phase ϕ_1 so that this is so. This would be done by an application at a discrete set of points that amply cover the cycle of the programmed guide field. Having these phases in hand we then have a first pass set of triple-harmonic voltages V_1 given by

$$V_1 = \left(\frac{\sin \phi_1'}{\sin \phi_1} \right) V_1'. \tag{226}$$

These would be the starting voltages in the V_1 voltage program for the triple-harmonic bucket. They could be subsequently tuned if necessary. In the next section it is shown that for any set of programmed values of V_1, the corresponding values of phases ϕ_s, ψ_2, ψ_3 and voltages V_2 and V_3 can be obtained from lookup tables. These then give the triple-harmonic voltage

$$V(\psi) = V_1 \sin \psi - V_2 \sin(2\psi - 2\psi_2) + V_3 \sin(3\psi - 3\psi_3) \tag{227}$$

throughout the guide field cycle.

19 Lookup tables for triple-harmonic phases and voltage ratios

We assume that voltage V_1 and guide field time-derivative dB/dt are given and require that phase ϕ_1 satisfy

$$V_1 \sin \phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c} \frac{dB}{dt} \right). \tag{228}$$

Obtaining the required phase ϕ_1 would be no different than what is currently done to obtain the synchronous phase ϕ_1' in the single-harmonic setup. As shown in Section 8 we must also have

$$0 \le \sin \phi_1 \le \frac{2}{3}. \tag{229}$$

In practice this is no different than requiring that the single-harmonic synchronous phase satisfy

$$0 \le \sin \phi_1' \le 1. \tag{230}$$

The constraint (229) is satisfied if

$$0 \le \phi_1 \le 41.8103^\circ \tag{231}$$

or

$$138.1897^\circ \leq \phi_1 < 180^\circ. \tag{232}$$

Starting with

$$\sin\phi_1 = \frac{1}{V_1}\left\{2\pi R_s \rho_s \left(\frac{1}{c}\,\frac{dB}{dt}\right)\right\} \tag{233}$$

we then have

$$\sin\phi_s = \frac{3}{2}\sin\phi_1 \tag{234}$$

$$\phi_s = \arcsin\left\{\frac{3}{2}\sin\phi_1\right\} \tag{235}$$

$$\tan\left(2\phi_s - 2\psi_2\right) = \frac{1}{2}\tan\phi_s \tag{236}$$

$$2\phi_s - 2\psi_2 = \arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{237}$$

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{238}$$

$$\tan\left(3\phi_s - 3\psi_2\right) = \frac{1}{3}\tan\phi_s \tag{239}$$

$$3\phi_s - 3\psi_3 = \arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{240}$$

$$\psi_3 = \phi_s - \frac{1}{3}\arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{241}$$

$$\frac{V_2}{V_1} = \frac{4}{5}\left\{\frac{\cos\phi_s}{\cos(2\phi_s - 2\psi_2)}\right\} = \frac{2}{5}\left\{\frac{\sin\phi_s}{\sin(2\phi_s - 2\psi_2)}\right\} \tag{242}$$

and

$$\frac{V_3}{V_1} = \frac{1}{5}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{15}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\}. \tag{243}$$

These formulae can be used to construct lookup tables that give ϕ_s, ψ_2, ψ_3, V_2/V_1, and V_3/V_1 for any ϕ_1 in range (231) or (232).

The triple-harmonic voltage is then

$$V(\psi) = V_1\sin\psi - V_2\sin(2\psi - 2\psi_2) + V_3\sin(3\psi - 3\psi_3). \tag{244}$$

This satisfies

$$V(\phi_s) = 2\pi R_s \rho_s \left(\frac{1}{c}\,\frac{dB}{dt}\right) \tag{245}$$

and produces a flattened RF bucket.

The phases ϕ_s, ψ_2, and ψ_3 are plotted as functions of ϕ_1 in Figure 1.

The ratios V_2/V_1 and V_3/V_1 are plotted as functions of ϕ_1 in Figure 2.

The phase ϕ_1 is given by (233).

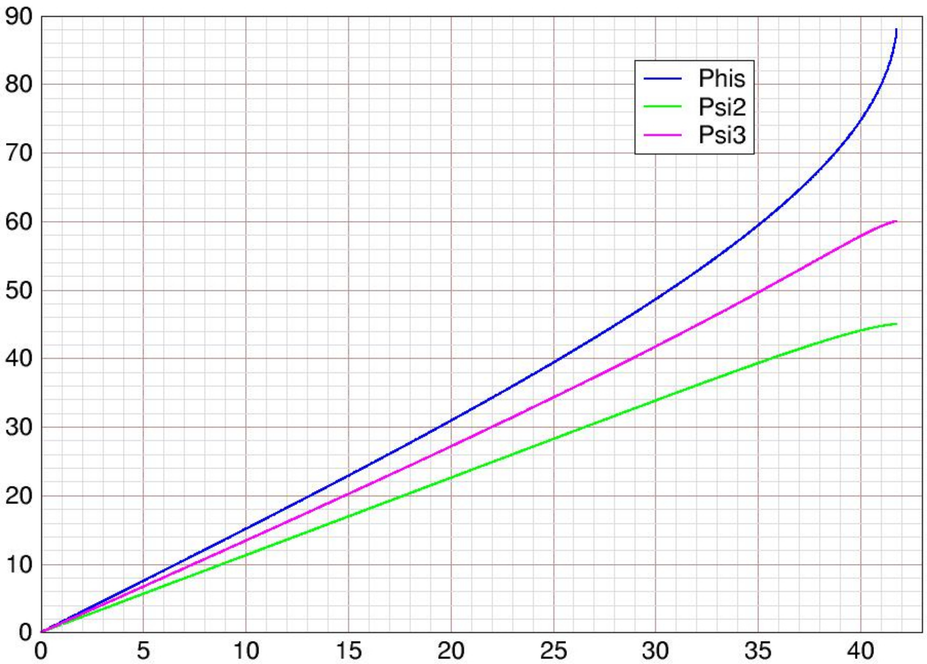

Figure 1: Triple-harmonic phases ϕ_s, ψ_2, and ψ_3 plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the phases in degrees. Starting with the lowest curve and going up, the green, pink, and blue curves are ψ_2, ψ_3, and ϕ_s, respectively. The phase ϕ_1 is given by (233). As $\sin\phi_1$ approaches $2/3$, the phases ψ_2, ψ_3, and ϕ_s approach $\pi/4$, $\pi/3$, and $\pi/2$ respectively.

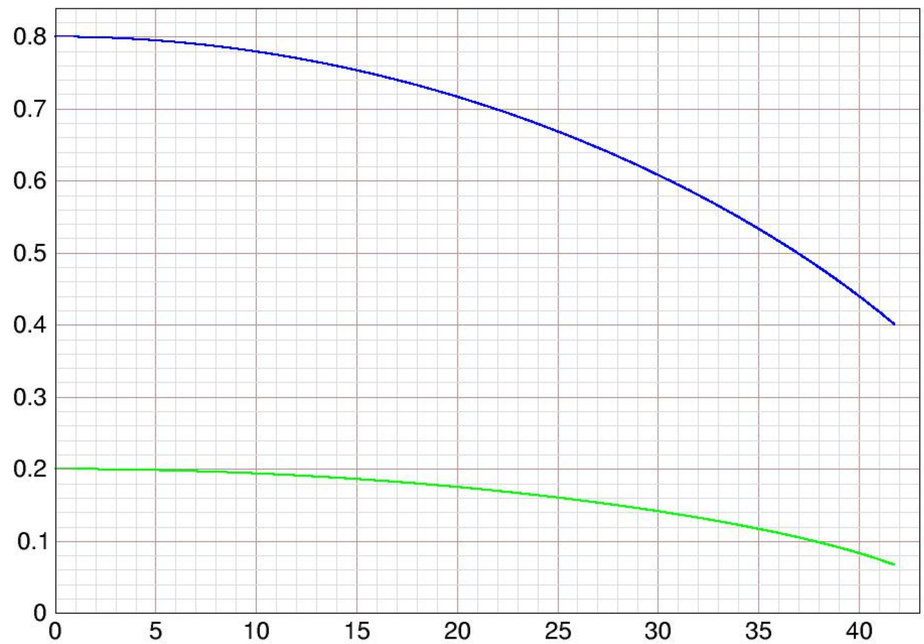

Figure 2: Triple-harmonic voltage ratios V_2/V_1 and V_3/V_1 plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the voltage ratios. The upper and lower curves are V_2/V_1 and V_3/V_1, respectively. The phase ϕ_1 is given by (233). As $\sin\phi_1$ approaches 2/3, the ratios V_2/V_1 and V_3/V_1 approach 2/5 and 1/15, respectively.

20 Triple-harmonic bucket turning point phases and normalized area

The turning point phases ψ_e and ψ_u and normalized area for the triple-harmonic bucket are plotted as functions of ϕ_1 in Figures 3 and 4.

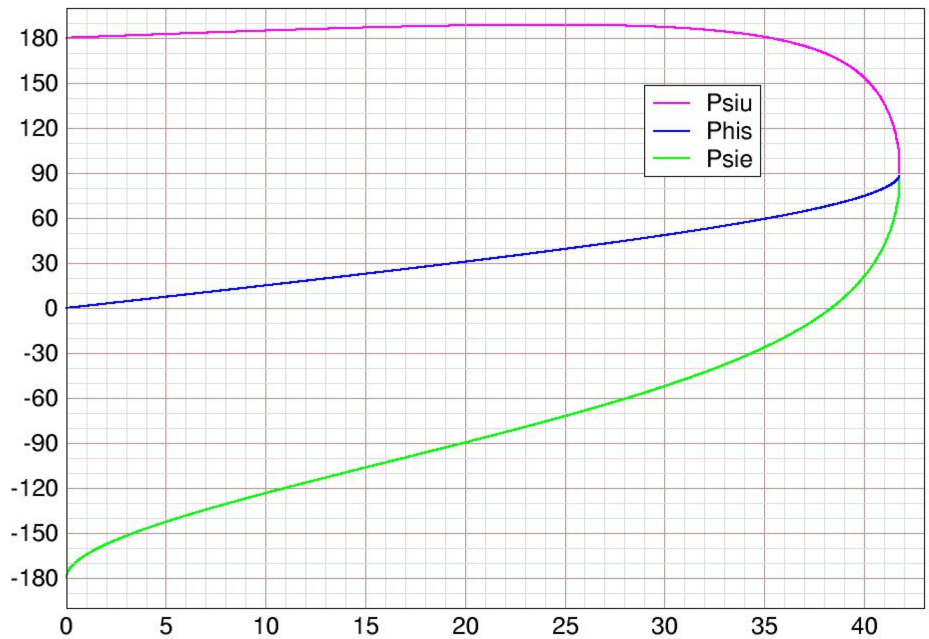

Figure 3: Triple-harmonic phases ψ_e, ϕ_s, and ψ_u plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the phases in degrees. The lower (green) and upper (pink) curves are ψ_e and ψ_u, respectively. The middle curve (blue) is the synchronous phase ϕ_s. The phase ϕ_1 is given by (233). The triple-harmonic bucket extends from turning point phase ψ_e to unstable fixed point phase ψ_u. As $\sin\phi_1$ approaches $2/3$, all three phases approach $\pi/2$, and the bucket phase width $\psi_u - \psi_e$ goes to zero.

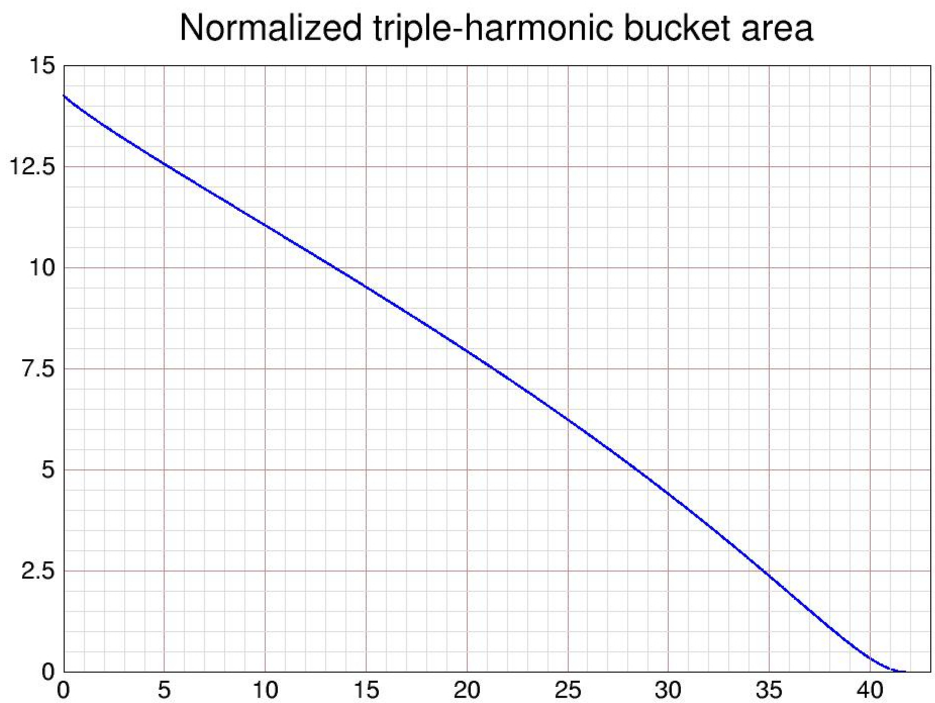

Figure 4: Normalized triple-harmonic bucket area plotted as a function of phase ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the normalized area $\mathcal{B}_3$ defined in Section 10. As $\sin\phi_1$ approaches $2/3$, the area goes to zero.

21 Triple-harmonic bucket and bunch example

Figure 5 shows an example of a normalized triple-harmonic bucket with matched bunch and corresponding potential. These were obtained with $\phi_1 = 35$ degrees. The normalized bucket area is 2.38. The width of the matched bunch has been adjusted so that the bunch area is half that of the bucket. The bunch width is 156 degrees.

Figure 6 shows the corresponding normalized RF voltages.

Figure 7 shows the voltages with the normalized potential and bucket superimposed.

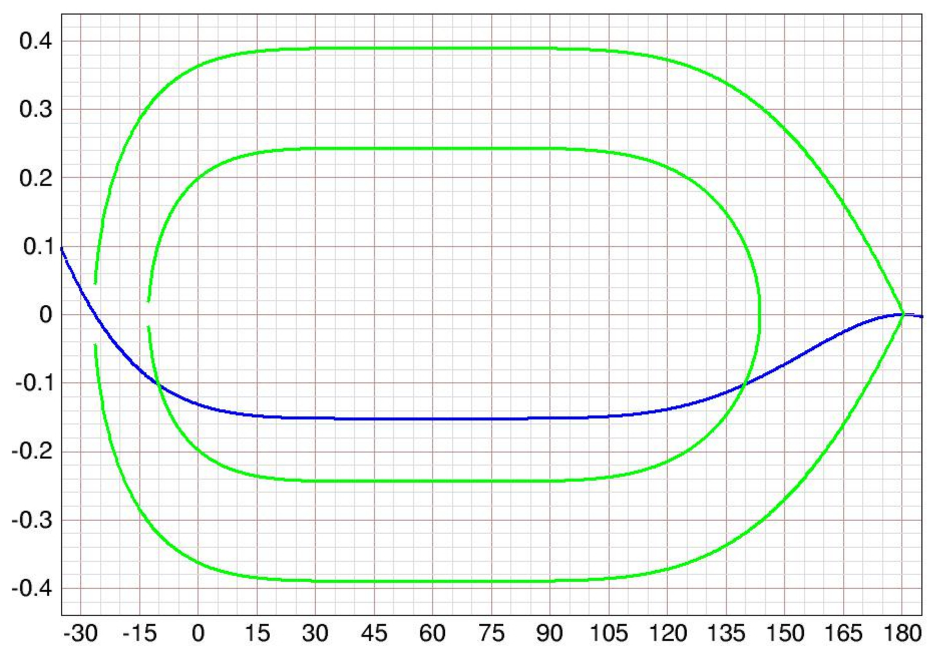

Figure 5: Normalized triple-harmonic bucket and bunch (green curves) and potential (blue curve) obtained with $\phi_1 = 35$ degrees. The normalized bucket area is 2.38. The width of the matched bunch has been adjusted so that the bunch area is half that of the bucket. The bunch width is 156 degrees. The horizontal axis gives the RF phase ψ in degrees.

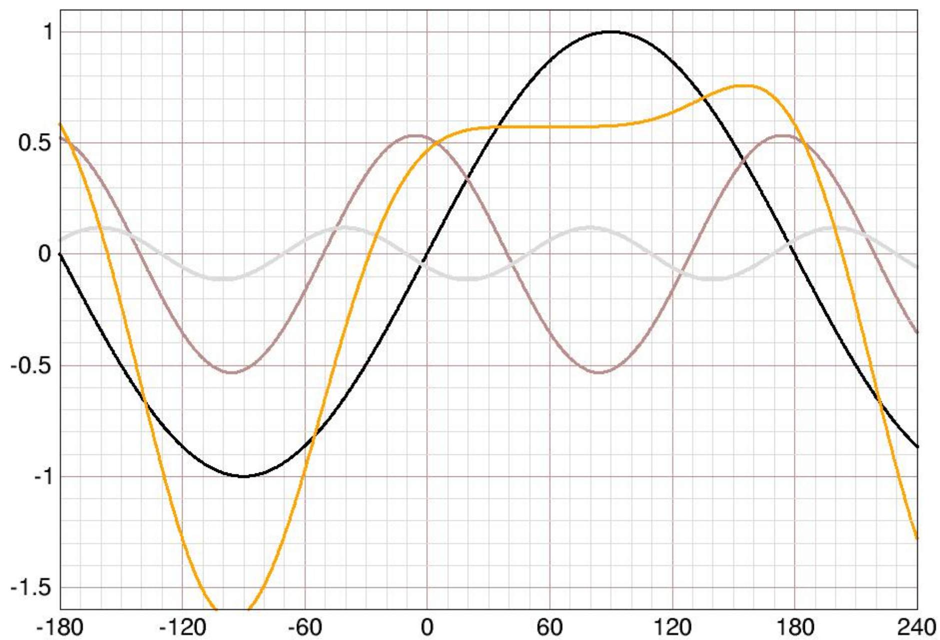

Figure 6: Normalized triple-harmonic voltages $\sin\psi$, $-(V_2/V_1)\sin(2\psi - 2\psi_2)$, and $(V_3/V_1)\sin(3\psi - 3\psi_3)$ as functions of phase ψ. These are the black, brown, and gray curves respectively. The orange curve is the sum of the three voltages. The horizontal axis gives the phase in degrees.

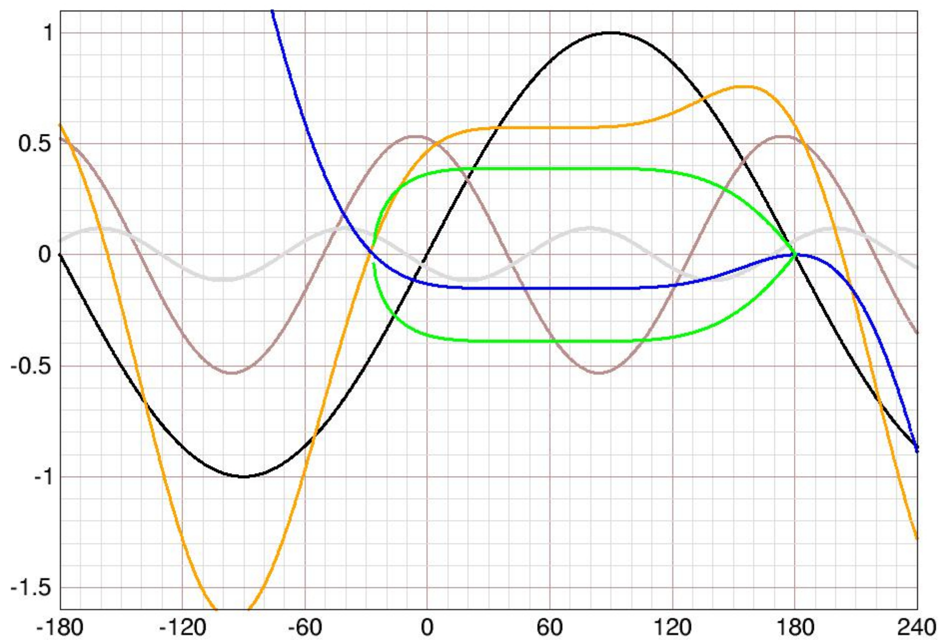

Figure 7: Normalized triple-harmonic voltages with the normalized potential and bucket superimposed. The horizontal axis gives the phase in degrees.

22 Triple-harmonic parameters for $G\gamma = 4.5$

For

$$G\gamma = 4.5 \tag{246}$$

we have

$$\beta\gamma^2 = 5.7783768, \quad B\rho = 7.2051786 \text{ Tm}, \quad B = 843.91166 \text{ G} \tag{247}$$

Taking

$$dB/dt = 0.01 \text{ G/ms} \tag{248}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{249}$$

then gives

$$V_1' = 7.498 \text{ kV}, \quad \phi_1' = 0.52657 \text{ degrees}. \tag{250}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 0.8399 \text{ degrees} \tag{251}$$

and therefore

$$\phi_s = 1.2599, \quad \psi_2 = 0.9449, \quad \psi_3 = 1.1199, \text{ degrees} \tag{252}$$

$$\frac{V_1}{V_1'} = 0.6270, \quad \frac{V_2}{V_1} = 0.7999, \quad \frac{V_3}{V_1} = 0.2000. \tag{253}$$

The resulting single and triple-harmonic buckets are shown in Figure 8 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 195.4 and 239.6 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.816.

The single and triple-harmonic bunch heights are 1.345 and 0.9458 respectively. The ratio of triple to single-harmonic bunch height is 0.703.

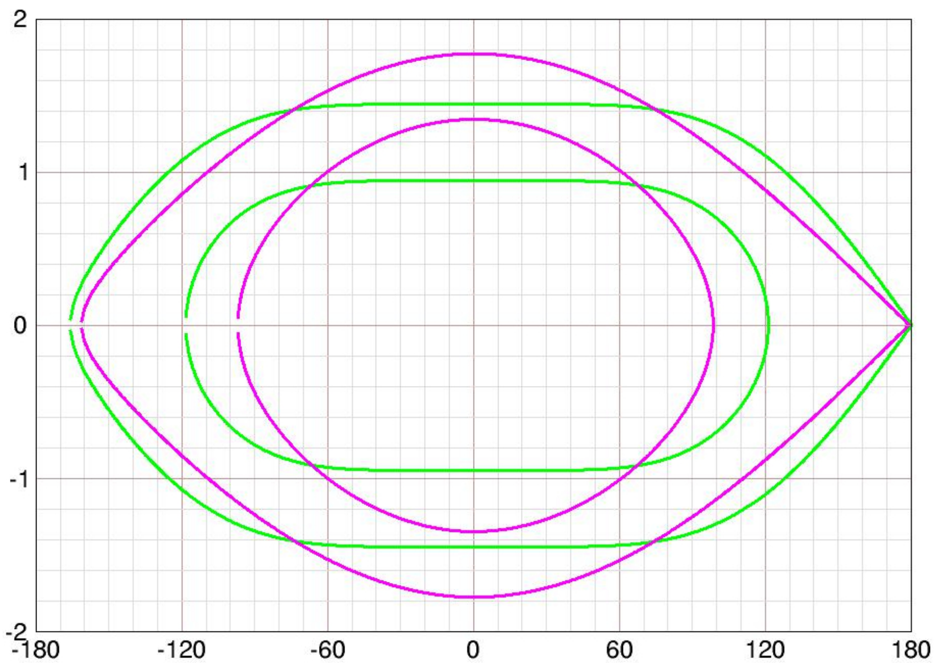

Figure 8: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 4.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 195.4 and 239.6 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.816. The single and triple-harmonic bunch heights are 1.345 and 0.9458 respectively. The ratio of triple to single-harmonic bunch height is 0.703. The horizontal axis gives the RF phase ψ in degrees.

23 Triple-harmonic parameters for $G\gamma = 6.0$

For

$$G\gamma = 6.0 \tag{254}$$

we have

$$\beta\gamma^2 = 10.688256, \quad B\rho = 9.9955569 \text{ Tm}, \quad B = 1170.7367 \text{ G} \tag{255}$$

Taking

$$dB/dt = 9.0 \text{ G/ms} \tag{256}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{257}$$

then gives

$$V_1' = 89.942 \text{ kV}, \quad \phi_1' = 43.59258 \text{ degrees}. \tag{258}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 36.2684 \text{ degrees} \tag{259}$$

and therefore

$$\phi_s = 62.5425, \quad \psi_2 = 40.5938, \quad \psi_3 = 51.6492, \text{ degrees} \tag{260}$$

$$\frac{V_1}{V_1'} = 1.1656, \quad \frac{V_2}{V_1} = 0.5119, \quad \frac{V_3}{V_1} = 0.1096. \tag{261}$$

The resulting single and triple-harmonic buckets are shown in Figure 9 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 86.8 and 147.8 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.587.

The single and triple-harmonic bunch heights are 0.3944 and 0.1992 respectively. The ratio of triple to single-harmonic bunch height is 0.505.

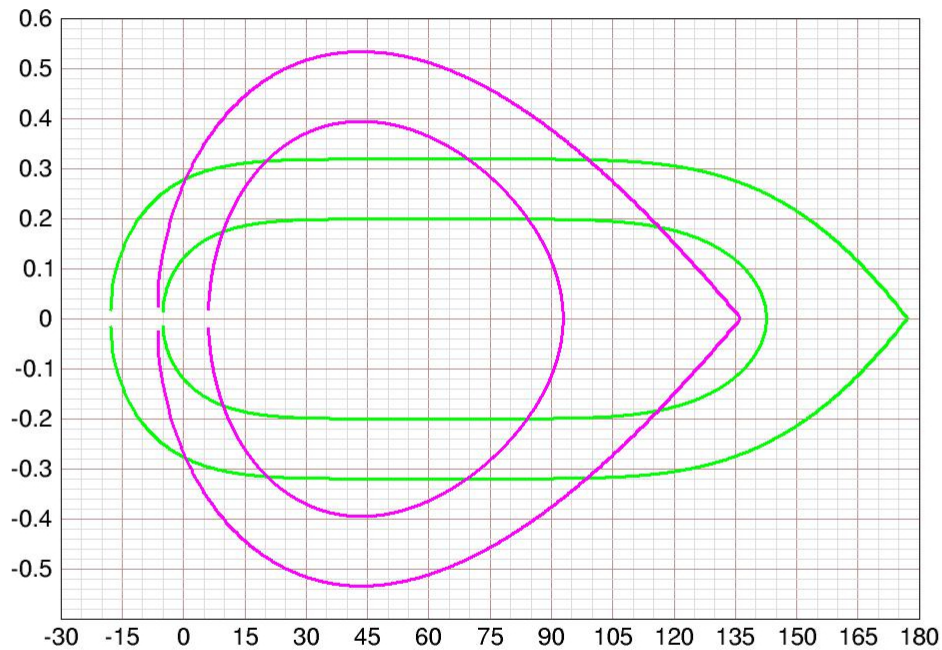

Figure 9: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 6.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 86.8 and 147.8 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.587. The single and triple-harmonic bunch heights are 0.3944 and 0.1992 respectively. The ratio of triple to single-harmonic bunch height is 0.505. The horizontal axis gives the RF phase ψ in degrees.

24 Triple-harmonic parameters for $G\gamma = 7.5$

For

$$G\gamma = 7.5 \tag{262}$$

we have

$$\beta\gamma^2 = 16.992559, \quad B\rho = 12.713026 \text{ Tm}, \quad B = 1489.0222 \text{ G} \tag{263}$$

Taking

$$dB/dt = 18.0 \text{ G/ms} \tag{264}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{265}$$

then gives

$$V_1' = 152.715 \text{ kV}, \quad \phi_1' = 54.3111 \text{ degrees.} \tag{266}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 38.4927 \text{ degrees} \tag{267}$$

and therefore

$$\phi_s = 69.0067, \quad \psi_2 = 42.7591, \quad \psi_3 = 55.3468, \text{ degrees} \tag{268}$$

$$\frac{V_1}{V_1'} = 1.3049, \quad \frac{V_2}{V_1} = 0.4708, \quad \frac{V_3}{V_1} = 0.0949. \tag{269}$$

The resulting single and triple-harmonic buckets are shown in Figure 10 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 66.0 and 126.5 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.522.

The single and triple-harmonic bunch heights are 0.2548 and 0.1142 respectively. The ratio of triple to single-harmonic bunch height is 0.448.

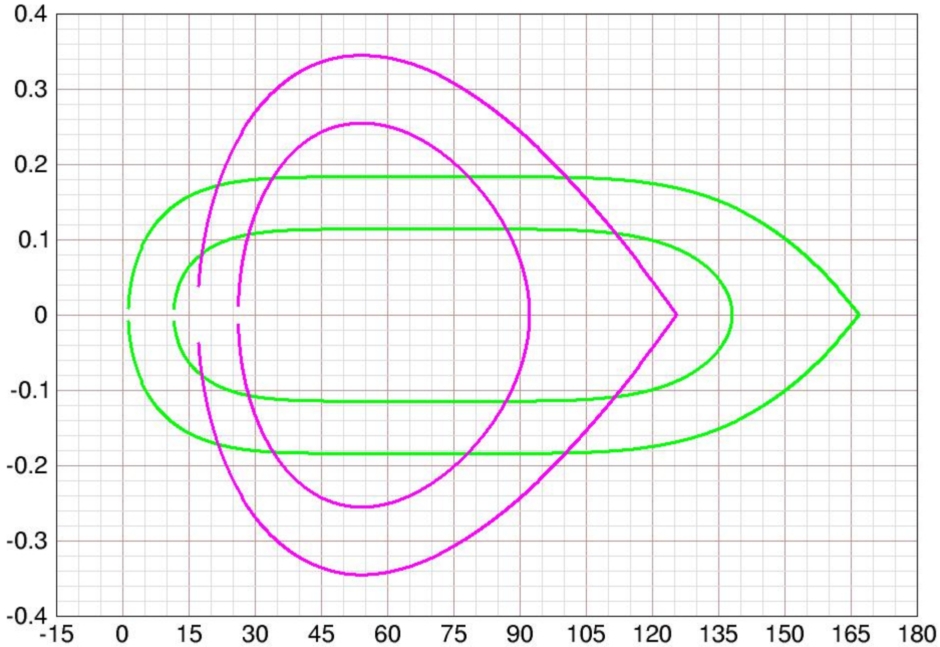

Figure 10: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 7.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 66.0 and 126.5 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.522. The single and triple-harmonic bunch heights are 0.2548 and 0.1142 respectively. The ratio of triple to single-harmonic bunch height is 0.448. The horizontal axis gives the RF phase ψ in degrees.

25 Triple-harmonic parameters for $G\gamma = 10.0$

For

$$G\gamma = 10.0 \tag{270}$$

we have

$$\beta\gamma^2 = 30.606873, \quad B\rho = 17.173957 \text{ Tm}, \quad B = 2011.5119 \text{ G} \tag{271}$$

Taking

$$dB/dt = 22.0 \text{ G/ms} \tag{272}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{273}$$

then gives

$$V_1' = 171.219 \text{ kV}, \quad \phi_1' = 62.30122 \text{ degrees}. \tag{274}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 39.63665 \text{ degrees} \tag{275}$$

and therefore

$$\phi_s = 73.1121, \quad \psi_2 = 43.7449, \quad \psi_3 = 57.2210, \text{ degrees} \tag{276}$$

$$\frac{V_1}{V_1'} = 1.3880, \quad \frac{V_2}{V_1} = 0.4478, \quad \frac{V_3}{V_1} = 0.0863. \tag{277}$$

The resulting single and triple-harmonic buckets are shown in Figure 11 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 50.9 and 109.4 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.466.

The single and triple-harmonic bunch heights are 0.1702 and 0.06801 respectively. The ratio of triple to single-harmonic bunch height is 0.400.

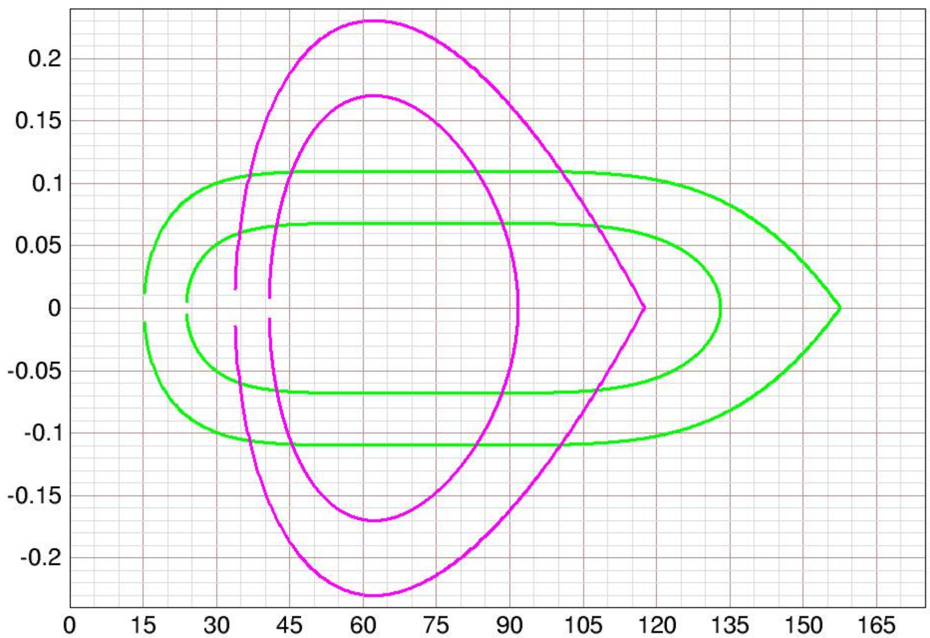

Figure 11: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 10.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 50.9 and 109.4 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.466. The single and triple-harmonic bunch heights are 0.1702 and 0.06801 respectively. The ratio of triple to single-harmonic bunch height is 0.400. The horizontal axis gives the RF phase ψ in degrees.

26 Triple-harmonic parameters for $G\gamma = 12.5$

For

$$G\gamma = 12.5 \tag{278}$$

we have

$$\beta\gamma^2 = 48.108272, \quad B\rho = 21.595395 \text{ Tm}, \quad B = 2529.3760 \text{ G} \tag{279}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{280}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{281}$$

then gives

$$V_1' = 184.954 \text{ kV}, \quad \phi_1' = 68.6580 \text{ degrees.} \tag{282}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 40.3377 \text{ degrees} \tag{283}$$

and therefore

$$\phi_s = 76.1527, \quad \psi_2 = 44.2743, \quad \psi_3 = 58.3137, \text{ degrees} \tag{284}$$

$$\frac{V_1}{V_1'} = 1.4390, \quad \frac{V_2}{V_1} = 0.4330, \quad \frac{V_3}{V_1} = 0.0805. \tag{285}$$

The resulting single and triple-harmonic buckets are shown in Figure 12 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 39.1 and 94.3 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.415.

The single and triple-harmonic bunch heights are 0.1136 and 0.04036 respectively. The ratio of triple to single-harmonic bunch height is 0.355.

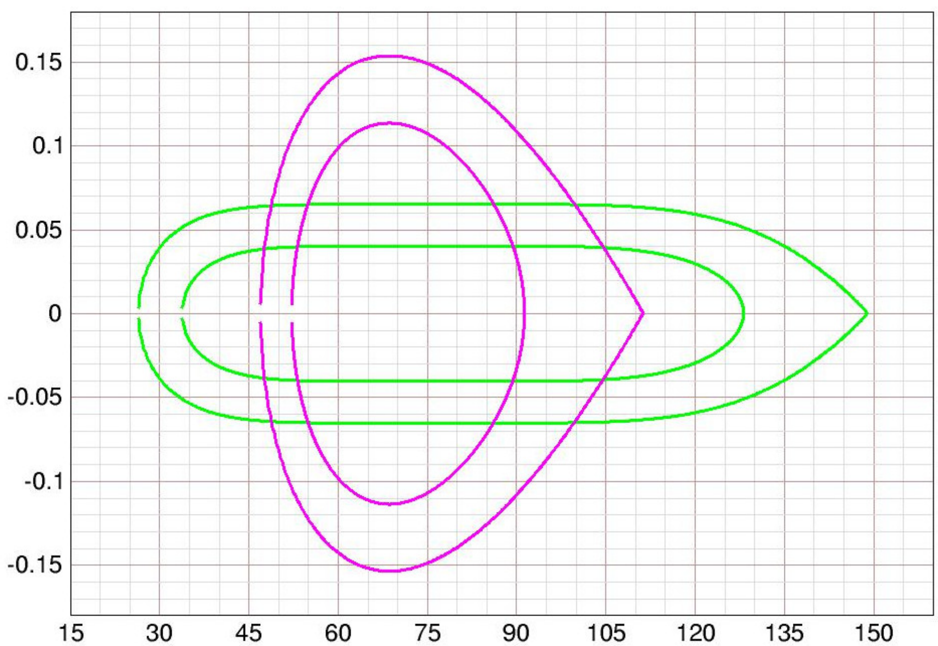

Figure 12: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 12.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 39.1 and 94.3 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.415. The single and triple-harmonic bunch heights are 0.1136 and 0.04036 respectively. The ratio of triple to single-harmonic bunch height is 0.355. The horizontal axis gives the RF phase ψ in degrees.

27 Triple-harmonic parameters for $G\gamma = 14.0$

For

$$G\gamma = 14.0 \tag{286}$$

we have

$$\beta\gamma^2 = 60.475409, \quad B\rho = 24.238302 \text{ Tm}, \quad B = 2838.9283 \text{ G} \tag{287}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{288}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{289}$$

then gives

$$V_1' = 180.410 \text{ kV}, \quad \phi_1' = 72.7235 \text{ degrees.} \tag{290}$$

We can then adjust ϕ_1 to give a normalized triple-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 40.7118 \text{ degrees} \tag{291}$$

and therefore

$$\phi_s = 78.0647, \quad \psi_2 = 44.5229, \quad \psi_3 = 58.8580, \text{ degrees} \tag{292}$$

$$\frac{V_1}{V_1'} = 1.4640, \quad \frac{V_2}{V_1} = 0.4249, \quad \frac{V_3}{V_1} = 0.0772. \tag{293}$$

The resulting single and triple-harmonic buckets are shown in Figure 13 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and triple-harmonic bunch widths are 31.6 and 83.8 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.377.

The single and triple-harmonic bunch heights are 0.08226 and 0.02659 respectively. The ratio of triple to single-harmonic bunch height is 0.323.

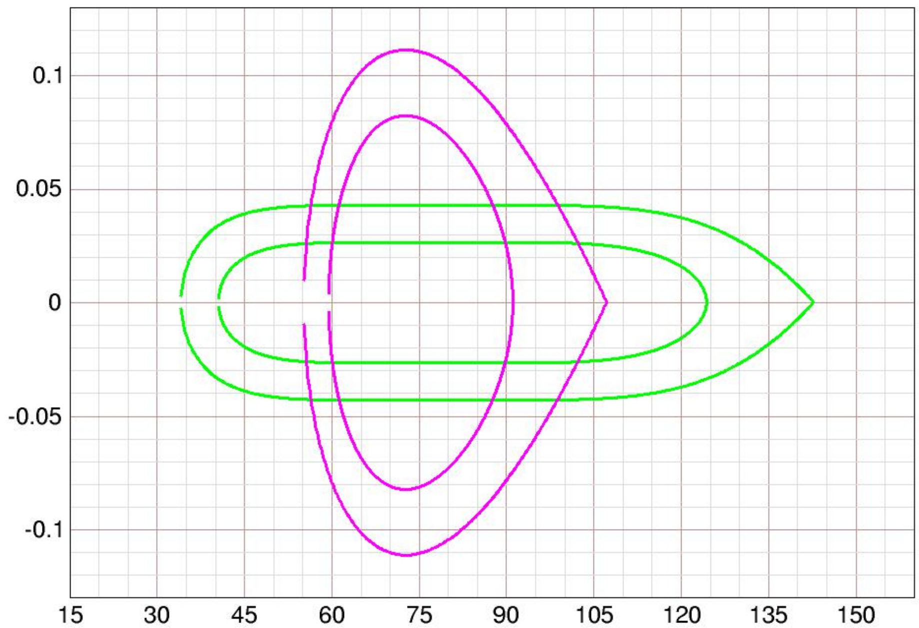

Figure 13: Normalized single and triple-harmonic buckets and matched bunches obtained for $G\gamma = 14.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and triple-harmonic bunch widths are 31.6 and 83.8 degrees respectively. The ratio of single to triple-harmonic bunch width is 0.377. The single and triple-harmonic bunch heights are 0.08226 and 0.02659 respectively. The ratio of triple to single-harmonic bunch height is 0.323. The horizontal axis gives the RF phase ψ in degrees.

28 Triple-harmonic parameter summary for acceleration of polarized protons in AGS

The following tables summarize the data of Sections 22 through 27. The guide field B and its time derivative are given in units of G and G/ms. The single-harmonic RF voltage V_1' is given in units of kV. The phases ϕ_1', ϕ_1, ϕ_s, ψ_2, ψ_3, and the single and triple-harmonic bunch widths W_1 and W_3 are given in degrees. The ratio H_3/H_1 is the ratio of triple to single-harmonic bunch height. Values of $\beta\gamma^2$, W_1, W_3, and H_3/H_1 are tabulated for comparison of the incoherent space charge tune shifts in the single and triple-harmonic bunches.

Table 10: Triple-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1	V_3/V_1
843.9	0.01	4.5	7.498	0.6270	0.7999	0.2000
1170.7	9.0	6.0	89.942	1.1656	0.5119	0.1096
1489.0	18.0	7.5	152.715	1.3049	0.4708	0.0949
2011.5	22.0	10.0	171.219	1.3880	0.4478	0.0863
2529.4	25.0	12.5	184.954	1.4390	0.4330	0.0805
2838.9	25.0	14.0	180.410	1.4640	0.4249	0.0772

Table 11: Triple-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2	ψ_3
0.01	4.5	0.52657	0.8399	1.2599	0.9449	1.1199
9.0	6.0	43.59258	36.2684	62.5425	40.5938	51.6492
18.0	7.5	54.3111	38.4927	69.0067	42.7591	55.3468
22.0	10.0	62.30122	39.63665	73.1121	43.7449	57.2210
25.0	12.5	68.6580	40.3377	76.1527	44.2743	58.3137
25.0	14.0	72.7235	40.7118	78.0647	44.5229	58.8580

Table 12: Triple-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_3	W_1/W_3	H_3/H_1
0.01	4.5	5.7784	195.4	239.6	0.816	0.703
9.0	6.0	10.6883	86.8	147.8	0.587	0.507
18.0	7.5	16.9926	66.0	126.5	0.522	0.448
22.0	10.0	30.6069	50.9	109.4	0.466	0.400
25.0	12.5	48.1083	39.1	94.3	0.415	0.355
25.0	14.0	60.4754	31.6	83.8	0.377	0.323

29 Incoherent space charge tune shift formula

We follow here the treatment of Courant [5].

Consider a beam of particles of mass m and charge eq circulating in a storage ring or accelerator. (Here e is the elementary charge and q is an integer.) Assume that the density of particles in the beam is uniform over longitudinal distances that are large compared to the transverse dimensions of the beam. Assume further that the beam cross section in the transverse plane is a uniformly populated ellipse of area πab, where a and b are the horizontal and vertical half widths of the ellipse.

Any given particle moving with the beam will experience a shift in betatron tune due to the presence of the other particles. It is assumed here that the effect of the image currents generated in the vacuum chamber can be ignored. The tune shift is then

$$\delta Q = -\frac{N}{\pi QB}\,\frac{r_0 R}{b(a+b)}\,\frac{1}{\beta^2\gamma}(1-\beta^2). \tag{294}$$

This is Courant's equation (1) with the tune denoted by Q instead of ν. Here N is number of particles in the beam, R is the ring radius (i.e. the ring circumference divided by 2π) and B, the bunching factor, is the fraction of the ring circumference occupied by the beam. The classical electrostatic radius of the particle is

$$r_0 = q^2\left(\frac{m_e}{m}\right) r_e \tag{295}$$

where

$$r_e = 2.817\,940\,3262(13)\times 10^{-15}\ \mathrm{m}\,[3] \tag{296}$$

is the classical electron radius. The electron mass-energy equivalent is

$$m_e c^2 = 0.510\,998\,950\,00(15) \text{ MeV [3]}. \tag{297}$$

As stated succinctly by Courant, the factor $1 - \beta^2$ in (294) arises from the combination of electric repulsion and magnetic attraction between the given particle and the rest of the beam; the strength of the latter is β^2 times the former.

30 Tune shift in terms of transverse emittance

Let $\pi\epsilon_H$ and $\pi\epsilon_V$ be the horizontal and vertical emittances of the beam and let $\langle\beta_H\rangle$ and $\langle\beta_V\rangle$ be the averages of the corresponding Courant-Snyder parameters of the ring lattice. Then

$$a = \sqrt{\epsilon_H \langle\beta_H\rangle}, \quad b = \sqrt{\epsilon_V \langle\beta_V\rangle} \tag{298}$$

and

$$b(a+b) = \epsilon_V \langle\beta_V\rangle + \sqrt{\epsilon_V \epsilon_H \langle\beta_V\rangle \langle\beta_H\rangle}. \tag{299}$$

We shall assume that

$$\epsilon_H = \epsilon_V = \epsilon \tag{300}$$

and that

$$\langle\beta_H\rangle = \langle\beta_V\rangle = \beta_{AV} \tag{301}$$

where

$$\beta_{AV} = \frac{R}{Q}. \tag{302}$$

We then have

$$b(a+b) = 2\epsilon\beta_{AV} \tag{303}$$

$$\frac{1}{\pi Q}\,\frac{r_0 R}{b(a+b)} = \frac{r_0}{2\pi\epsilon} \tag{304}$$

and (294) becomes

$$\delta Q = -\frac{N}{B}\,\frac{r_0}{2\pi\epsilon}\,\frac{1}{\beta^2\gamma}(1-\beta^2). \tag{305}$$

Since the normalized emittance is

$$\pi\epsilon_N = \beta\gamma\,\pi\epsilon \tag{306}$$

we then have

$$\delta Q = -\frac{N}{B}\frac{r_0}{2\pi\epsilon_N}\frac{1}{\beta}(1-\beta^2) \tag{307}$$

which we can write as

$$\delta Q = -\frac{r_0}{2\pi\epsilon_N}\left(\frac{N}{B\beta\gamma^2}\right). \tag{308}$$

31 Tune shift for a round gaussian beam

For a beam distribution that is cylindrically symmetric and Gaussian, Conte and MacKay [6] obtain

$$(\delta Q)_G = -\frac{r_0}{4\pi\mathcal{E}_N}\left(\frac{N}{B\beta\gamma^2}\right) \tag{309}$$

where $\pi\mathcal{E}_N$ is the rms normalized emittance of the distribution. This is to be compared with (308). The relationship between the two is

$$(\delta Q)_G = \left(\frac{\epsilon_N}{2\mathcal{E}_N}\right)\delta Q \tag{310}$$

where $\pi\epsilon_N$ is the normalized emittance of a uniform distribution. If

$$\epsilon_N = 2\mathcal{E}_N \tag{311}$$

then the two tune shifts are the same.

32 Comparison of the incoherent tune shifts in the single and triple-harmonic bunches

Let δQ_1 and δQ_3 be the incoherent space charge tune shifts in the single and triple-harmonic bunches, respectively. Then the ratio

$$\frac{\delta Q_3}{\delta Q_1} = \frac{B_1}{B_3} \tag{312}$$

where B_1 and B_3 are the corresponding bunching factors. Here one may simply take

$$B_1 = \frac{W_1}{2\pi h}, \quad B_3 = \frac{W_3}{2\pi h} \tag{313}$$

where the bunch widths W_1 and W_3 are given in radians. This gives

$$\frac{\delta Q_3}{\delta Q_1} = \frac{W_1}{W_3}. \tag{314}$$

This ratio is the reduction of space charge tune shift due to the lengthing of the bunch in the triple-harmonic bucket. It is tabulated in the sixth column of Table 12 and goes from 0.816 to 0.377 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

The height of the bunch in the triple-harmonic bucket is also reduced, thereby reducing the peak current of the circulating beam. This suggests a refinement of what we take to be the bunching factor. Let H_1 and H_3 be the heights of the bunches in the single and triple-harmonic buckets. Each bunch has the same area, A say. We can then define new bunch widths W_1' and W_3' such that

$$W_1' H_1 = W_3' H_3 = A. \tag{315}$$

This is illustrated in Figure 14 where boxes have been drawn around the bunches. The widths of the boxes are W_1' and W_3'. If we now take the bunching factors to be

$$B_1 = \frac{W_1'}{2\pi h}, \quad B_3 = \frac{W_3'}{2\pi h} \tag{316}$$

then we have

$$\frac{\delta Q_3}{\delta Q_1} = \frac{W_1'}{W_3'} = \frac{H_3}{H_1}. \tag{317}$$

This ratio is tabulated in the last column of Table 12 and goes from 0.703 to 0.323 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

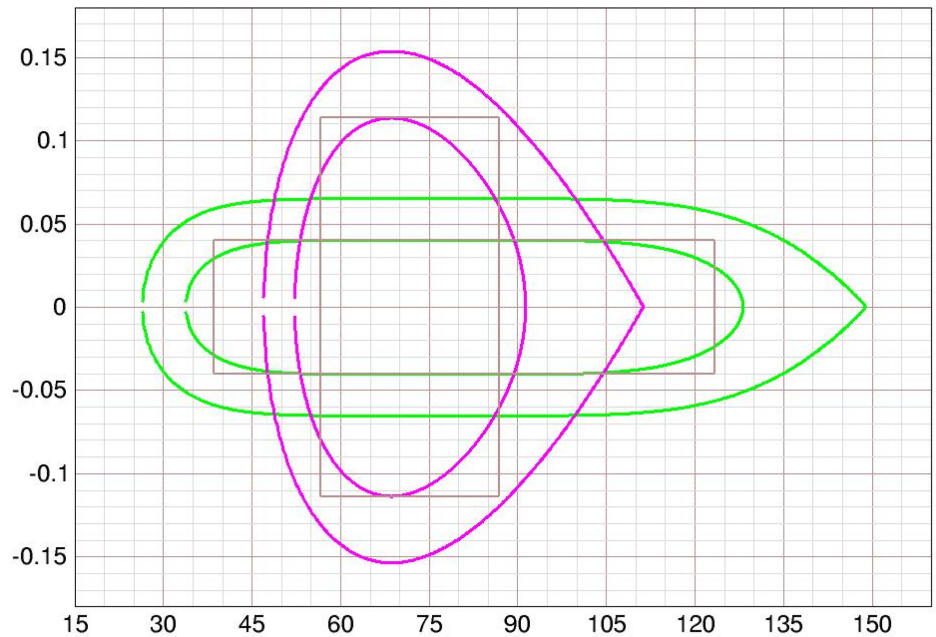

Figure 14: This is the same as Figure 12, but with boxes drawn around the bunches to illustrate the widths W_1' and W_3'. The bunches and boxes all have the same area. W_1' and W_3' are the widths of the single and triple-harmonic boxes, respectively. The box heights H_1 and H_3 are equal to the corresponding bunch heights. The ratio W_1'/W_3' is equal to the ratio H_3/H_1.

33 Reduction of incoherent tune shift with increasing gamma

Let $(\delta Q)_I$ and $(\delta Q)_F$ be the initial and final incoherent tune shifts as proton bunches are accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$. Then the reduction in tune shift is given by the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{B_I\,(\beta\gamma^2)_I}{B_F\,(\beta\gamma^2)_F} \tag{318}$$

where bunching factors

$$B_I = \frac{W_I}{2\pi h}, \quad B_F = \frac{W_F}{2\pi h} \tag{319}$$

and W_I and W_F are the initial and final bunch widths (in radians). Thus the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{W_I\,(\beta\gamma^2)_I}{W_F\,(\beta\gamma^2)_F}. \tag{320}$$

Putting in numbers from columns three and four of Table 12 we obtain

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.591 \tag{321}$$

for bunches in the single-harmonic bucket.

Putting in numbers from columns three and five of the table gives

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.273 \tag{322}$$

for bunches in the triple-harmonic bucket.

This shows that the triple-harmonic bucket gives a significantly greater reduction in incoherent tune shift with increasing gamma.

34 Potential for a double-harmonic bucket

Defining

$$\psi = \phi + \phi_s \tag{323}$$

we have

$$U = A_1 \cos\psi - \frac{1}{2} A_2 \cos(2\psi - 2\psi_2) + C\psi \tag{324}$$

$$\frac{\partial U}{\partial \psi} = -A_1 \sin\psi + A_2 \sin(2\psi - 2\psi_2) + C \tag{325}$$

$$\frac{\partial^2 U}{\partial \psi^2} = -A_1 \cos\psi + 2A_2 \cos(2\psi - 2\psi_2) \tag{326}$$

$$\frac{\partial^3 U}{\partial \psi^3} = A_1 \sin\psi - 4A_2 \sin(2\psi - 2\psi_2) \tag{327}$$

$$\frac{\partial^4 U}{\partial \psi^4} = A_1 \cos\psi - 8A_2 \cos(2\psi - 2\psi_2) \tag{328}$$

and so on, where

$$C = A_1 \sin\phi_s - A_2 \sin(2\phi_s - 2\psi_2). \tag{329}$$

Using integer superscripts to denote the number of differentiations with respect to ψ, we have

$$U^1(\phi_s) = 0 \tag{330}$$

$$U^2(\phi_s) = -A_1 \cos\phi_s + 2A_2 \cos(2\phi_s - 2\psi_2) \tag{331}$$

$$U^3(\phi_s) = A_1 \sin\phi_s - 4A_2 \sin(2\phi_s - 2\psi_2) \tag{332}$$

$$U^4(\phi_s) = A_1 \cos\phi_s - 8A_2 \cos(2\phi_s - 2\psi_2). \tag{333}$$

35 Conditions for a flattened double-harmonic bucket

For a flattened RF bucket we want derivatives

$$U^1(\phi_s) = U^2(\phi_s) = U^3(\phi_s) = 0 \tag{334}$$

which gives

$$A_1 C_1 = 2A_2 C_2 \tag{335}$$

$$A_1 S_1 = 4A_2 S_2 \tag{336}$$

where

$$C_1 = \cos\phi_s, \quad C_2 = \cos(2\phi_s - 2\psi_2) \tag{337}$$

$$S_1 = \sin\phi_s, \quad S_2 = \sin(2\phi_s - 2\psi_2). \tag{338}$$

It then follows that

$$\tan(2\phi_s - 2\psi_2) = \frac{1}{2}\tan\phi_s \tag{339}$$

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{340}$$

and

$$\frac{A_2}{A_1} = \frac{1}{2}\frac{C_1}{C_2} = \frac{1}{4}\frac{S_1}{S_2}. \tag{341}$$

Thus, given synchronous phase ϕ_s, one obtains ψ_2 and the ratio A_2/A_1.

36 Synchronous phase for double-harmonic bucket

As a function of phase, the RF voltage must satisfy

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right)U^1(\psi) \tag{342}$$

where

$$-U^1(\psi) = A_1\sin\psi - A_2\sin(2\psi - 2\psi_2) - C \tag{343}$$

$$C = A_1\sin\phi_s - A_2\sin(2\phi_s - 2\psi_2). \tag{344}$$

Thus, using

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h} \tag{345}$$

we have

$$V(\psi) = V_1\sin\psi - V_2\sin(2\psi - 2\psi_2). \tag{346}$$

The synchronous phase ϕ_s must satisfy

$$V(\phi_s) = 2\pi R_s\rho_s\left(\frac{1}{c}\frac{dB}{dt}\right) \tag{347}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. Here it is useful to define phase ϕ_1 such that

$$V_1 \sin \phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c} \frac{dB}{dt} \right). \tag{348}$$

We then have

$$V_1 \sin \phi_s - V_2 \sin(2\phi_s - 2\psi_2) = V_1 \sin \phi_1 \tag{349}$$

which gives

$$\sin \phi_s = \sin \phi_1 + \frac{V_2}{V_1} \sin(2\phi_s - 2\psi_2). \tag{350}$$

Using

$$\frac{V_2}{V_1} = \frac{A_2}{A_1} \tag{351}$$

we then have

$$S_1 = \sin \phi_1 + \frac{A_2}{A_1} S_2 \tag{352}$$

where, according to (341),

$$\frac{A_2}{A_1} = \frac{1}{4} \frac{S_1}{S_2}. \tag{353}$$

Thus

$$S_1 = \sin \phi_1 + \frac{1}{4} S_1 \tag{354}$$

$$S_1 \left\{ 1 - \frac{1}{4} \right\} = \sin \phi_1 \tag{355}$$

$$S_1 = \frac{4}{3} \sin \phi_1 \tag{356}$$

and therefore

$$\phi_s = \arcsin \left(\frac{4}{3} \sin \phi_1 \right). \tag{357}$$

Here we see that in order to have a real synchronous phase we must have

$$0 < \sin \phi_1 \leq \frac{3}{4}. \tag{358}$$

This constraint is satisfied if

$$0 \leq \phi_1 \leq 48.5904^\circ \tag{359}$$

or

$$131.4096^\circ \le \phi_1 < 180^\circ. \tag{360}$$

If a particular ϕ_1 is given, we can obtain ϕ_s from (357). This in turn can be used in (340) to obtain ψ_2. Finally, ϕ_s and ψ_2 can be used in (341) to obtain the ratio V_2/V_1. The voltage V_1 is given by (348). We therefore have the result that the phases ϕ_s and ψ_2, and the ratio V_2/V_1 are completely determined by the phase ϕ_1.

37 Normalized double-harmonic voltage and potential

It is convenient to normalize the voltage and potential so that they are dimensionless and completely determined by the phase ϕ_1. We define normalized voltage

$$\mathcal{V}(\psi) = \frac{1}{V_1}\, V(\psi) \tag{361}$$

and normalized potential

$$\mathcal{U}(\psi) = \frac{1}{A_1}\left\{U(\psi_u) - U(\psi)\right\} \tag{362}$$

with derivatives

$$\mathcal{U}^m(\psi) = -\frac{1}{A_1}\, U^m(\psi). \tag{363}$$

Here

$$V(\psi) = V_1 \sin\psi - V_2 \sin(2\psi - 2\psi_2) \tag{364}$$

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right) U^1(\psi) \tag{365}$$

$$U(\psi) = A_1 \cos\psi - \frac{1}{2} A_2 \cos(2\psi - 2\psi_2) + C\psi \tag{366}$$

$$U^1(\psi) = -A_1 \sin\psi + A_2 \sin(2\psi - 2\psi_2) + C \tag{367}$$

$$U^2(\psi) = -A_1 \cos\psi + 2A_2 \cos(2\psi - 2\psi_2) \tag{368}$$

$$C = A_1 \sin\phi_s - A_2 \sin(2\phi_s - 2\psi_2) \tag{369}$$

and therefore

$$\mathcal{V}(\psi) - \mathcal{V}(\phi_s) = \mathcal{U}^1(\psi) \tag{370}$$

$$\mathcal{U}(\psi) = \mathcal{D} - \cos\psi + \frac{1}{2} R_2 \, \cos(2\psi - 2\psi_2) - \mathcal{C}\psi \tag{371}$$

$$\mathcal{U}^1(\psi) = \sin\psi - R_2\,\sin(2\psi - 2\psi_2) - \mathcal{C} \tag{372}$$

$$\mathcal{U}^2(\psi) = -\cos\psi + 2R_2\,\cos(2\psi - 2\psi_2) \tag{373}$$

where

$$\mathcal{D} = \cos\psi_u - \frac{1}{2}R_2\,\cos(2\psi_u - 2\psi_2) + \mathcal{C}\psi_u \tag{374}$$

$$\mathcal{C} = \sin\phi_s - R_2\,\sin(2\phi_s - 2\psi_2) \tag{375}$$

and

$$R_2 = \frac{A_2}{A_1}. \tag{376}$$

The phase ψ_u in these equations is the unstable fixed point phase associated with oscillations about the synchronous phase ϕ_s. It satisfies

$$\mathcal{U}(\psi_u) = 0, \quad \mathcal{U}^1(\psi_u) = 0. \tag{377}$$

Below transition one has

$$0 < \phi_s < \psi_u \tag{378}$$

and

$$\mathcal{U}^2(\psi_u) < 0. \tag{379}$$

There is an additional phase

$$\psi_e < \phi_s < \psi_u \tag{380}$$

that satisfies

$$\mathcal{U}(\psi_e) = \mathcal{U}(\psi_u) = 0. \tag{381}$$

The equations of this and the previous section show that the normalized voltage and potential are completely determined by the phase ϕ_1. The phases ψ_u and ψ_e are also completely determined by ϕ_1.

38 Double-harmonic bucket width and area

The RF bucket associated with the stable fixed point phase ϕ_s extends from ψ_e to ψ_u. It is defined by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_u) - U(\psi)\right\} \tag{382}$$

$$a = \frac{h^2c^2\eta_s}{R_s^2E_s}, \quad \eta_s = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}, \quad E_s = mc^2\gamma_s. \tag{383}$$

The phase width of the bucket is

$$\Delta\psi = |\psi_u - \psi_e| \,. \tag{384}$$

In terms of the normalized potential (362) we have

$$W^2(\psi) = \frac{2}{a}\, A_1\, \mathcal{U}(\psi) \tag{385}$$

and

$$W(\psi) = \pm \left(\frac{2A_1}{|a|}\right)^{1/2} |\mathcal{U}(\psi)|^{1/2} \,. \tag{386}$$

Here

$$A_1 = \frac{eQV_1}{2\pi h}, \qquad \frac{1}{|a|} = \frac{R_s^2 E_s}{h^2 c^2 |\eta_s|} \tag{387}$$

which gives

$$\left(\frac{2A_1}{|a|}\right)^{1/2} = \frac{R_s}{hc} \left\{\frac{eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} = \frac{B_1}{8\sqrt{2}} \tag{388}$$

where

$$B_1 = 8\, \frac{R_s}{hc} \left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{389}$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1. Thus we have

$$W(\psi) = \pm \frac{B_1}{8\sqrt{2}}\; |\mathcal{U}(\psi)|^{1/2} \,. \tag{390}$$

The normalized double-harmonic RF bucket is defined by the curves

$$\mathcal{W}(\psi) = \pm\, |\mathcal{U}(\psi)|^{1/2} \tag{391}$$

which are dimensionless and completely determined by the phase ϕ_1.

The area of the double-harmonic bucket is

$$\mathcal{A}_2 = 2 \int_{\psi_e}^{\psi_u} |W(\psi)|\; d\psi \tag{392}$$

where

$$|W(\psi)| = \frac{B_1}{8\sqrt{2}}\; |\mathcal{U}(\psi)|^{1/2} \,. \tag{393}$$

Defining normalized bucket area

$$\mathcal{B}_2 = 2 \int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\, d\psi \tag{394}$$

we then have

$$\mathcal{A}_2 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_2. \tag{395}$$

The normalized bucket area is dimensionless and is completely determined by the phase ϕ_1.

39 Bunch matched to double-harmonic bucket

Consider a particle moving along the boundary of a bunch matched to the double-harmonic bucket and let ψ_R be the right turning point phase of the boundary. Below transition one has

$$\psi_e < \phi_s < \psi_R < \psi_u. \tag{396}$$

The corresponding left turning point phase ψ_L satisfies

$$\psi_e < \psi_L < \phi_s < \psi_R < \psi_u \tag{397}$$

and

$$\mathcal{U}(\psi_L) = \mathcal{U}(\psi_R). \tag{398}$$

Here, as shown in Section 37,

$$\mathcal{U}(\psi) = \frac{1}{A_1}\left\{U(\psi_u) - U(\psi)\right\} \tag{399}$$

where

$$\frac{1}{A_1}\,U(\psi_u) = \cos\psi_u - \frac{1}{2}R_2\cos(2\psi_u - 2\psi_2) + \mathcal{C}\psi_u \tag{400}$$

$$\frac{1}{A_1}\,U(\psi) = \cos\psi - \frac{1}{2}R_2\cos(2\psi - 2\psi_2) + \mathcal{C}\psi \tag{401}$$

$$\mathcal{C} = \sin\phi_s - R_2\sin(2\phi_s - 2\psi_2) \tag{402}$$

and

$$R_2 = \frac{A_2}{A_1}. \tag{403}$$

Given either turning point phase, one can solve (398) numerically to obtain the other. The phase width of the bunch is then

$$\Delta\psi = \psi_R - \psi_L. \tag{404}$$

Alternatively, if $\Delta\psi$ is given, one can solve

$$\mathcal{U}(\psi_R - \Delta\psi) = \mathcal{U}(\psi_R) \tag{405}$$

to obtain ψ_R and $\psi_L = \psi_R - \Delta\psi$.

Below transition the boundary of the matched bunch is given by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_R) - U(\psi)\right\} \tag{406}$$

and

$$\psi_L \leq \psi \leq \psi_R. \tag{407}$$

In terms of the normalized potential (399) we have

$$W^2(\psi) = \frac{2}{a}\,A_1\left\{\mathcal{U}(\psi) - \mathcal{U}(\psi_R)\right\} \tag{408}$$

and

$$W(\psi) = \pm\left(\frac{2A_1}{|a|}\right)^{1/2}|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}\,. \tag{409}$$

According to (388) and (389) we then have

$$W(\psi) = \pm\frac{B_1}{8\sqrt{2}}\,|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2} \tag{410}$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{411}$$

is the single-harmonic stationary bucket area one would have with RF voltage V_1.

The normalized bunch boundary is defined by the curves

$$\mathcal{W}(\psi) = \pm\,|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2} \tag{412}$$

where

$$\psi_L \leq \psi \leq \psi_R. \tag{413}$$

The area of the bunch is

$$B = \frac{B_1}{8\sqrt{2}}\,\mathcal{B} \tag{414}$$

where

$$\mathcal{B} = 2\int_{\psi_L}^{\psi_R}|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}\,d\psi \tag{415}$$

is defined to be the normalized area. The normalized bunch boundary and area are dimensionless and completely determined by the turning point phase ψ_R and the phase ϕ_1.

40 Adjusting the ratio of double to single-harmonic bucket area

For the double-harmonic bucket we have area

$$\mathcal{A}_2 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_2 \tag{416}$$

and normalized area

$$\mathcal{B}_2 = 2\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\,d\psi \tag{417}$$

where, as shown in Sections 36 and 37, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1.

Similarly, for the single-harmonic bucket we have area

$$\mathcal{A}_1 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_1 \tag{418}$$

and normalized area

$$\mathcal{B}_1 = 2\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\,d\psi \tag{419}$$

where, as shown in Sections 14 and 15, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1'.

Thus, the ratio

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} = \frac{\mathcal{B}_2}{\mathcal{B}_1} \tag{420}$$

is completely determined by ϕ_1 and ϕ_1'. If either of these phases is given, the other can be adjusted to give a desired value for the ratio.

If, for example, we start with a given single-harmonic bucket having synchronous phase ϕ_1' and want a double-harmonic bucket with the same area, the phase ϕ_1 can be adjusted to give

$$\mathcal{B}_2 = \mathcal{B}_1 \tag{421}$$

and therefore

$$\mathcal{A}_2 = \mathcal{A}_1. \tag{422}$$

Since

$$V_1\sin\phi_1 = V_1'\sin\phi_1' \tag{423}$$

we also have voltage ratio

$$\frac{V_1}{V_1'} = \frac{\sin\phi_1'}{\sin\phi_1}. \tag{424}$$

If the single-harmonic voltage V_1' is given, we then have the required double-harmonic voltage V_1.

41 Lookup tables for double-harmonic phases and voltage ratio

Proceeding as in Sections 18 and 19, we start with

$$\sin\phi_1 = \frac{1}{V_1}\left\{2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right)\right\} \tag{425}$$

and calculate

$$\sin\phi_s = \frac{4}{3}\sin\phi_1 \tag{426}$$

$$\phi_s = \arcsin\left\{\frac{4}{3}\sin\phi_1\right\} \tag{427}$$

$$\tan\left(2\phi_s - 2\psi_2\right) = \frac{1}{2}\tan\phi_s \tag{428}$$

$$2\phi_s - 2\psi_2 = \arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{429}$$

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{430}$$

and

$$\frac{V_2}{V_1} = \frac{1}{2}\left\{\frac{\cos\phi_s}{\cos(2\phi_s - 2\psi_2)}\right\} = \frac{1}{4}\left\{\frac{\sin\phi_s}{\sin(2\phi_s - 2\psi_2)}\right\}. \tag{431}$$

These formulae can be used to construct lookup tables that give ϕ_s, ψ_2, and V_2/V_1 for any ϕ_1 in range (359) or (360).

The double-harmonic voltage is then

$$V(\psi) = V_1\sin\psi - V_2\sin(2\psi - 2\psi_2). \tag{432}$$

This satisfies

$$V(\phi_s) = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{433}$$

and produces a flattened RF bucket.

The phases ϕ_s and ψ_2 are plotted as functions of ϕ_1 in Figure 15.

The ratio V_2/V_1 is plotted as a function of ϕ_1 in Figure 16.

The phase ϕ_1 is given by (425).

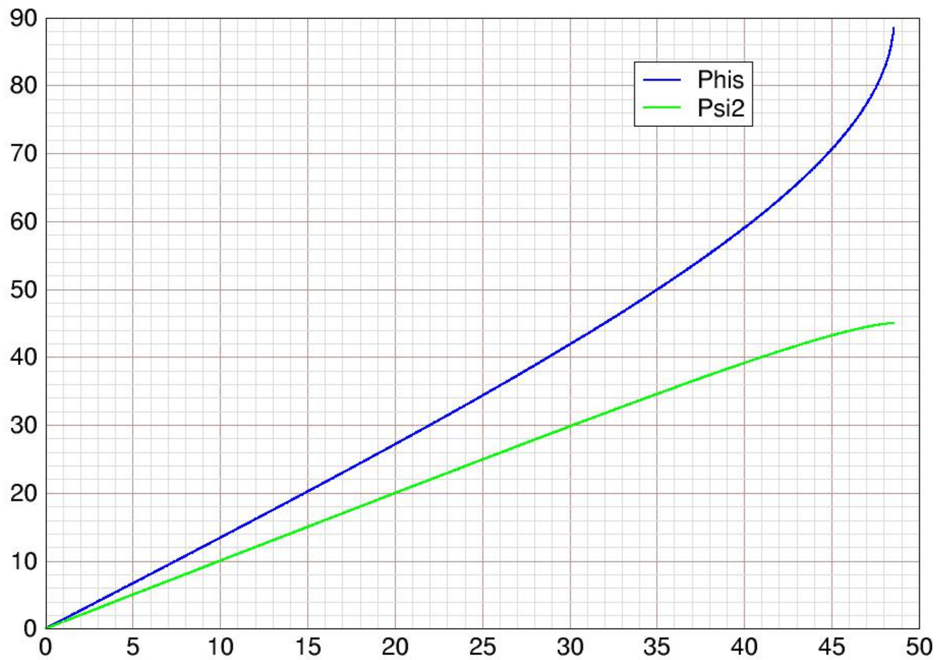

Figure 15: Double-harmonic phases ϕ_s and ψ_2 plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives ϕ_s and ψ_2 in degrees. The upper (blue) and lower (green) curves are ϕ_s and ψ_2, respectively. The phase ϕ_1 is given by (425). As $\sin\phi_1$ approaches $3/4$, the phases ϕ_s and ψ_2 approach $\pi/2$ and $\pi/4$, respectively.

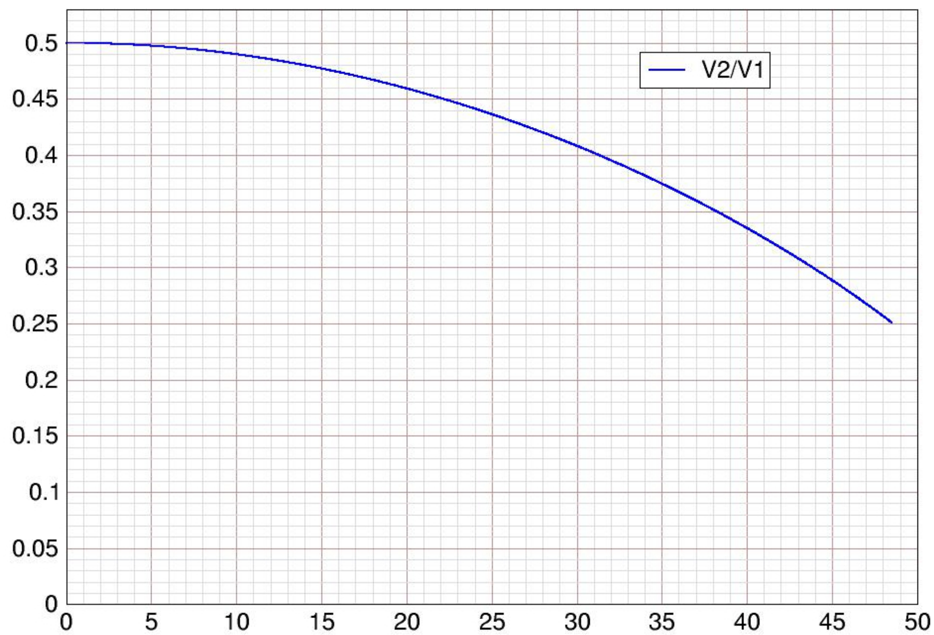

Figure 16: Double-harmonic voltage ratio V_2/V_1 plotted as a function of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives V_2/V_1. The phase ϕ_1 is given by (425). As $\sin\phi_1$ approaches 3/4, the voltage ratio approaches 1/4.

42 Double-harmonic bucket turning point phases and normalized area

The turning point phases ψ_e and ψ_u and normalized area for the double-harmonic bucket are plotted as functions of ϕ_1 in Figures 17 and 18.

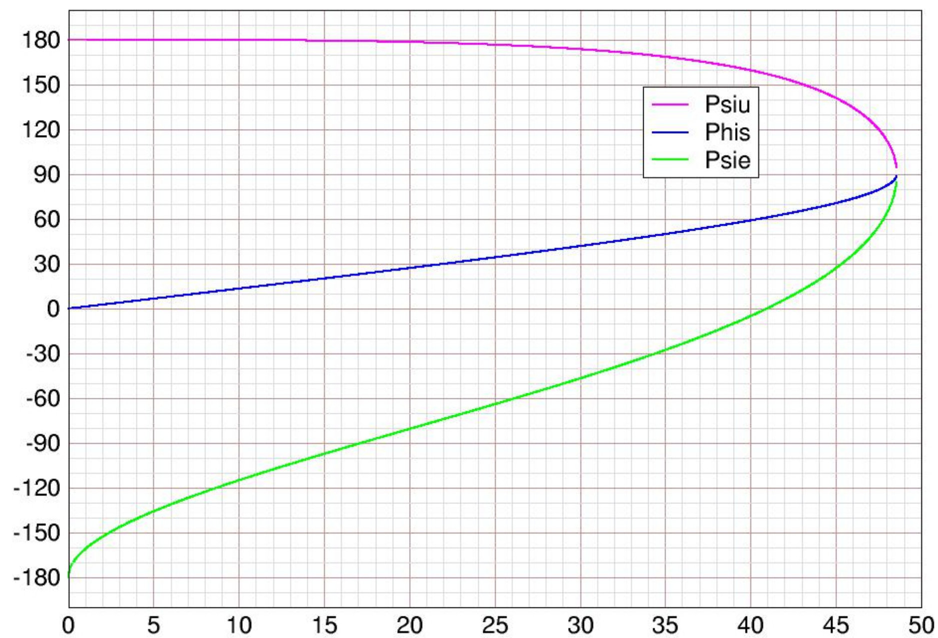

Figure 17: Double-harmonic phases ψ_e, ϕ_s, and ψ_u plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the phases in degrees. The lower (green) and upper (pink) curves are ψ_e and ψ_u, respectively. The middle curve (blue) is the synchronous phase ϕ_s. The phase ϕ_1 is given by (425). The double-harmonic bucket extends from turning point phase ψ_e to unstable fixed point phase ψ_u. As $\sin\phi_1$ approaches $3/4$, all three phases approach $\pi/2$, and the bucket phase width $\psi_u - \psi_e$ goes to zero.

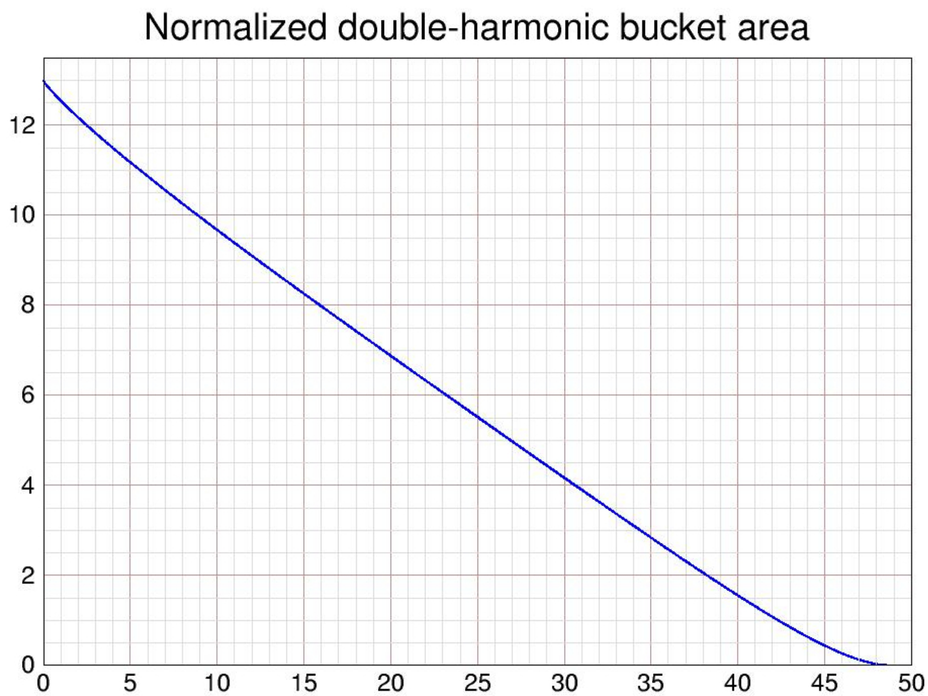

Figure 18: Normalized double-harmonic bucket area plotted as a function of phase ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the normalized area $\mathcal{B}_2$ defined in Section 38. As $\sin\phi_1$ approaches $3/4$, the area goes to zero.

43 Double-harmonic parameters for $G\gamma = 4.5$

For

$$G\gamma = 4.5 \tag{434}$$

we have

$$\beta\gamma^2 = 5.7783768, \quad B\rho = 7.2051786 \text{ Tm}, \quad B = 843.91166 \text{ G} \tag{435}$$

Taking

$$dB/dt = 0.01 \text{ G/ms} \tag{436}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{437}$$

then gives

$$V_1' = 7.498 \text{ kV}, \quad \phi_1' = 0.52657 \text{ degrees}. \tag{438}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 0.69601 \text{ degrees} \tag{439}$$

and therefore

$$\phi_s = 0.9280, \quad \psi_2 = 0.6960 \text{ degrees} \tag{440}$$

$$\frac{V_1}{V_1'} = 0.7566, \quad \frac{V_2}{V_1} = 0.49995. \tag{441}$$

The resulting single and double-harmonic buckets are shown in Figure 19 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 195.4 and 222.9 degrees respectively. The ratio of single to double-harmonic bunch width is 0.877.

The single and double-harmonic bunch heights are 1.224 and 0.9654 respectively. The ratio of double to single-harmonic bunch height is 0.7885.

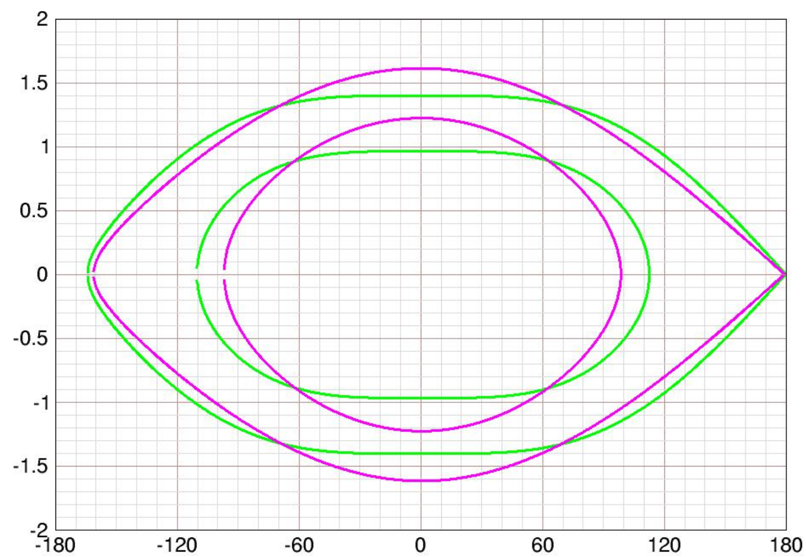

Figure 19: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 4.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 195.4 and 222.9 degrees respectively. The ratio of single to double-harmonic bunch width is 0.877. The single and double-harmonic bunch heights are 1.224 and 0.9654 respectively. The ratio of double to single-harmonic bunch height is 0.7885. The horizontal axis gives the RF phase ψ in degrees.

44 Double-harmonic parameters for $G\gamma = 6.0$

For

$$G\gamma = 6.0 \tag{442}$$

we have

$$\beta\gamma^2 = 10.688256, \quad B\rho = 9.9955569 \text{ Tm}, \quad B = 1170.7367 \text{ G} \tag{443}$$

Taking

$$dB/dt = 9.0 \text{ G/ms} \tag{444}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{445}$$

then gives

$$V_1' = 89.942 \text{ kV}, \quad \phi_1' = 43.59258 \text{ degrees.} \tag{446}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 38.6424 \text{ degrees} \tag{447}$$

and therefore

$$\phi_s = 56.3678, \quad \psi_2 = 37.90275 \text{ degrees} \tag{448}$$

$$\frac{V_1}{V_1'} = 1.1042, \quad \frac{V_2}{V_1} = 0.3464. \tag{449}$$

The resulting single and double-harmonic buckets are shown in Figure 20 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 86.79 and 122.8 degrees respectively. The ratio of single to double-harmonic bunch width is 0.707.

The single and double-harmonic bunch heights are 0.4052 and 0.2569 respectively. The ratio of double to single-harmonic bunch height is 0.634.

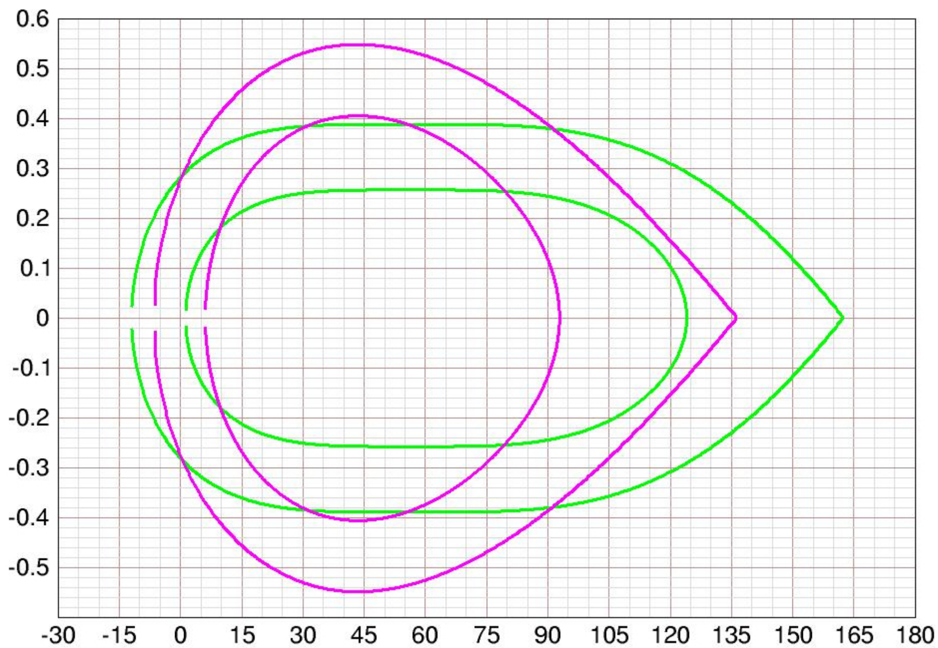

Figure 20: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 6.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 86.79 and 122.8 degrees respectively. The ratio of single to double-harmonic bunch width is 0.707. The single and double-harmonic bunch heights are 0.4052 and 0.2569 respectively. The ratio of double to single-harmonic bunch height is 0.634. The horizontal axis gives the RF phase ψ in degrees.

45 Double-harmonic parameters for $G\gamma = 7.5$

For

$$G\gamma = 7.5 \tag{450}$$

we have

$$\beta\gamma^2 = 16.992559, \quad B\rho = 12.713026 \text{ Tm}, \quad B = 1489.0222 \text{ G} \tag{451}$$

Taking

$$dB/dt = 18.0 \text{ G/ms} \tag{452}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{453}$$

then gives

$$V_1' = 152.715 \text{ kV}, \quad \phi_1' = 54.3111 \text{ degrees.} \tag{454}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 42.56825 \text{ degrees} \tag{455}$$

and therefore

$$\phi_s = 64.4165, \quad \psi_2 = 41.2951, \text{ degrees} \tag{456}$$

$$\frac{V_1}{V_1'} = 1.2006, \quad \frac{V_2}{V_1} = 0.3122. \tag{457}$$

The resulting single and double-harmonic buckets are shown in Figure 21 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 66.02 and 100.92 degrees respectively. The ratio of single to double-harmonic bunch width is 0.654.

The single and double-harmonic bunch heights are 0.2656 and 0.1556 respectively. The ratio of double to single-harmonic bunch height is 0.586.

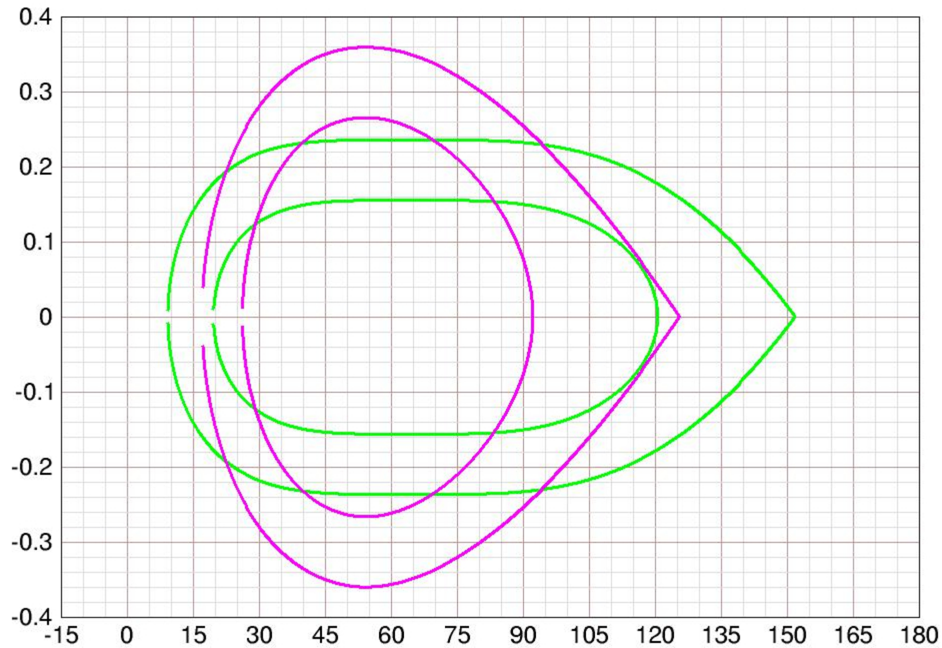

Figure 21: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 7.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 66.02 and 100.92 degrees respectively. The ratio of single to double-harmonic bunch width is 0.654. The single and double-harmonic bunch heights are 0.2656 and 0.1556 respectively. The ratio of double to single-harmonic bunch height is 0.586. The horizontal axis gives the RF phase ψ in degrees.

46 Double-harmonic parameters for $G\gamma = 10.0$

For

$$G\gamma = 10.0 \tag{458}$$

we have

$$\beta\gamma^2 = 30.606873, \quad B\rho = 17.173957 \text{ Tm}, \quad B = 2011.5119 \text{ G} \tag{459}$$

Taking

$$dB/dt = 22.0 \text{ G/ms} \tag{460}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{461}$$

then gives

$$V_1' = 171.219 \text{ kV}, \quad \phi_1' = 62.30122 \text{ degrees.} \tag{462}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 44.719265 \text{ degrees} \tag{463}$$

and therefore

$$\phi_s = 69.7478, \quad \psi_2 = 42.9610, \text{ degrees} \tag{464}$$

$$\frac{V_1}{V_1'} = 1.2583, \quad \frac{V_2}{V_1} = 0.2915. \tag{465}$$

The resulting single and double-harmonic buckets are shown in Figure 22 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 50.94156 and 83.8858 degrees respectively. The ratio of single to double-harmonic bunch width is 0.607.

The single and double-harmonic bunch heights are 0.17876 and 0.09717 respectively. The ratio of double to single-harmonic bunch height is 0.544.

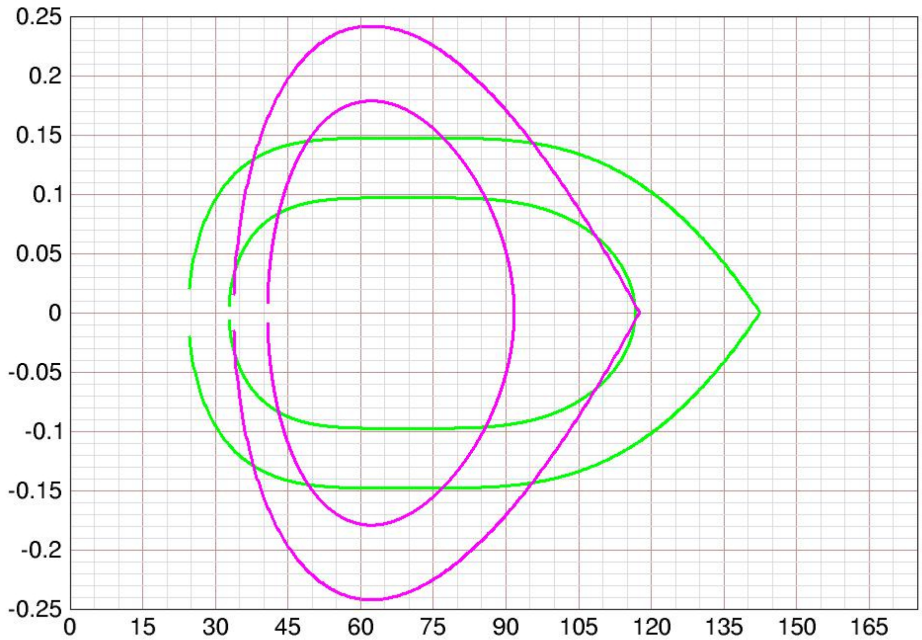

Figure 22: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 10.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 50.94156 and 83.8858 degrees respectively. The ratio of single to double-harmonic bunch width is 0.607. The single and double-harmonic bunch heights are 0.17876 and 0.09717 respectively. The ratio of double to single-harmonic bunch height is 0.544. The horizontal axis gives the RF phase ψ in degrees.

47 Double-harmonic parameters for $G\gamma = 12.5$

For

$$G\gamma = 12.5 \tag{466}$$

we have

$$\beta\gamma^2 = 48.108272, \quad B\rho = 21.595395 \text{ Tm}, \quad B = 2529.3760 \text{ G} \tag{467}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{468}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{469}$$

then gives

$$V_1' = 184.954 \text{ kV}, \quad \phi_1' = 68.6577 \text{ degrees.} \tag{470}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 46.0637 \text{ degrees} \tag{471}$$

and therefore

$$\phi_s = 73.7703, \quad \psi_2 = 43.8738 \text{ degrees} \tag{472}$$

$$\frac{V_1}{V_1'} = 1.2934, \quad \frac{V_2}{V_1} = 0.27775. \tag{473}$$

The resulting single and double-harmonic buckets are shown in Figure 23 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 39.1183 and 69.5079 degrees respectively. The ratio of single to double-harmonic bunch width is 0.563.

The single and double-harmonic bunch heights are 0.1198 and 0.06032 respectively. The ratio of double to single-harmonic bunch height is 0.503.

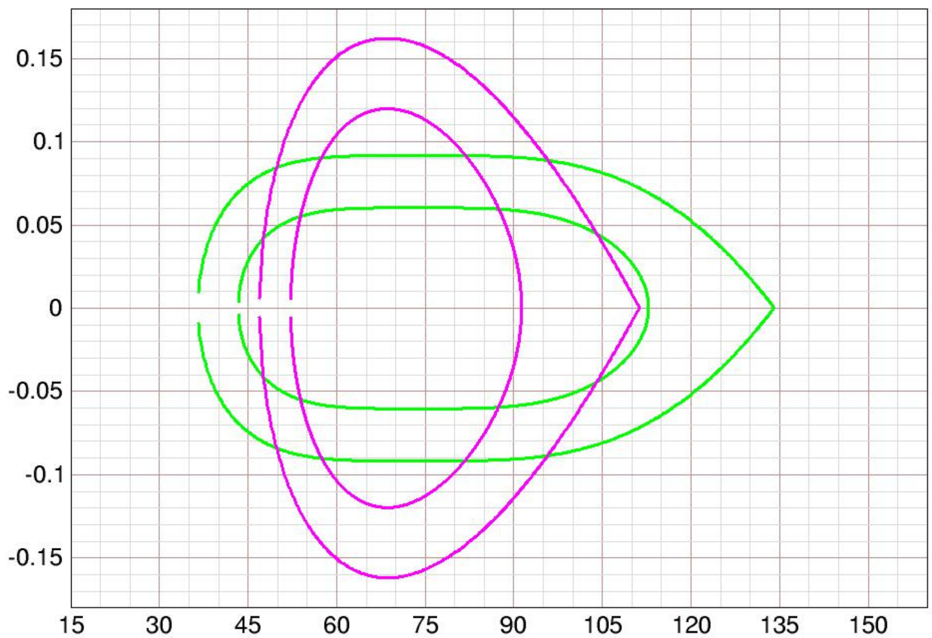

Figure 23: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 12.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 39.1183 and 69.5079 degrees respectively. The ratio of single to double-harmonic bunch width is 0.563. The single and double-harmonic bunch heights are 0.1198 and 0.06032 respectively. The ratio of double to single-harmonic bunch height is 0.503. The horizontal axis gives the RF phase ψ in degrees.

48 Double-harmonic parameters for $G\gamma = 14.0$

For

$$G\gamma = 14.0 \tag{474}$$

we have

$$\beta\gamma^2 = 60.475409, \quad B\rho = 24.238302 \text{ Tm}, \quad B = 2838.9283 \text{ G} \tag{475}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{476}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{477}$$

then gives

$$V_1' = 180.410 \text{ kV}, \quad \phi_1' = 72.72346 \text{ degrees.} \tag{478}$$

We can then adjust ϕ_1 to give a normalized double-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 46.7777 \text{ degrees} \tag{479}$$

and therefore

$$\phi_s = 76.3130, \quad \psi_2 = 44.2976 \text{ degrees} \tag{480}$$

$$\frac{V_1}{V_1'} = 1.3104, \quad \frac{V_2}{V_1} = 0.2702. \tag{481}$$

The resulting single and double-harmonic buckets are shown in Figure 24 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and double-harmonic bunch widths are 31.6137 and 59.7323 degrees respectively. The ratio of single to double-harmonic bunch width is 0.529.

The single and double-harmonic bunch heights are 0.08691 and 0.04113 respectively. The ratio of double to single-harmonic bunch height is 0.473.

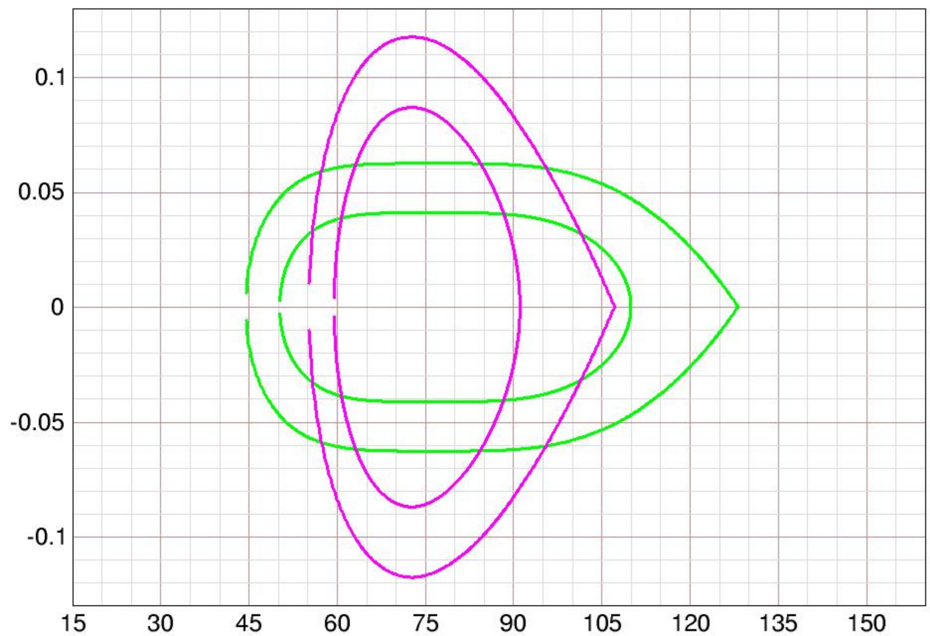

Figure 24: Normalized single and double-harmonic buckets and matched bunches obtained for $G\gamma = 14.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and double-harmonic bunch widths are 31.6137 and 59.7323 degrees respectively. The ratio of single to double-harmonic bunch width is 0.529. The single and double-harmonic bunch heights are 0.08691 and 0.04113 respectively. The ratio of double to single-harmonic bunch height is 0.473. The horizontal axis gives the RF phase ψ in degrees.

49 Double-harmonic parameter summary for acceleration of polarized protons in AGS

The following tables summarize the data of Sections 43 through 48. The guide field B and its time derivative are given in units of G and G/ms. The single-harmonic RF voltage V_1' is given in units of kV. The phases ϕ_1', ϕ_1, ϕ_s, ψ_2, and the single and double-harmonic bunch widths W_1 and W_2 are given in degrees. The ratio H_2/H_1 is the ratio of double to single-harmonic bunch height. Values of $\beta\gamma^2$, W_1, W_2, and H_2/H_1 are tabulated for comparison of the incoherent space charge tune shifts in the single and double-harmonic bunches.

Table 13: Double-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1
843.9	0.01	4.5	7.498	0.7566	0.49995
1170.7	9.0	6.0	89.942	1.1042	0.3464
1489.0	18.0	7.5	152.715	1.2006	0.3122
2011.5	22.0	10.0	171.219	1.2583	0.2915
2529.4	25.0	12.5	184.954	1.2934	0.27775
2838.9	25.0	14.0	180.410	1.3104	0.2702

Table 14: Double-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2
0.01	4.5	0.52657	0.69601	0.9280	0.6960
9.0	6.0	43.59258	38.6424	56.3678	37.90275
18.0	7.5	54.3111	42.5683	64.4165	41.2951
22.0	10.0	62.30122	44.7193	69.7478	42.9610
25.0	12.5	68.6577	46.0637	73.7703	43.8738
25.0	14.0	72.72346	46.7777	76.3130	44.2976

Table 15: Double-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_2	W_1/W_2	H_2/H_1
0.01	4.5	5.7784	195.4	222.9	0.877	0.7885
9.0	6.0	10.6883	86.79	122.8	0.707	0.634
18.0	7.5	16.9926	66.02	100.92	0.654	0.586
22.0	10.0	30.6069	50.94	83.89	0.607	0.544
25.0	12.5	48.1083	39.12	69.51	0.563	0.503
25.0	14.0	60.4754	31.61	59.73	0.529	0.473

50 Comparison of the incoherent tune shifts in the single and double-harmonic bunches

Let δQ_1 and δQ_2 be the incoherent space charge tune shifts in the single and double-harmonic bunches, respectively. Then the ratio

$$\frac{\delta Q_2}{\delta Q_1} = \frac{B_1}{B_2} \tag{482}$$

where B_1 and B_2 are the corresponding bunching factors. Here one may simply take

$$B_1 = \frac{W_1}{2\pi h}, \quad B_2 = \frac{W_2}{2\pi h} \tag{483}$$

where the bunch widths W_1 and W_2 are given in radians. This gives

$$\frac{\delta Q_2}{\delta Q_1} = \frac{W_1}{W_2}. \tag{484}$$

This ratio is the reduction of space charge tune shift due to the lengthing of the bunch in the double-harmonic bucket. It is tabulated in the sixth column of Table 15 and goes from 0.877 to 0.529 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

As shown in Section 32, one may also take

$$\frac{\delta Q_2}{\delta Q_1} = \frac{H_2}{H_1}. \tag{485}$$

This ratio is tabulated in the last column of Table 15 and goes from 0.789 to 0.473 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

51 Reduction of incoherent tune shift with increasing gamma

Let $(\delta Q)_I$ and $(\delta Q)_F$ be the initial and final incoherent tune shifts as proton bunches are accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$. Then the reduction in tune shift is given by the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{B_I\,(\beta\gamma^2)_I}{B_F\,(\beta\gamma^2)_F} \tag{486}$$

where bunching factors

$$B_I = \frac{W_I}{2\pi h}, \quad B_F = \frac{W_F}{2\pi h} \tag{487}$$

and W_I and W_F are the initial and final bunch widths (in radians). Thus the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{W_I\,(\beta\gamma^2)_I}{W_F\,(\beta\gamma^2)_F}. \tag{488}$$

Putting in numbers from columns three and four of Table 15 we obtain

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.591 \tag{489}$$

for bunches in the single-harmonic bucket.

Putting in numbers from columns three and five of the table gives

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.357 \tag{490}$$

for bunches in the double-harmonic bucket.

This shows that the double-harmonic bucket gives a significantly greater reduction in incoherent tune shift with increasing gamma. For the triple-harmonic bucket one has

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.273 \tag{491}$$

as shown in Section 33.

52 Potential for a quad-harmonic bucket

Defining

$$\psi = \phi + \phi_s \tag{492}$$

$$C_1(\psi) = \cos\psi, \quad C_2(\psi) = \cos(2\psi - 2\psi_2) \tag{493}$$

$$C_3(\psi) = \cos(3\psi - 3\psi_3), \quad C_4(\psi) = \cos(4\psi - 4\psi_4) \tag{494}$$

and

$$S_1(\psi) = \sin\psi, \quad S_2(\psi) = \sin(2\psi - 2\psi_2) \tag{495}$$

$$S_3(\psi) = \sin(3\psi - 3\psi_3), \quad S_4(\psi) = \sin(4\psi - 4\psi_4) \tag{496}$$

we have

$$U = A_1C_1(\psi) - \frac{1}{2}A_2C_2(\psi) + \frac{1}{3}A_3C_3(\psi) - \frac{1}{4}A_4C_4(\psi) + C\psi \tag{497}$$

$$\frac{\partial U}{\partial \psi} = -A_1S_1(\psi) + A_2S_2(\psi) - A_3S_3(\psi) + A_4S_4(\psi) + C \tag{498}$$

$$\frac{\partial^2 U}{\partial \psi^2} = -A_1C_1(\psi) + 2A_2C_2(\psi) - 3A_3C_3(\psi) + 4A_4C_4(\psi) \tag{499}$$

$$\frac{\partial^3 U}{\partial \psi^3} = A_1S_1(\psi) - 4A_2S_2(\psi) + 9A_3S_3(\psi) - 16A_4S_4(\psi) \tag{500}$$

$$\frac{\partial^4 U}{\partial \psi^4} = A_1C_1(\psi) - 8A_2C_2(\psi) + 27A_3C_3(\psi) - 64A_4C_4(\psi) \tag{501}$$

$$\frac{\partial^5 U}{\partial \psi^5} = -A_1S_1(\psi) + 16A_2S_2(\psi) - 81A_3S_3(\psi) + 256A_4S_4(\psi) \tag{502}$$

$$\frac{\partial^6 U}{\partial \psi^6} = -A_1C_1(\psi) + 32A_2C_2(\psi) - 243A_3C_3(\psi) + 1024A_4C_4(\psi) \tag{503}$$

$$\frac{\partial^7 U}{\partial \psi^7} = A_1S_1(\psi) - 64A_2S_2(\psi) + 729A_3S_3(\psi) - 4096A_4S_4(\psi) \tag{504}$$

and so on, where

$$C = A_1S_1(\phi_s) - A_2S_2(\phi_s) + A_3S_3(\phi_s) - A_4S_4(\phi_s). \tag{505}$$

Using integer superscripts to denote the number of differentiations with respect to ψ, we have

$$U^1(\phi_s) = 0 \tag{506}$$

$$U^2(\phi_s) = -A_1C_1 + 2A_2C_2 - 3A_3C_3 + 4A_4C_4 \tag{507}$$
$$U^3(\phi_s) = A_1S_1 - 4A_2S_2 + 9A_3S_3 - 16A_4S_4 \tag{508}$$
$$U^4(\phi_s) = A_1C_1 - 8A_2C_2 + 27A_3C_3 - 64A_4C_4 \tag{509}$$
$$U^5(\phi_s) = -A_1S_1 + 16A_2S_2 - 81A_3S_3 + 256A_4S_4 \tag{510}$$
$$U^6(\phi_s) = -A_1C_1 + 32A_2C_2 - 243A_3C_3 + 1024A_4C_4 \tag{511}$$
$$U^7(\phi_s) = A_1S_1 - 64A_2S_2 + 729A_3S_3 - 4096A_4S_4 \tag{512}$$

where

$$C_1 = C_1(\phi_s), \quad C_2 = C_2(\phi_s), \quad C_3 = C_3(\phi_s), \quad C_4 = C_4(\phi_s) \tag{513}$$
$$S_1 = S_1(\phi_s), \quad S_2 = S_2(\phi_s), \quad S_3 = S_3(\phi_s), \quad S_4 = S_4(\phi_s). \tag{514}$$

53 Conditions for a flattened quad-harmonic bucket

For a flattened RF bucket we want derivatives

$$U^2(\phi_s) = U^3(\phi_s) = U^4(\phi_s) = U^5(\phi_s) = U^6(\phi_s) = U^7(\phi_s) = 0 \tag{515}$$

which gives

$$A_1C_1 = 2A_2C_2 - 3A_3C_3 + 4A_4C_4 \tag{516}$$
$$-A_1C_1 = -8A_2C_2 + 27A_3C_3 - 64A_4C_4 \tag{517}$$
$$A_1C_1 = 32A_2C_2 - 243A_3C_3 + 1024A_4C_4 \tag{518}$$

and

$$-A_1S_1 = -4A_2S_2 + 9A_3S_3 - 16A_4S_4 \tag{519}$$
$$A_1S_1 = 16A_2S_2 - 81A_3S_3 + 256A_4S_4 \tag{520}$$
$$-A_1S_1 = -64A_2S_2 + 729A_3S_3 - 4096A_4S_4. \tag{521}$$

These equations imply

$$0 = -6A_2C_2 + 24A_3C_3 - 60A_4C_4 \tag{522}$$
$$0 = 24A_2C_2 - 216A_3C_3 + 960A_4C_4 \tag{523}$$
$$0 = 6A_2C_2 - 54A_3C_3 + 240A_4C_4 \tag{524}$$
$$0 = -30A_3C_3 + 180A_4C_4 \tag{525}$$

and

$$0 = 12A_2S_2 - 72A_3S_3 + 240A_4S_4 \tag{526}$$

$$0 = -48A_2S_2 + 648A_3S_3 - 3840A_4S_4 \tag{527}$$

$$0 = -12A_2S_2 + 162A_3S_3 - 960A_4S_4 \tag{528}$$

$$0 = 90A_3S_3 - 720A_4S_4. \tag{529}$$

One then finds that

$$A_4C_4 = \frac{1}{6}\,A_3C_3, \quad A_3C_3 = \frac{3}{7}\,A_2C_2, \quad A_2C_2 = A_1C_1 \tag{530}$$

$$A_4S_4 = \frac{1}{8}\,A_3S_3, \quad A_3S_3 = \frac{2}{7}\,A_2S_2, \quad A_2S_2 = \frac{1}{2}\,A_1S_1 \tag{531}$$

which give

$$A_4C_4 = \frac{1}{14}\,A_1C_1, \quad A_3C_3 = \frac{3}{7}\,A_1C_1, \quad A_2C_2 = A_1C_1 \tag{532}$$

$$A_4S_4 = \frac{1}{56}\,A_1S_1, \quad A_3S_3 = \frac{1}{7}\,A_1S_1, \quad A_2S_2 = \frac{1}{2}\,A_1S_1 \tag{533}$$

and therefore

$$\frac{S_4}{C_4} = \frac{1}{4}\,\frac{S_1}{C_1}, \quad \frac{S_3}{C_3} = \frac{1}{3}\,\frac{S_1}{C_1}, \quad \frac{S_2}{C_2} = \frac{1}{2}\,\frac{S_1}{C_1} \tag{534}$$

$$\frac{A_4}{A_1} = \frac{1}{14}\,\frac{C_1}{C_4} = \frac{1}{56}\,\frac{S_1}{S_4} \tag{535}$$

$$\frac{A_3}{A_1} = \frac{3}{7}\,\frac{C_1}{C_3} = \frac{1}{7}\,\frac{S_1}{S_3} \tag{536}$$

$$\frac{A_2}{A_1} = \frac{C_1}{C_2} = \frac{1}{2}\,\frac{S_1}{S_2}. \tag{537}$$

Thus, if the value of ϕ_s is given, we can calculate values for the phases ψ_2, ψ_3, and ψ_4, and the ratios A_2/A_1, A_3/A_1, and A_4/A_1. Specifically one has

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{538}$$

$$\psi_3 = \phi_s - \frac{1}{3}\arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{539}$$

$$\psi_4 = \phi_s - \frac{1}{4}\arctan\left\{\frac{1}{4}\tan\phi_s\right\} \tag{540}$$

and

$$\frac{A_2}{A_1} = \left\{\frac{\cos\phi_s}{\cos(2\phi_s - 2\psi_2)}\right\} = \frac{1}{2}\left\{\frac{\sin\phi_s}{\sin(2\phi_s - 2\psi_2)}\right\} \tag{541}$$

$$\frac{A_3}{A_1} = \frac{3}{7}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{7}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\} \tag{542}$$

$$\frac{A_4}{A_1} = \frac{1}{14}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{56}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\}. \tag{543}$$

54 Synchronous phase for flattened quad-harmonic bucket

As a function of phase, the RF voltage satisfies

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right) U^1(\psi) \tag{544}$$

where

$$-U^1(\psi) = A_1 S_1(\psi) - A_2 S_2(\psi) + A_3 S_3(\psi) - A_4 S_4(\psi) - C \tag{545}$$

$$C = A_1 S_1 - A_2 S_2 + A_3 S_3 - A_4 S_4. \tag{546}$$

Thus, using

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h}, \quad A_3 = \frac{eQV_3}{2\pi h}, \quad A_4 = \frac{eQV_4}{2\pi h} \tag{547}$$

we have

$$V(\psi) = V_1 S_1(\psi) - V_2 S_2(\psi) + V_3 S_3(\psi) - V_4 S_4(\psi). \tag{548}$$

The synchronous phase ϕ_s must satisfy

$$V(\phi_s) = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{549}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. Here it is useful to define phase ϕ_1 such that

$$V_1 \sin\phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right). \tag{550}$$

We then have

$$V_1 S_1 - V_2 S_2 + V_3 S_3 - V_4 S_4 = V_1 \sin\phi_1 \tag{551}$$

which gives

$$S_1 = \sin\phi_1 + \frac{V_2}{V_1}\, S_2 - \frac{V_3}{V_1}\, S_3 + \frac{V_4}{V_1}\, S_4. \tag{552}$$

Using

$$\frac{V_2}{V_1} = \frac{A_2}{A_1}, \quad \frac{V_3}{V_1} = \frac{A_3}{A_1}, \quad \frac{V_4}{V_1} = \frac{A_4}{A_1} \tag{553}$$

we then have

$$S_1 = \sin\phi_1 + \frac{A_2}{A_1}\, S_2 - \frac{A_3}{A_1}\, S_3 + \frac{A_4}{A_1}\, S_4 \tag{554}$$

where, as shown in the previous section,

$$\frac{A_2}{A_1} = \frac{1}{2}\,\frac{S_1}{S_2}, \quad \frac{A_3}{A_1} = \frac{1}{7}\,\frac{S_1}{S_3}, \quad \frac{A_4}{A_1} = \frac{1}{56}\,\frac{S_1}{S_4}. \tag{555}$$

Thus

$$S_1 = \sin\phi_1 + \frac{1}{2}\, S_1 - \frac{1}{7}\, S_1 + \frac{1}{56}\, S_1 \tag{556}$$

$$S_1 \left\{ 1 - \frac{1}{2} + \frac{1}{7} - \frac{1}{56} \right\} = \sin\phi_1 \tag{557}$$

$$S_1 = \frac{8}{5}\, \sin\phi_1 \tag{558}$$

and therefore

$$\phi_s = \arcsin\left(\frac{8}{5}\, \sin\phi_1 \right). \tag{559}$$

Here we see that in order to have a real synchronous phase we must have

$$0 \le \sin\phi_1 \le \frac{5}{8}. \tag{560}$$

This constraint satisfied if

$$0 \le \phi_1 \le 38.6822^\circ \tag{561}$$

or

$$141.3178^\circ \le \phi_1 < 180^\circ. \tag{562}$$

If a particular ϕ_1 is given, we can obtain ϕ_s from (559). This in turn can be used in (538), (539), and (540) to obtain ψ_2, ψ_3, and ψ_4. Finally, ϕ_s, ψ_2, ψ_3, and ψ_4 can be used in (541), (542), and (543) to obtain the ratios V_2/V_1, V_3/V_1, and V_4/V_1. The voltage V_1 is given by (550).

55 Normalized quad-harmonic voltage and potential

It is convenient to normalize the voltage and potential so that they are dimensionless and completely determined by the phase ϕ_1. We define normalized voltage

$$\mathcal{V}(\psi) = \frac{1}{V_1}\, V(\psi) \tag{563}$$

and normalized potential

$$\mathcal{U}(\psi) = \frac{1}{A_1}\left\{U(\psi_u) - U(\psi)\right\} \tag{564}$$

with derivatives

$$\mathcal{U}^m(\psi) = -\frac{1}{A_1}\, U^m(\psi). \tag{565}$$

Here

$$V(\psi) = V_1 S_1(\psi) - V_2 S_2(\psi) + V_3 S_3(\psi) - V_4 S_4(\psi) \tag{566}$$

$$V(\psi) - V(\phi_s) = -\left(\frac{2\pi h}{eQ}\right) U^1(\psi) \tag{567}$$

$$U(\psi) = A_1 C_1(\psi) - \frac{1}{2} A_2 C_2(\psi) + \frac{1}{3} A_3 C_3(\psi) - \frac{1}{4} A_4 C_4(\psi) + C\psi \tag{568}$$

$$U^1(\psi) = -A_1 S_1(\psi) + A_2 S_2(\psi) - A_3 S_3(\psi) + A_4 S_4(\psi) + C \tag{569}$$

$$U^2(\psi) = -A_1 C_1(\psi) + 2A_2 C_2(\psi) - 3A_3 C_3(\psi) + 4A_4 C_4(\psi) \tag{570}$$

$$C = A_1 S_1 - A_2 S_2 + A_3 S_3 - A_4 S_4 \tag{571}$$

and therefore

$$\mathcal{V}(\psi) - \mathcal{V}(\phi_s) = \mathcal{U}^1(\psi) \tag{572}$$

$$\mathcal{U}(\psi) = \mathcal{D} - C_1(\psi) + \frac{1}{2} R_2 C_2(\psi) - \frac{1}{3} R_3 C_3(\psi) + \frac{1}{4} R_4 C_4(\psi) - \mathcal{C}\psi \tag{573}$$

$$\mathcal{U}^1(\psi) = S_1(\psi) - R_2 S_2(\psi) + R_3 S_3(\psi) - R_4 S_4(\psi) - \mathcal{C} \tag{574}$$

$$\mathcal{U}^2(\psi) = -C_1(\psi) + 2R_2 C_2(\psi) - 3R_3 C_3(\psi) + 4R_4 C_4(\psi) \tag{575}$$

where

$$\mathcal{D} = C_1(\psi_u) - \frac{1}{2} R_2 C_2(\psi_u) + \frac{1}{3} R_3 C_3(\psi_u) - \frac{1}{4} R_4 C_4(\psi_u) + \mathcal{C}\psi_u \tag{576}$$

$$\mathcal{C} = S_1 - R_2 S_2 + R_3 S_3 - R_4 S_4 \tag{577}$$

and

$$R_2 = \frac{A_2}{A_1}, \quad R_3 = \frac{A_3}{A_1}, \quad R_4 = \frac{A_4}{A_1}. \tag{578}$$

The phase ψ_u in these equations is the unstable fixed point phase associated with oscillations about the synchronous phase ϕ_s. It satisfies

$$\mathcal{U}(\psi_u) = 0, \quad \mathcal{U}^1(\psi_u) = 0. \tag{579}$$

Below transition one has

$$0 < \phi_s < \psi_u \tag{580}$$

and

$$\mathcal{U}^2(\psi_u) < 0. \tag{581}$$

There is an additional phase

$$\psi_e < \phi_s < \psi_u \tag{582}$$

that satisfies

$$\mathcal{U}(\psi_e) = \mathcal{U}(\psi_u) = 0. \tag{583}$$

The equations of this and the previous section show that the normalized voltage and potential are completely determined by the phase ϕ_1. The phases ψ_u and ψ_e are also completely determined by ϕ_1.

56 Quad-harmonic bucket width and area

The RF bucket associated with the stable fixed point phase ϕ_s extends from ψ_e to ψ_u. It is defined by the curves $W(\psi)$ where

$$W^2(\psi) = \frac{2}{a}\left\{U(\psi_u) - U(\psi)\right\} \tag{584}$$

$$a = \frac{h^2 c^2 \eta_s}{R_s^2 E_s}, \quad \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}, \quad E_s = mc^2\gamma_s. \tag{585}$$

The phase width of the bucket is

$$\Delta\psi = |\psi_u - \psi_e|. \tag{586}$$

In terms of the normalized potential (564) we have

$$W^2(\psi) = \frac{2}{a} A_1 \mathcal{U}(\psi) \tag{587}$$

and

$$W(\psi) = \pm \left(\frac{2A_1}{|a|} \right)^{1/2} |\mathcal{U}(\psi)|^{1/2} . \tag{588}$$

Here

$$A_1 = \frac{eQV_1}{2\pi h}, \quad \frac{1}{|a|} = \frac{R_s^2 E_s}{h^2 c^2 |\eta_s|} \tag{589}$$

which gives

$$\left(\frac{2A_1}{|a|} \right)^{1/2} = \frac{R_s}{hc} \left\{ \frac{eQV_1 E_s}{\pi h |\eta_s|} \right\}^{1/2} = \frac{B_1}{8\sqrt{2}} \tag{590}$$

where

$$B_1 = 8 \, \frac{R_s}{hc} \left\{ \frac{2eQV_1 E_s}{\pi h |\eta_s|} \right\}^{1/2} \tag{591}$$

is the single-harmonic stationary bucket area. Thus we have

$$W(\psi) = \pm \frac{B_1}{8\sqrt{2}} \, |\mathcal{U}(\psi)|^{1/2} . \tag{592}$$

The normalized quad-harmonic RF bucket is defined by the curves

$$\mathcal{W}(\psi) = \pm \, |\mathcal{U}(\psi)|^{1/2} \tag{593}$$

which are dimensionless and completely determined by the phase ϕ_1.

The area of the quad-harmonic bucket is

$$\mathcal{A}_4 = 2 \int_{\psi_e}^{\psi_u} |W(\psi)| \, d\psi \tag{594}$$

where

$$|W(\psi)| = \frac{B_1}{8\sqrt{2}} \, |\mathcal{U}(\psi)|^{1/2} . \tag{595}$$

Defining normalized bucket area

$$\mathcal{B}_4 = 2 \int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2} \, d\psi \tag{596}$$

we then have

$$\mathcal{A}_4 = \frac{B_1}{8\sqrt{2}} \, \mathcal{B}_4 . \tag{597}$$

The normalized bucket area is dimensionless and is completely determined by the phase ϕ_1.

57 Bunch matched to quad-harmonic bucket

Consider a particle moving along the boundary of a bunch matched to the quad-harmonic bucket and let ψ_R be the right turning point phase of the boundary. Below transition one has

$$\psi_e < \phi_s < \psi_R < \psi_u. \tag{598}$$

The corresponding left turning point phase ψ_L satisfies

$$\psi_e < \psi_L < \phi_s < \psi_R < \psi_u \tag{599}$$

and

$$\mathcal{U}(\psi_L) = \mathcal{U}(\psi_R). \tag{600}$$

As in section 11, the normalized bunch boundary is defined by the curves

$$\mathcal{W}(\psi) = \pm\,|\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2} \tag{601}$$

where

$$\psi_L \le \psi \le \psi_R. \tag{602}$$

The area of the bunch is

$$B = \frac{B_1}{8\sqrt{2}}\,\mathcal{B} \tag{603}$$

where

$$B_1 = 8\,\frac{R_s}{hc}\left\{\frac{2eQV_1E_s}{\pi h|\eta_s|}\right\}^{1/2} \tag{604}$$

and

$$\mathcal{B} = 2\int_{\psi_L}^{\psi_R} |\mathcal{U}(\psi) - \mathcal{U}(\psi_R)|^{1/2}\,d\psi \tag{605}$$

is defined to be the normalized area. The normalized bunch boundary and area are dimensionless and completely determined by the turning point phase ψ_R and the phase ϕ_1.

58 Adjusting the ratio of quad to single-harmonic bucket area

For the quad-harmonic bucket we have area

$$\mathcal{A}_4 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_4 \tag{606}$$

and normalized area

$$\mathcal{B}_4 = 2\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\, d\psi \tag{607}$$

where, as shown in Sections 54 and 55, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1.

Similarly, for the single-harmonic bucket we have area

$$\mathcal{A}_1 = \frac{B_1}{8\sqrt{2}}\,\mathcal{B}_1 \tag{608}$$

and normalized area

$$\mathcal{B}_1 = 2\left(\frac{\sin\phi_1}{\sin\phi_1'}\right)^{1/2}\int_{\psi_e}^{\psi_u} |\mathcal{U}(\psi)|^{1/2}\, d\psi \tag{609}$$

where, as shown in Sections 14 and 15, the phases ψ_e and ψ_u, and the normalized potential $\mathcal{U}(\psi)$ are completely determined by the phase ϕ_1'.

Thus, the ratio

$$\frac{\mathcal{A}_4}{\mathcal{A}_1} = \frac{\mathcal{B}_4}{\mathcal{B}_1} \tag{610}$$

is completely determined by ϕ_1 and ϕ_1'. If either of these phases is given, the other can be adjusted to give a desired value for the ratio.

If, for example, we start with a given single-harmonic bucket having synchronous phase ϕ_1' and want a quad-harmonic bucket with the same area, the phase ϕ_1 can be adjusted to give

$$\mathcal{B}_4 = \mathcal{B}_1 \tag{611}$$

and therefore

$$\mathcal{A}_4 = \mathcal{A}_1. \tag{612}$$

Since

$$V_1 \sin\phi_1 = V_1' \sin\phi_1' \tag{613}$$

we also have voltage ratio

$$\frac{V_1}{V_1'} = \frac{\sin\phi_1'}{\sin\phi_1}. \tag{614}$$

If the single-harmonic voltage V_1' is given, we then have the required quad-harmonic voltage V_1.

59 Application of quad-harmonic bucket to acceleration of polarized protons in AGS

For protons

$$mc^2 = 938.272\,088\,16(29) \text{ MeV} \tag{615}$$

$$g = 5.585\,694\,6893(16) \tag{616}$$

$$G = (g-2)/2 = 1.792\,8473\,4465 \tag{617}$$

and in AGS

$$R_s = 128.4526 \text{ m}, \quad \rho_s = 85.378351 \text{ m} \tag{618}$$

$$\gamma_t = 8.5, \quad h = 6. \tag{619}$$

Suppose we have acceleration of polarized protons in AGS set up using just a single RF harmonic and wish to move to a quad-harmonic setup. Let h, V_1', and ϕ_1' be the harmonic number, voltage, and synchronous phase, respectively, for the single-harmonic setup. For a given single-harmonic voltage V_1' the synchronous phase ϕ_1' must satisfy

$$V_1' \sin\phi_1' = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{620}$$

where R_s and ρ_s are the radius and radius-of-curvature of the orbit followed by the synchronous particle, and B is the programmed guide field. According to (550) we also have, for the quad-harmonic bucket,

$$V_1 \sin\phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right) \tag{621}$$

and therefore

$$V_1 \sin\phi_1 = V_1' \sin\phi_1'. \tag{622}$$

As a starting point for moving to a quad-harmonic setup, we would like the quad-harmonic bucket to have the same area as that of the single-harmonic bucket. As shown in the previous section, we can adjust the phase ϕ_1 so that this is so. This would be done by an application at a discrete set of times that amply cover the cycle of the programmed guide field. Having these phases in hand we then have a first pass set of quad-harmonic voltages V_1 given by

$$V_1 = \left(\frac{\sin\phi_1'}{\sin\phi_1}\right) V_1'. \tag{623}$$

These would be the starting voltages in the V_1 voltage program for the quad-harmonic bucket. They could be subsequently tuned if necessary. In the next section it is shown that for any set of programmed values of V_1, the corresponding values of phases ϕ_s, ψ_2, ψ_3, ψ_4 and voltages V_2, V_3, V_4 can be obtained from lookup tables. These then give the quad-harmonic voltage

$$V(\psi) = V_1 \sin\psi - V_2 \sin(2\psi - 2\psi_2) + V_3 \sin(3\psi - 3\psi_3) - V_4 \sin(4\psi - 4\psi_4) \quad (624)$$

throughout the guide field cycle.

60 Lookup tables for quad-harmonic phases and voltage ratios

We assume that voltage V_1 and guide field time-derivative dB/dt are given and require that phase ϕ_1 satisfy

$$V_1 \sin\phi_1 = 2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right). \quad (625)$$

Obtaining the required phase ϕ_1 would be no different than what is currently done to obtain the synchronous phase ϕ_1' in the single-harmonic setup. As shown in Section 54 we must also have

$$0 \le \sin\phi_1 \le \frac{5}{8}. \quad (626)$$

In practice this is no different than requiring that the single-harmonic synchronous phase satisfy

$$0 \le \sin\phi_1' \le 1. \quad (627)$$

The constraint (626) is satisfied if

$$0 \le \phi_1 \le 38.6822^\circ \quad (628)$$

or

$$141.3178^\circ \le \phi_1 < 180^\circ. \quad (629)$$

Starting with

$$\sin\phi_1 = \frac{1}{V_1}\left\{2\pi R_s \rho_s \left(\frac{1}{c}\frac{dB}{dt}\right)\right\} \quad (630)$$

we then have

$$\sin\phi_s = \frac{8}{5}\sin\phi_1 \quad (631)$$

$$\phi_s = \arcsin\left\{\frac{8}{5}\sin\phi_1\right\} \tag{632}$$

$$\tan(2\phi_s - 2\psi_2) = \frac{1}{2}\tan\phi_s \tag{633}$$

$$2\phi_s - 2\psi_2 = \arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{634}$$

$$\psi_2 = \phi_s - \frac{1}{2}\arctan\left\{\frac{1}{2}\tan\phi_s\right\} \tag{635}$$

$$\tan(3\phi_s - 3\psi_2) = \frac{1}{3}\tan\phi_s \tag{636}$$

$$3\phi_s - 3\psi_3 = \arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{637}$$

$$\psi_3 = \phi_s - \frac{1}{3}\arctan\left\{\frac{1}{3}\tan\phi_s\right\} \tag{638}$$

$$\tan(4\phi_s - 4\psi_2) = \frac{1}{4}\tan\phi_s \tag{639}$$

$$4\phi_s - 4\psi_4 = \arctan\left\{\frac{1}{4}\tan\phi_s\right\} \tag{640}$$

$$\psi_4 = \phi_s - \frac{1}{4}\arctan\left\{\frac{1}{4}\tan\phi_s\right\} \tag{641}$$

$$\frac{V_2}{V_1} = \left\{\frac{\cos\phi_s}{\cos(2\phi_s - 2\psi_2)}\right\} = \frac{1}{2}\left\{\frac{\sin\phi_s}{\sin(2\phi_s - 2\psi_2)}\right\} \tag{642}$$

$$\frac{V_3}{V_1} = \frac{3}{7}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{7}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\} \tag{643}$$

and

$$\frac{V_4}{V_1} = \frac{1}{14}\left\{\frac{\cos\phi_s}{\cos(3\phi_s - 3\psi_3)}\right\} = \frac{1}{56}\left\{\frac{\sin\phi_s}{\sin(3\phi_s - 3\psi_3)}\right\}. \tag{644}$$

These formulae can be used to construct lookup tables that give ϕ_s, ψ_2, ψ_3, ψ_4, V_2/V_1, V_3/V_1, and V_4/V_1 for any ϕ_1 in range (628) or (629).

The quad-harmonic voltage is then

$$V(\psi) = V_1\sin\psi - V_2\sin(2\psi - 2\psi_2) + V_3\sin(3\psi - 3\psi_3) - V_4\sin(4\psi - 4\psi_4) \tag{645}$$

which satisfies

$$V(\phi_s) = 2\pi R_s \rho_s \left(\frac{1}{c}\,\frac{dB}{dt}\right) \tag{646}$$

and produces a flattened RF bucket.

The phases ϕ_s, ψ_2, ψ_3, ψ_4 are plotted as functions of ϕ_1 in Figure 25.

The ratios V_2/V_1, V_3/V_1, V_4/V_1 are plotted as functions of ϕ_1 in Figure 26.

The phase ϕ_1 is given by (630).

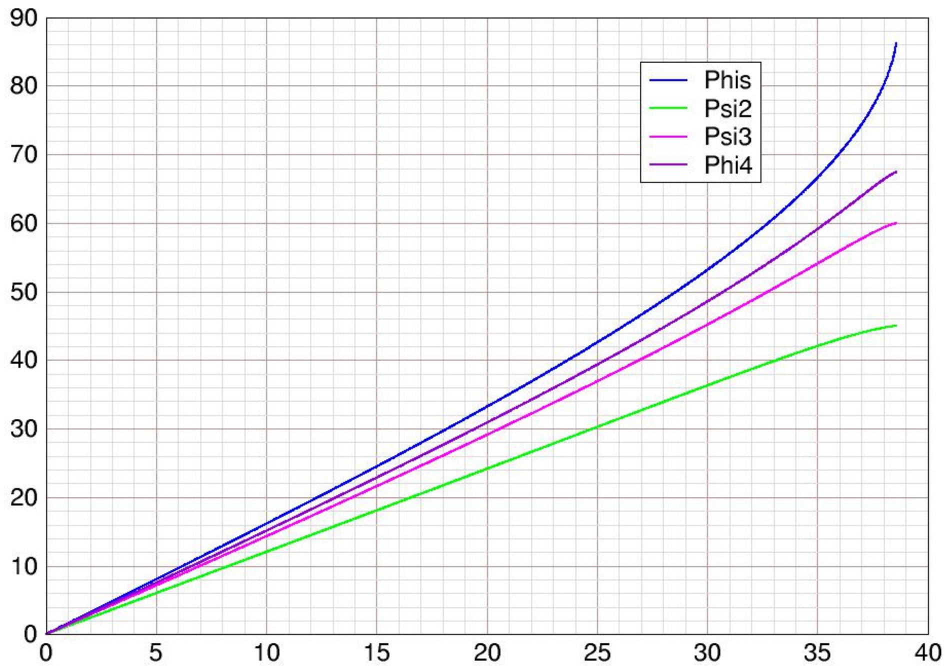

Figure 25: Quad-harmonic phases ϕ_s, ψ_2, and ψ_3 plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the phases in degrees. Starting with the lowest curve and going up, the green, pink, violet, and blue curves are ψ_2, ψ_3, ψ_4, and ϕ_s, respectively. The phase ϕ_1 is given by (630). As $\sin\phi_1$ approaches 5/8, the phases ψ_2, ψ_3, ψ_4, and ϕ_s approach $\pi/4$, $\pi/3$, $3\pi/8$, and $\pi/2$, respectively.

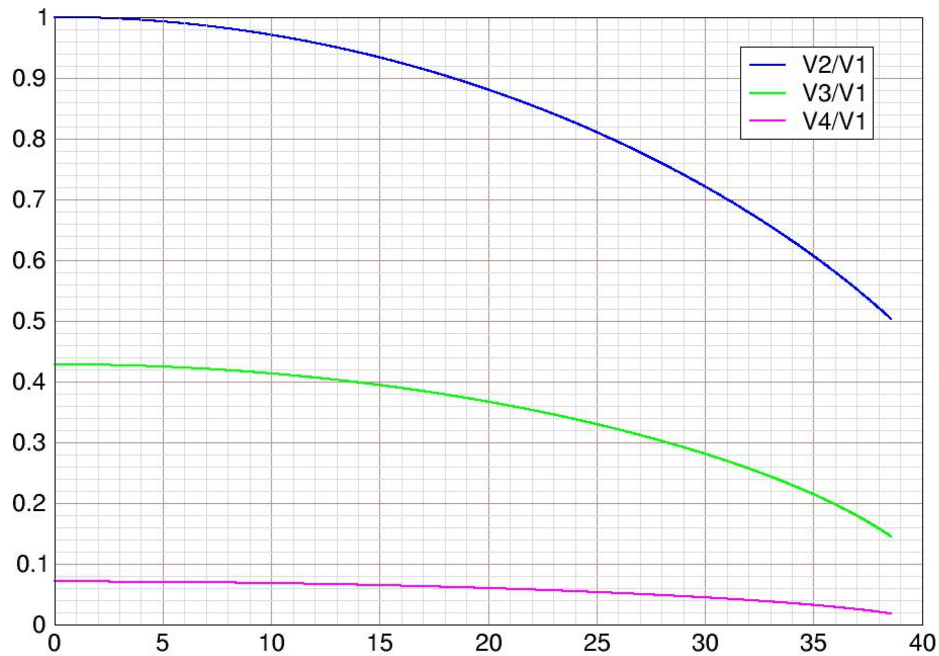

Figure 26: Quad-harmonic voltage ratios V_2/V_1, V_3/V_1, and V_4/V_1 plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the voltage ratios. Starting with the upper curve and going down, the blue, green, and pink curves are V_2/V_1, V_3/V_1, and V_4/V_1, respectively. The phase ϕ_1 is given by (630). As $\sin\phi_1$ approaches 5/8, the ratios V_2/V_1, V_3/V_1, and V_4/V_1 approach 1/2, 1/7, and 1/56, respectively.

61 Quad-harmonic bucket turning point phases and normalized area

The turning point phases ψ_e and ψ_u and normalized area for the quad-harmonic bucket are plotted as functions of ϕ_1 in Figures 27 and 28.

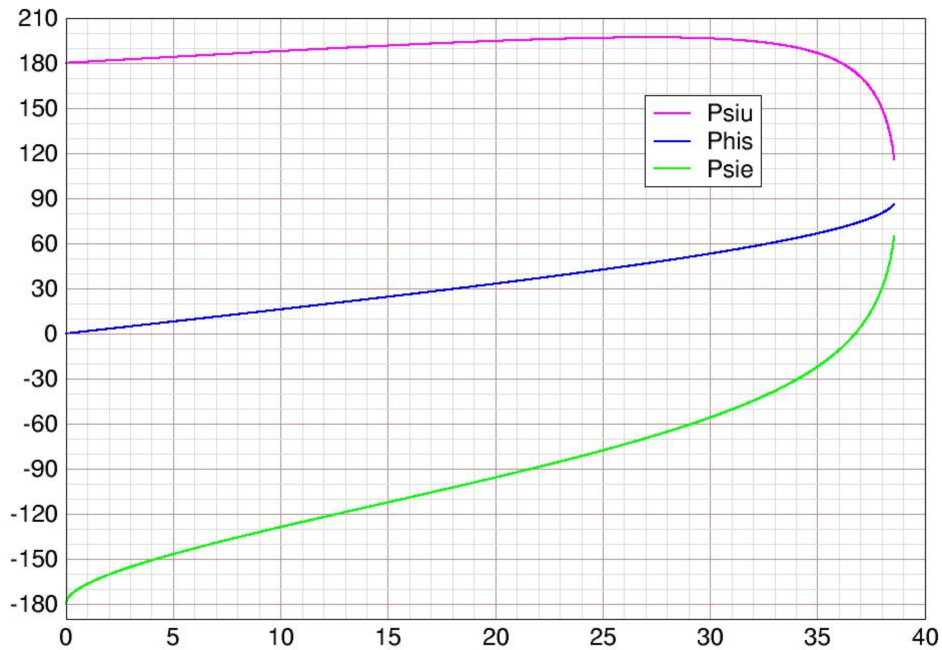

Figure 27: Quad-harmonic phases ψ_e, ϕ_s, and ψ_u plotted as functions of ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the phases in degrees. The lower (green) and upper (pink) curves are ψ_e and ψ_u, respectively. The middle curve (blue) is the synchronous phase ϕ_s. The phase ϕ_1 is given by (630). The quad-harmonic bucket extends from turning point phase ψ_e to unstable fixed point phase ψ_u. As $\sin\phi_1$ approaches $5/8$, all three phases approach $\pi/2$, and the bucket phase width $\psi_u - \psi_e$ goes to zero.

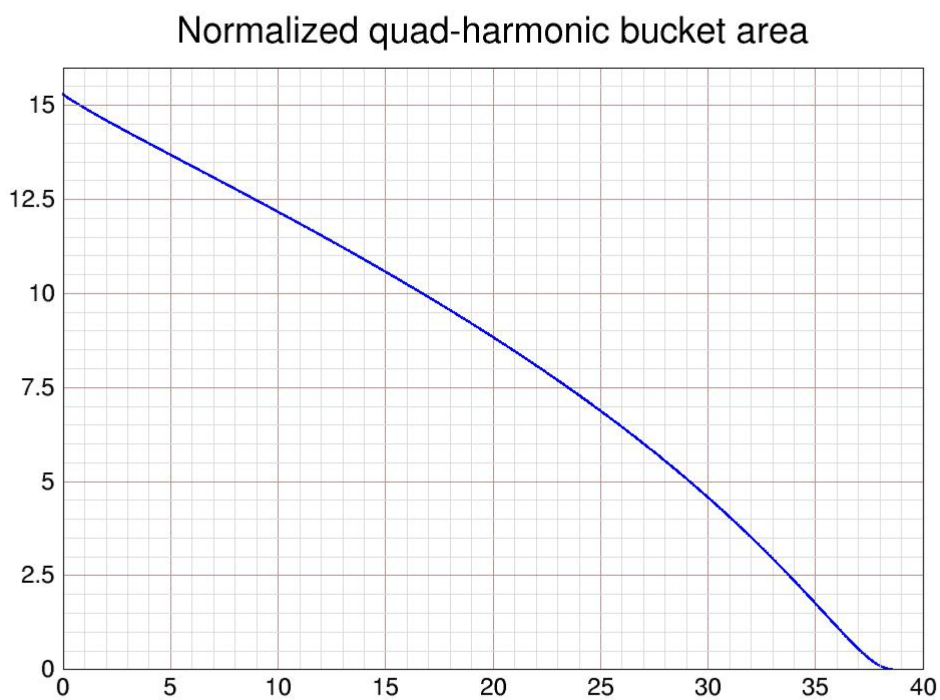

Figure 28: Normalized quad-harmonic bucket area plotted as a function of phase ϕ_1. The horizontal axis gives ϕ_1 in degrees. The vertical axis gives the normalized area $\mathcal{B}_3$ defined in Section 56. As $\sin\phi_1$ approaches $5/8$, the area goes to zero.

62 Quad-harmonic parameters for $G\gamma = 4.5$

For

$$G\gamma = 4.5 \tag{647}$$

we have

$$\beta\gamma^2 = 5.7783768, \quad B\rho = 7.2051786 \text{ Tm}, \quad B = 843.91166 \text{ G} \tag{648}$$

Taking

$$dB/dt = 0.01 \text{ G/ms} \tag{649}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{650}$$

then gives

$$V_1' = 7.498 \text{ kV}, \quad \phi_1' = 0.526567 \text{ degrees.} \tag{651}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 0.96790882 \text{ degrees} \tag{652}$$

and therefore

$$\phi_s = 1.5488, \quad \psi_2 = 1.1615, \quad \psi_3 = 1.3766, \quad \psi_4 = 1.4519, \text{ degrees} \tag{653}$$

$$\frac{V_1}{V_1'} = 0.5440, \quad \frac{V_2}{V_1} = 0.9997, \quad \frac{V_3}{V_1} = 0.4284, \quad \frac{V_4}{V_1} = 0.07140. \tag{654}$$

The resulting single and quad-harmonic buckets are shown in Figure 29 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 195.4 and 251.2 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.778.

The single and quad-harmonic bunch heights are 1.444 and 0.9462 respectively. The ratio of triple to single-harmonic bunch height is 0.655.

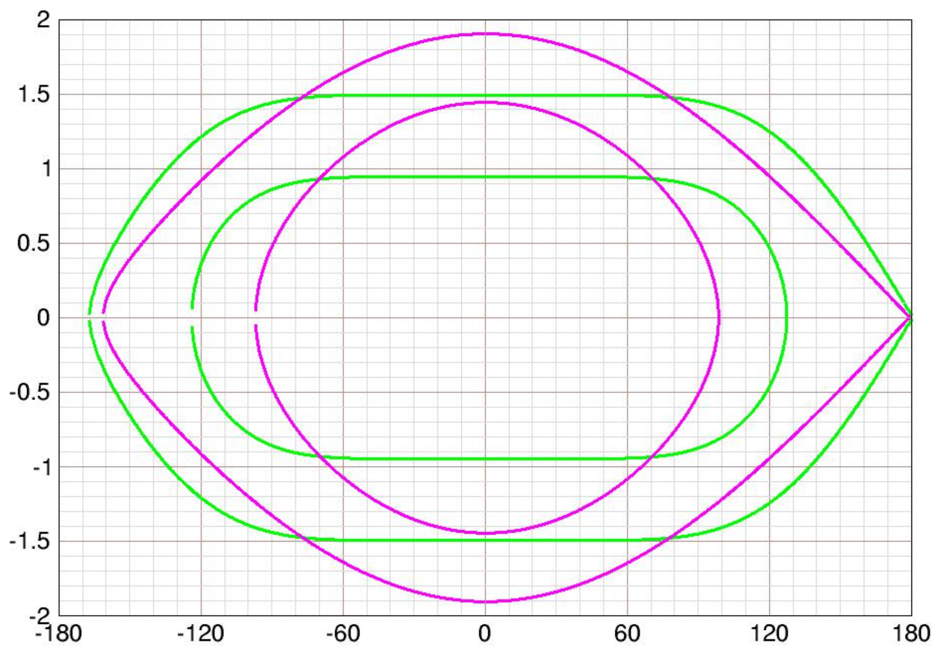

Figure 29: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 4.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 195.4 and 251.2 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.778. The single and quad-harmonic bunch heights are 1.444 and 0.9462 respectively. The ratio of quad to single-harmonic bunch height is 0.655. The horizontal axis gives the RF phase ψ in degrees.

63 Quad-harmonic parameters for $G\gamma = 6.0$

For

$$G\gamma = 6.0 \tag{655}$$

we have

$$\beta\gamma^2 = 10.688256, \quad B\rho = 9.9955569 \text{ Tm}, \quad B = 1170.7367 \text{ G} \tag{656}$$

Taking

$$dB/dt = 9.0 \text{ G/ms} \tag{657}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{658}$$

then gives

$$V_1' = 89.942 \text{ kV}, \quad \phi_1' = 43.59258 \text{ degrees.} \tag{659}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 34.9114831 \text{ degrees} \tag{660}$$

and therefore

$$\phi_s = 66.3048, \quad \psi_2 = 41.9421, \quad \psi_3 = 53.8989, \quad \psi_4 = 58.8879 \tag{661}$$

$$\frac{V_1}{V_1'} = 1.2048, \quad \frac{V_2}{V_1} = 0.6092, \quad \frac{V_3}{V_1} = 0.2163, \quad \frac{V_4}{V_1} = 0.03304. \tag{662}$$

The resulting single and quad-harmonic buckets are shown in Figure 30 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 86.79 and 166.3 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.522.

The single and quad-harmonic bunch heights are 0.3879 and 0.1704 respectively. The ratio of quad to single-harmonic bunch height is 0.439.

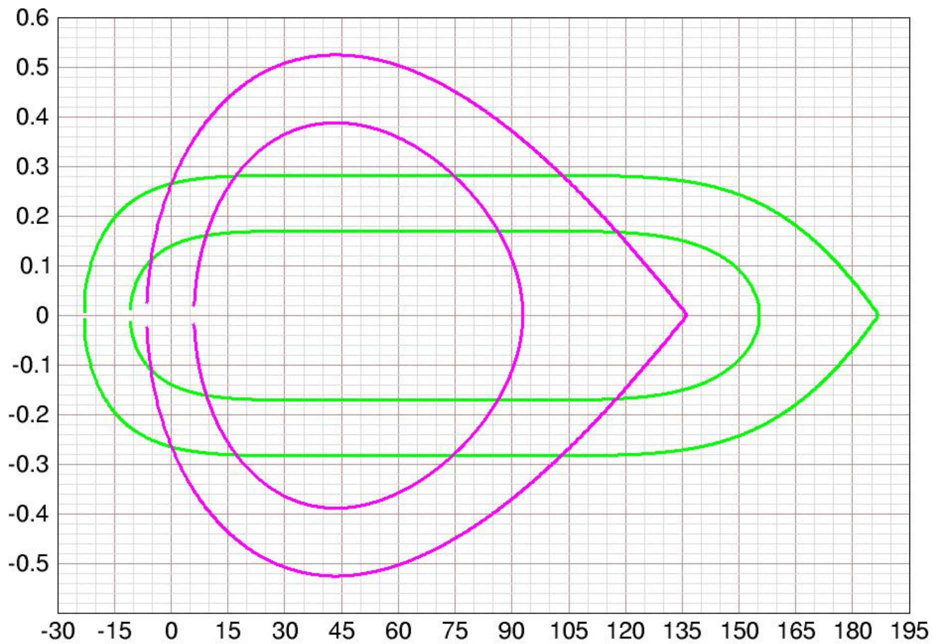

Figure 30: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 6.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 86.79 and 166.3 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.522. The single and quad-harmonic bunch heights are 0.3879 and 0.1704 respectively. The ratio of quad to single-harmonic bunch height is 0.439. The horizontal axis gives the RF phase ψ in degrees.

64 Quad-harmonic parameters for $G\gamma = 7.5$

For

$$G\gamma = 7.5 \tag{663}$$

we have

$$\beta\gamma^2 = 16.992559, \quad B\rho = 12.713026 \text{ Tm}, \quad B = 1489.0222 \text{ G} \tag{664}$$

Taking

$$dB/dt = 18.0 \text{ G/ms} \tag{665}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{666}$$

then gives

$$V_1' = 152.715 \text{ kV}, \quad \phi_1' = 54.311135 \text{ degrees.} \tag{667}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 36.417402 \text{ degrees} \tag{668}$$

and therefore

$$\phi_s = 71.7797, \quad \psi_2 = 43.4592, \quad \psi_3 = 56.6599, \quad \psi_4 = 62.4758 \tag{669}$$

$$\frac{V_1}{V_1'} = 1.3681, \quad \frac{V_2}{V_1} = 0.5686, \quad \frac{V_3}{V_1} = 0.1907, \quad \frac{V_4}{V_1} = 0.02804. \tag{670}$$

The resulting single and quad-harmonic buckets are shown in Figure 31 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 66.0154 and 145.918 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.452.

The single and quad-harmonic bunch heights are 0.2488 and 0.09453 respectively. The ratio of quad to single-harmonic bunch height is 0.380.

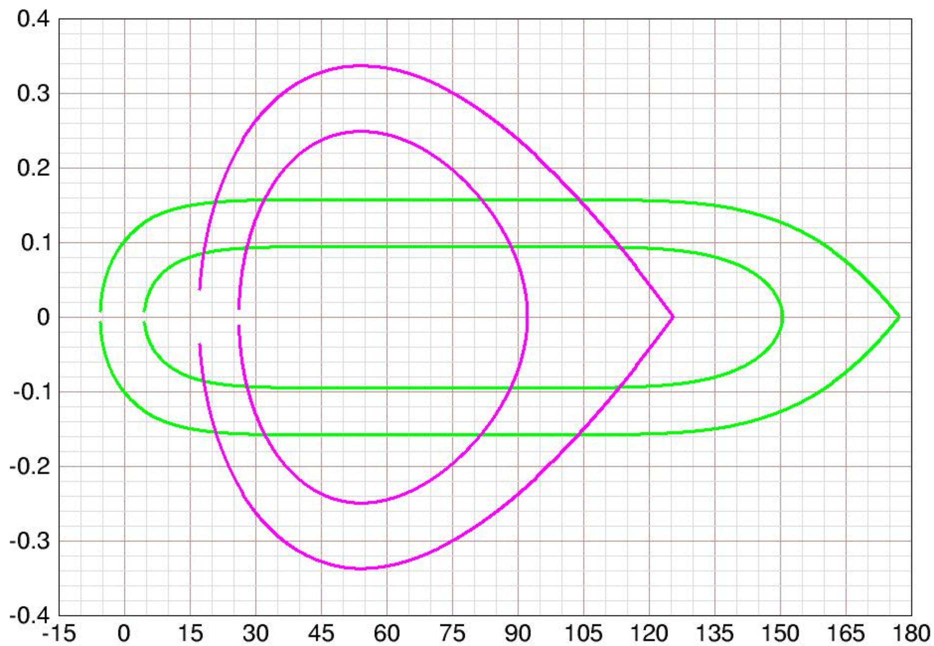

Figure 31: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 7.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 66.0154 and 145.918 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.452. The single and quad-harmonic bunch heights are 0.2488 and 0.09453 respectively. The ratio of quad to single-harmonic bunch height is 0.380. The horizontal axis gives the RF phase ψ in degrees.

65 Quad-harmonic parameters for $G\gamma = 10.0$

For

$$G\gamma = 10.0 \tag{671}$$

we have

$$\beta\gamma^2 = 30.606873, \quad B\rho = 17.173957 \text{ Tm}, \quad B = 2011.5119 \text{ G} \tag{672}$$

Taking

$$dB/dt = 22.0 \text{ G/ms} \tag{673}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{674}$$

then gives

$$V_1' = 171.219 \text{ kV}, \quad \phi_1' = 62.301222 \text{ degrees.} \tag{675}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 37.172555 \text{ degrees} \tag{676}$$

and therefore

$$\phi_s = 75.1831, \quad \psi_2 = 44.1237, \quad \psi_3 = 57.9947, \quad \psi_4 = 64.3374 \tag{677}$$

$$\frac{V_1}{V_1'} = 1.4654, \quad \frac{V_2}{V_1} = 0.54685, \quad \frac{V_3}{V_1} = 0.1763, \quad \frac{V_4}{V_1} = 0.02513. \tag{678}$$

The resulting single and quad-harmonic buckets are shown in Figure 32 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 50.940481 and 129.199 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.394.

The single and quad-harmonic bunch heights are 0.1656 and 0.05478 respectively. The ratio of quad to single-harmonic bunch height is 0.331.

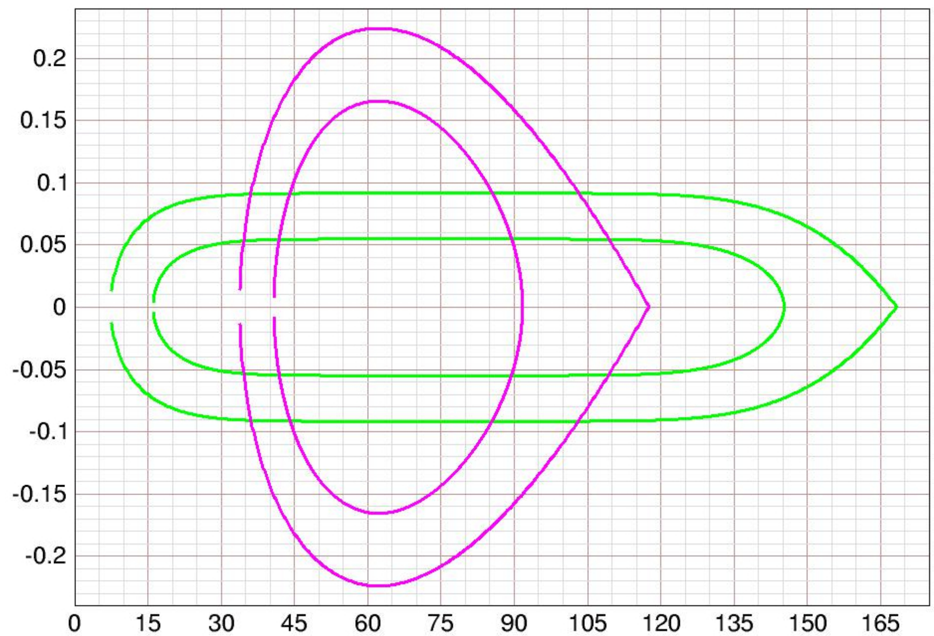

Figure 32: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 10.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 50.940481 and 129.199 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.394. The single and quad-harmonic bunch heights are 0.1656 and 0.05478 respectively. The ratio of quad to single-harmonic bunch height is 0.331. The horizontal axis gives the RF phase ψ in degrees.

66 Quad-harmonic parameters for $G\gamma = 12.5$

For

$$G\gamma = 12.5 \tag{679}$$

we have

$$\beta\gamma^2 = 48.108272, \quad B\rho = 21.595395 \text{ Tm}, \quad B = 2529.3760 \text{ G} \tag{680}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{681}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{682}$$

then gives

$$V_1' = 184.954 \text{ kV}, \quad \phi_1' = 68.657683 \text{ degrees.} \tag{683}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 37.632985 \text{ degrees} \tag{684}$$

and therefore

$$\phi_s = 77.6775, \quad \psi_2 = 44.4776, \quad \psi_3 = 58.7570, \quad \psi_4 = 65.4641 \tag{685}$$

$$\frac{V_1}{V_1'} = 1.5254, \quad \frac{V_2}{V_1} = 0.5331, \quad \frac{V_3}{V_1} = 0.1669, \quad \frac{V_4}{V_1} = 0.02317. \tag{686}$$

The resulting single and quad-harmonic buckets are shown in Figure 33 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 39.118263 and 114.2815 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.342.

The single and quad-harmonic bunch heights are 0.1103 and 0.03164 respectively. The ratio of quad to single-harmonic bunch height is 0.287.

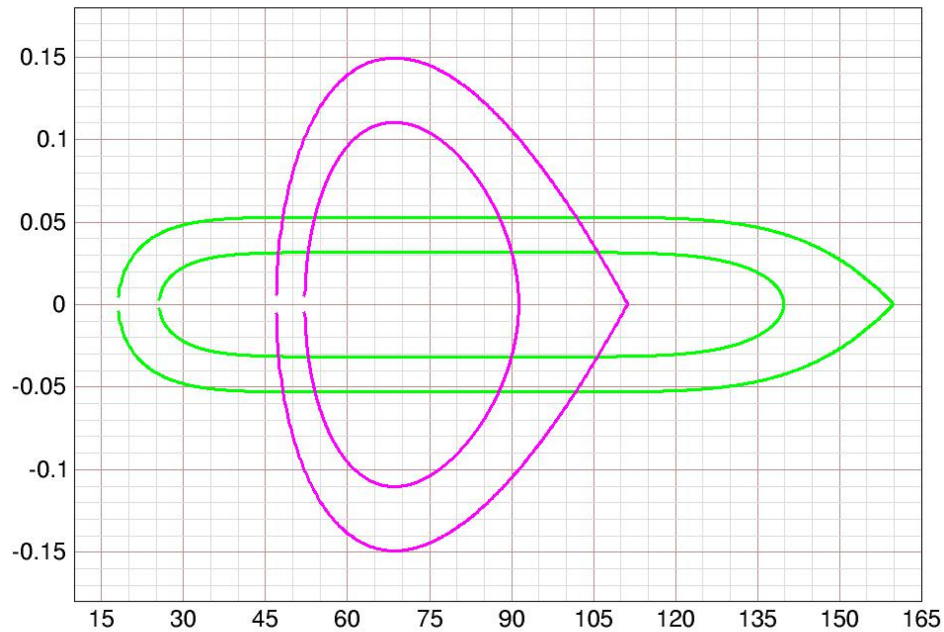

Figure 33: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 12.5$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 39.118263 and 114.2815 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.342. The single and quad-harmonic bunch heights are 0.1103 and 0.03164 respectively. The ratio of quad to single-harmonic bunch height is 0.287. The horizontal axis gives the RF phase ψ in degrees.

67 Quad-harmonic parameters for $G\gamma = 14.0$

For

$$G\gamma = 14.0 \tag{687}$$

we have

$$\beta\gamma^2 = 60.475409, \quad B\rho = 24.238302 \text{ Tm}, \quad B = 2838.9283 \text{ G} \tag{688}$$

Taking

$$dB/dt = 25.0 \text{ G/ms} \tag{689}$$

and adjusting V_1' to give single-harmonic bucket area

$$\mathcal{A}_1 = 2.0 \text{ eVs} \tag{690}$$

then gives

$$V_1' = 180.410 \text{ kV}, \quad \phi_1' = 72.723463 \text{ degrees}. \tag{691}$$

We can then adjust ϕ_1 to give a normalized quad-harmonic bucket with the same area as the normalized single-harmonic bucket. This gives

$$\phi_1 = 37.880106 \text{ degrees} \tag{692}$$

and therefore

$$\phi_s = 79.2400, \quad \psi_2 = 44.6452, \quad \psi_3 = 59.1360, \quad \psi_4 = 66.0500 \tag{693}$$

$$\frac{V_1}{V_1'} = 1.5552, \quad \frac{V_2}{V_1} = 0.5255, \quad \frac{V_3}{V_1} = 0.1616, \quad \frac{V_4}{V_1} = 0.02204 \tag{694}$$

The resulting single and quad-harmonic buckets are shown in Figure 13 along with their matched bunches. The two buckets have the same area, and each bunch has half the area of the bucket that holds it.

The single and quad-harmonic bunch widths are 31.609889 and 103.5727 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.305.

The single and quad-harmonic bunch heights are 0.07977 and 0.02038 respectively. The ratio of quad to single-harmonic bunch height is 0.256.

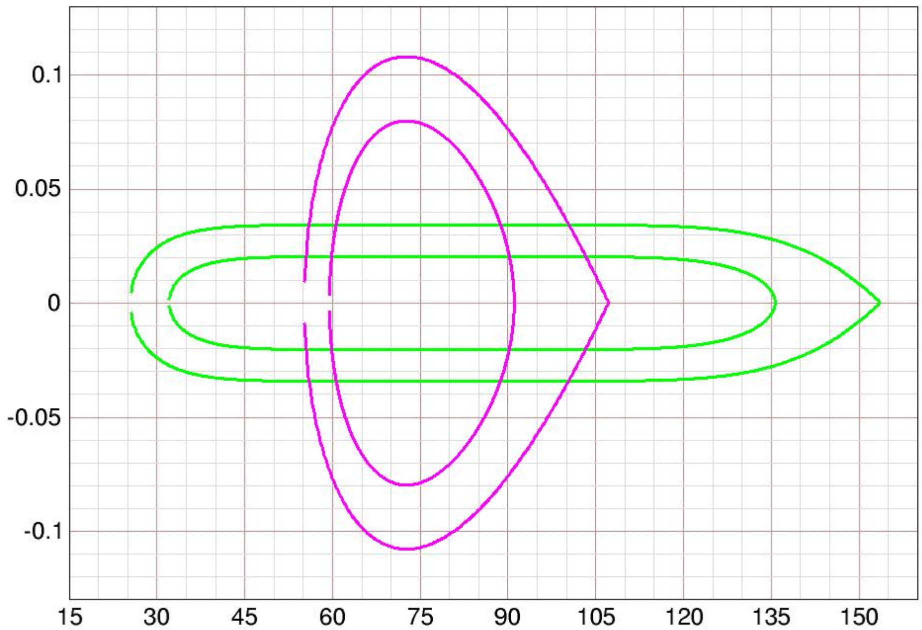

Figure 34: Normalized single and quad-harmonic buckets and matched bunches obtained for $G\gamma = 14.0$. The two buckets have the same area (2 eV s) and each bunch has half the area of the bucket that holds it. The single and quad-harmonic bunch widths are 31.609889 and 103.5727 degrees respectively. The ratio of single to quad-harmonic bunch width is 0.305. The single and quad-harmonic bunch heights are 0.07977 and 0.02038 respectively. The ratio of quad to single-harmonic bunch height is 0.256. The horizontal axis gives the RF phase ψ in degrees.

68 Quad-harmonic parameter summary for acceleration of polarized protons in AGS

The following tables summarize the data of Sections 62 through 67. The guide field B and its time derivative are given in units of G and G/ms. The single-harmonic RF voltage V_1' is given in units of kV. The phases ϕ_1', ϕ_1, ϕ_s, ψ_2, ψ_3, ψ_4, and the single and quad-harmonic bunch widths W_1 and W_4 are given in degrees. The ratio H_4/H_1 is the ratio of quad to single-harmonic bunch height. Values of $\beta\gamma^2$, W_1, W_4, and H_4/H_1 are tabulated for comparison of the incoherent space charge tune shifts in the single and quad-harmonic bunches.

Table 16: Quad-harmonic RF voltages

B	dB/dt	$G\gamma$	V_1'	V_1/V_1'	V_2/V_1	V_3/V_1	V_4/V_1
843.9	0.01	4.5	7.498	0.5440	0.9997	0.4284	0.07140
1170.7	9.0	6.0	89.942	1.2048	0.6092	0.2163	0.03304
1489.0	18.0	7.5	152.715	1.3681	0.5686	0.1907	0.02804
2011.5	22.0	10.0	171.219	1.4654	0.5469	0.1763	0.02513
2529.4	25.0	12.5	184.954	1.5254	0.5331	0.1669	0.02317
2838.9	25.0	14.0	180.410	1.5552	0.5255	0.1616	0.02204

Table 17: Quad-harmonic RF phases

dB/dt	$G\gamma$	ϕ_1'	ϕ_1	ϕ_s	ψ_2	ψ_3	ψ_4
0.01	4.5	0.52657	0.9679	1.5488	1.1615	1.3766	1.4519
9.0	6.0	43.59258	34.9115	66.3048	41.9421	53.8989	58.8879
18.0	7.5	54.3111	36.4174	71.7797	43.4592	56.6599	62.4758
22.0	10.0	62.3012	37.1726	75.1831	44.1237	57.9947	64.3374
25.0	12.5	68.6577	37.6330	77.6775	44.4776	58.7570	65.4641
25.0	14.0	72.7235	37.8801	79.2400	44.6452	59.1360	66.0500

Table 18: Quad-harmonic matched bunch parameters

dB/dt	$G\gamma$	$\beta\gamma^2$	W_1	W_4	W_1/W_4	H_4/H_1
0.01	4.5	5.7784	195.4	251.2	0.778	0.655
9.0	6.0	10.6883	86.79	166.3	0.522	0.439
18.0	7.5	16.9926	66.02	145.9	0.452	0.380
22.0	10.0	30.6069	50.94	129.2	0.394	0.331
25.0	12.5	48.1083	39.12	114.3	0.342	0.287
25.0	14.0	60.4754	31.61	103.6	0.305	0.256

69 Comparison of the incoherent tune shifts in the single and quad-harmonic bunches

Let δQ_1 and δQ_4 be the incoherent space charge tune shifts in the single and quad-harmonic bunches, respectively. Then the ratio

$$\frac{\delta Q_4}{\delta Q_1} = \frac{B_1}{B_4} \tag{695}$$

where B_1 and B_4 are the corresponding bunching factors. Here one may simply take

$$B_1 = \frac{W_1}{2\pi h}, \quad B_4 = \frac{W_4}{2\pi h} \tag{696}$$

where the bunch widths W_1 and W_4 are given in radians. This gives

$$\frac{\delta Q_4}{\delta Q_1} = \frac{W_1}{W_4}. \tag{697}$$

This ratio is the reduction of space charge tune shift due to the lengthing of the bunch in the quad-harmonic bucket. It is tabulated in the sixth column of Table 18 and goes from 0.778 to 0.305 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

As shown in Section 32, one may also take

$$\frac{\delta Q_4}{\delta Q_1} = \frac{H_4}{H_1}. \tag{698}$$

This ratio is tabulated in the last column of Table 18 and goes from 0.655 to 0.256 as the bunch is accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$.

70 Reduction of incoherent tune shift with increasing gamma

Let $(\delta Q)_I$ and $(\delta Q)_F$ be the initial and final incoherent tune shifts as proton bunches are accelerated from $G\gamma = 4.5$ to $G\gamma = 14.0$. Then the reduction in tune shift is given by the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{B_I\,(\beta\gamma^2)_I}{B_F\,(\beta\gamma^2)_F} \tag{699}$$

where bunching factors

$$B_I = \frac{W_I}{2\pi h}, \quad B_F = \frac{W_F}{2\pi h} \tag{700}$$

and W_I and W_F are the initial and final bunch widths (in radians). Thus the ratio

$$\frac{(\delta Q)_F}{(\delta Q)_I} = \frac{W_I\,(\beta\gamma^2)_I}{W_F\,(\beta\gamma^2)_F}. \tag{701}$$

Putting in numbers from columns three and four of Table 18 we obtain

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.591 \tag{702}$$

for bunches in the single-harmonic bucket.

Putting in numbers from columns three and five of the table gives

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.232 \tag{703}$$

for bunches in the quad-harmonic bucket.

This shows that the quad-harmonic bucket gives a significantly greater reduction in incoherent tune shift with increasing gamma. For the triple-harmonic bucket one has

$$\frac{(\delta Q)_F}{(\delta Q)_I} = 0.273 \tag{704}$$

as shown in Section 33.

References

[1] C.J. Gardner, "Simulations of Bunch Merging in Booster," C-A/AP Note 176, October 2004, pp. 12–22.

[2] These useful expressions are obtained by judicious use of standard trigonometric identities. I do not know who first wrote them down. I learned them from J.M. Brennan.

[3] https://physics.nist.gov/cuu/Constants/

[4] C.J. Gardner, "FY2016 Parameters for gold ions in Booster, AGS, and RHIC," C-A/AP/Note 574, October 2016, p. 5.

[5] E.D. Courant, "Beam Instabilities in Accelerators," IEEE Transactions on Nuclear Science, June 1965, pp. 550–555.

[6] W.W. MacKay and M. Conte, "An Introduction to the Physics of Particle Accelerators," Second Edition, World Scientific, 2008, pp. 231–233.