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H. Zhao

October 2020

Collider Accelerator Department  
**Brookhaven National Laboratory**

**U.S. Department of Energy**

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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## Dispersion effect on the cooling rates

H. Zhao, M. Blaskiewicz  
BNL, Upton NY 11973, USA

It is known that dispersion can redistribute the electron cooling rate between longitudinal and transverse planes [1-2]. This note presents a simple analytical model to explain and estimate this redistribution effect. It shows that a small horizontal dispersion on ions can enhance horizontal cooling at the expense of longitudinal cooling rate. The electron dispersion is also beneficial to this effect.

### I. Cooling force

For non-magnetized cooling, the cooling force on an ion is given by

$$\mathbf{F}(\mathbf{r}, \mathbf{u}_i) = 4\pi Z^2 r_e^2 m_e c^4 \int \ln \Lambda f_e(\mathbf{r}, \mathbf{u}_e) \frac{\mathbf{u}_e - \mathbf{u}_i}{|\mathbf{u}_e - \mathbf{u}_i|^3} d\mathbf{u}_e \quad (1)$$

where  $Z$  is the atomic number of the ion,  $m_e$  is the electron mass,  $r_e$  is the classical electron radius,  $c$  is the speed of light,  $\ln \Lambda$  is the coulomb logarithm,  $\mathbf{u}$  is the beam velocity in the particle reference frame, and  $f_e(\mathbf{r}, \mathbf{u}_e)$  is the electron beam distribution in 6-D phase space. In the cooling section, electron beam distribution can be simply written as  $f_e(\mathbf{r}, \mathbf{u}_e) = n_e(\mathbf{r}) f_{v_e}(\mathbf{u}_e)$ , where  $n_e(\mathbf{r})$  is the local charge density of electrons in the particle reference frame. Because the dispersion on the ion beam will only introduce the coupling between the transverse position and energy spread of ions, the integral in eq. (1) will not change. To simplify the model, we assume the rms velocity spreads of electron beam are the same in all directions, and we only work on the leader order of the velocity. Then we can get the change of  $\mathbf{u}_i$  at location  $\mathbf{r}$  is [2].

$$\Delta \mathbf{u}_i(\mathbf{r}, \mathbf{u}_i) = -\frac{2\sqrt{2}\pi Z^2 r_e^2 m_e \ln \Lambda L n_e(\mathbf{r})}{3\beta\gamma\sigma_{v_e}^3 m_i} \mathbf{u}_i = -C n_e(\mathbf{r}) \mathbf{u}_i \quad (2)$$

where  $L$  is the length of the cooling section and we assume the cooling section is short so that the change of beam distribution along cooling section is negligible.

### II. Ion dispersion

Now we consider an ion with the initial energy spread  $\delta_0$  and horizontal dispersion  $D$ , the initial emittance is

$$\epsilon_{x0} = (x_0 - D\delta_0)^2 / 2\beta_x + \beta_x x'_0{}^2 / 2$$

After passing the cooling section

$$\epsilon_{x1} = (x_1 - D\delta_1)^2 / 2\beta_x + \beta_x x'_1{}^2 / 2$$

For a short cooling section  $x_0 = x_1 = x$ ,  $\delta_1 - \delta_0 = -C n_e(\mathbf{r}) \delta_0$  and  $x'_1 - x'_0 = -C n_e(\mathbf{r}) x'$ , the changes of emittance and energy due to cooling for a single particle (only first order) are

$$\Delta \epsilon_x = -C n_e(x, y, s) \left( \beta_x x'^2 - \frac{D\delta x_\beta}{\beta_x} \right) \quad (3)$$

$$\Delta\delta^2 = -2Cn_e(x, y, s)\delta^2 \quad (4)$$

where  $x_\beta = x - D\delta$  is the betatron amplitude. The local density of electron beam at the position  $(x, y, s) = (x_\beta + D\delta, y_\beta, s)$  is given by

$$n_e(x, y, s) = \frac{eN_{e0}}{(2\pi)^{3/2}\sigma_{ex}\sigma_{ey}\sigma_{es}} \exp\left[-\frac{(x_\beta + D\delta)^2}{2\sigma_{ex}^2} - \frac{y_\beta^2}{2\sigma_{ey}^2} - \frac{s^2}{2\sigma_{es}^2}\right] \quad (5)$$

where  $N_{e0}$  is the total number of electrons. We know that the expected value of a function of two random variables can be calculated by:  $E[g(X_1, X_2, \dots)] = \int g(x_1, x_2, \dots) f_{X_1, X_2, \dots}(x_1, x_2, \dots) dx_1 dx_2 \dots$ , in which  $f_{X_1, X_2, \dots}(x_1, x_2, \dots)$  is the joint PDF of these variables. We assume the ion beam also has a Gaussian distribution ( $x_\beta \sim N(0, \sigma_{ix}^2), \delta \sim N(0, \delta_p^2)$ ), then we can get the average values of the cooling rate in two planes:

$$\frac{\langle \Delta\epsilon_x \rangle}{\epsilon_{x,rms}} = -\frac{eN_{e0}}{(2\pi)^{3/2} \sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2)}} \left( C + \frac{CD^2\delta_p^2}{\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2} \right) \quad (6)$$

$$\frac{\langle \Delta\delta \rangle}{\delta_p} = -\frac{eN_{e0}}{(2\pi)^{3/2} \sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2)}} \left( C - \frac{CD^2\delta_p^2}{\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2} \right) \quad (7)$$

It is clear that the transverse and longitudinal cooling rates are redistributed by dispersion. To estimate the dispersion effect on the cooling rates, we define the gain factor as the ratio of the cooling rates with dispersion and without dispersion,

$$k_x = \frac{\langle \Delta\epsilon_x \rangle}{\langle \Delta\epsilon_x \rangle_{D=0}} = \frac{\sigma_{ex}^2 + \sigma_{ix}^2 + 2D^2\delta_p^2}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2)^{3/2}} \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2} \quad (8)$$

$$k_p = \frac{\langle \Delta\delta \rangle}{\langle \Delta\delta \rangle_{D=0}} = \frac{(\sigma_{ex}^2 + \sigma_{ix}^2)^{3/2}}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2\delta_p^2)^{3/2}} \quad (9)$$

It's clear that  $k_p \leq 1$ , which means the longitudinal cooling will always be weakened by the dispersion. The dependency of  $k_x$  on dispersion is not obvious from eq. (8), we calculated the evolution of the gain factors with the dispersion under several conditions, as shown in Fig. 1. It shows that for a small dispersion, the horizontal cooling rate is increased. When the dispersion becomes larger, both horizontal and longitudinal cooling will be reduced because the ion beam size will be larger than the electron beam. Based on eq. (3) and eq. (4), a statistical method was also applied to get the gain factors, in which a group of particles with Gaussian distribution was used. As shown in Fig. 1, the statistical method gives the same results with the analytical model. The condition for the maximum of  $k_x$  is  $2D^2\delta_p^2 = \sigma_{ex}^2 + \sigma_{ix}^2$ , at which the horizontal cooling rate is maximized by the dispersion. The maximum value is  $k_{x,max} \sim 1.088$ , which is independent of beam parameters.

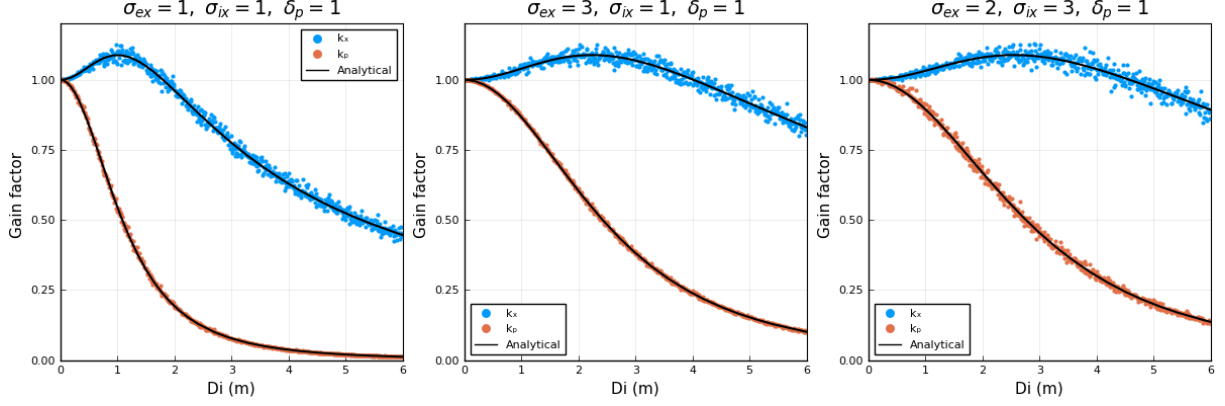


Fig. 1. The statistical and analytical results of the gain factor under different beam conditions.

In above, we just assume that the linear cooling forces in three dimensions are the same, which makes the increase of horizontal cooling not very obvious (up to 10.88%). In fact, this redistribution effect is much suitable for the high energy beam cooling. At high energy, the horizontal temperature of electron beam in rest frame is usually higher than in longitudinal plane, which causes the unbalanced cooling rates between horizontal and longitudinal cooling. The redistribution effect can be used to increase the horizontal cooling rate at the expense of the longitudinal cooling. Here, we still consider the linear cooling force but different strength in horizontal and longitudinal planes:  $\Delta \mathbf{u}_{x,p} = -C_{x,p} n_e \mathbf{u}_{x,p}$ . Using the same method, we get the new cooling rates:

$$\frac{\langle \Delta \epsilon_x \rangle}{\epsilon_{x,rms}} = - \frac{eN_{e0}}{(2\pi)^{3/2} \sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2)}} \left( C_x + \frac{C_p D^2 \delta_p^2}{\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2} \right) \quad (10)$$

$$\frac{\langle \Delta \delta \rangle}{\delta_p} = - \frac{eN_{e0}}{(2\pi)^{3/2} \sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2)}} \left( C_p - \frac{C_p D^2 \delta_p^2}{\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2} \right) \quad (11)$$

And the gain factors:

$$k_x = \frac{\sigma_{ex}^2 + \sigma_{ix}^2 + (1 + C_p/C_x) D^2 \delta_p^2}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2)^{3/2}} \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2} \quad (12)$$

$$k_p = \frac{(\sigma_{ex}^2 + \sigma_{ix}^2)^{3/2}}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D^2 \delta_p^2)^{3/2}} \quad (13)$$

Now, the maximum value of  $k_x$  is  $k_{x,max} = \frac{2}{3\sqrt{3}} \left( 1 + \frac{C_p}{C_x} \right) \sqrt{1 + \frac{C_x}{C_p}}$ , which only depends on the ratio of the cooling gradient in transverse and longitudinal planes. The new calculation results with  $C_p/C_x = 3$  are shown in Fig. 2. It shows that the increase of horizontal cooling rate is significant (up to ~80%), which is due to the strong longitudinal cooling rate.

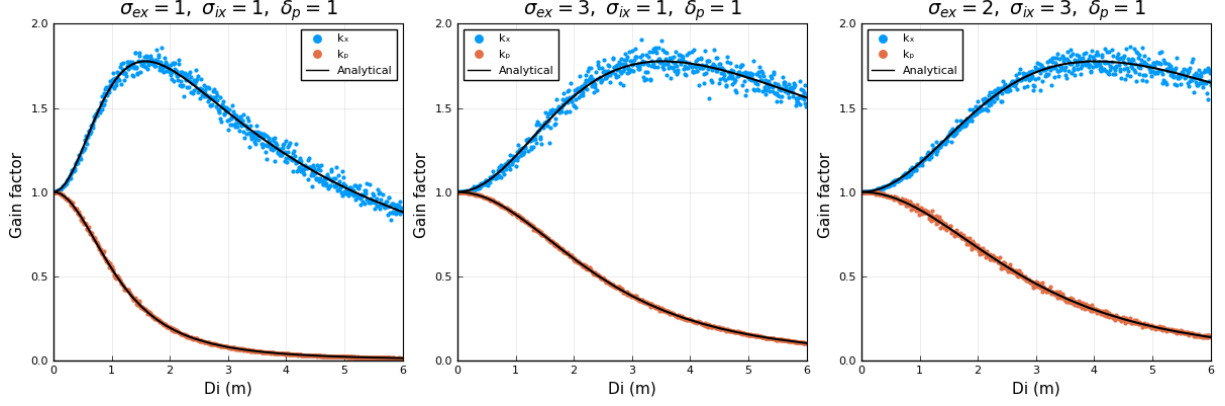


Fig. 2. The statistical and analytical results of the gain factor with  $C_p/C_x = 3$ .

### III. Electron dispersion

The effect of electron dispersion on cooling is well described in Ref. [2], but the contribution of the electron beam density to the cooling rate is not included. In this note we take it into consideration. From Ref. [2], we know that the new electron density distribution in lab frame with dispersion  $D_e$ , horizontal offset  $x_{off}$  and energy offset  $\delta_{off}$  is

$$n_e(r) = n_{e0} \exp \left\{ -\frac{(x - x_{off} - D_e \delta_{off})^2}{2(\epsilon_x \beta_x + D_e^2 \sigma_p^2)} - \frac{y^2}{2\epsilon_y \beta_y} - \frac{s_0^2}{2\sigma_s^2} \right\} \quad (14)$$

here we still assume a Gaussian density distribution. The standard form of the velocity distribution of electron beam in rest frame is

$$f_v(u_x, u_y, u_z) = \frac{\exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(u_x - \bar{u}_x)^2}{\sigma_1^2} + \frac{(u_z - \bar{u}_z)^2}{\sigma_3^2} - 2\rho \frac{(u_x - \bar{u}_x)(u_z - \bar{u}_z)}{\sigma_1 \sigma_3} \right] - \frac{(u_y - \bar{u}_y)^2}{2\sigma_2^2} \right\}}{(2\pi)^{3/2} \sigma_1 \sigma_2 \sigma_3 \sqrt{1-\rho^2}} \quad (15)$$

The various parameters are given by:

$$\bar{u}_x = -\frac{\gamma \alpha_x \epsilon_x (x - x_{off} - D_e \delta_{off})}{\epsilon_x \beta_x + D_e^2 \sigma_p^2}$$

$$\bar{u}_y = -\frac{\gamma \alpha_y y}{\beta_y}$$

$$\bar{u}_z = \frac{D_e \sigma_p^2 (x - x_{off}) + \epsilon_x \beta_x \delta_{off}}{\epsilon_x \beta_x + D_e^2 \sigma_p^2}$$

$$\sigma_1^2 = \frac{\epsilon_x \gamma^2}{\beta_x} \left( 1 + \frac{\alpha_x^2 D_e^2 \sigma_p^2}{\epsilon_x \beta_x + D_e^2 \sigma_p^2} \right)$$

$$\sigma_2^2 = \frac{\epsilon_y \gamma^2}{\beta_y}$$

$$\sigma_3^2 = \frac{\sigma_p^2 \epsilon_x \beta_x}{\epsilon_x \beta_x + D_e^2 \sigma_p^2}$$

$$\rho = \frac{\alpha_x D_e \sigma_p}{\sqrt{\epsilon_x \beta_x + D_e^2 \sigma_p^2 (1 + \alpha_x^2)}}$$

The equations above for cooling force calculation are good for computer work but difficult analytically. To simplify the model, we just consider the linear cooling force with dispersion and a short cooling section ( $\alpha = 0, x_{off} = 0, \delta_{off} = 0$ ), then the cooling force can be described by:

$$\Delta \mathbf{u}_{x,p} = -C_{x,p} n_e (\hat{\mathbf{p}} k x - \mathbf{u}_{x,p}), \quad k = \frac{D_e \sigma_p^2}{\epsilon_x \beta_x + D_e^2 \sigma_p^2}. \quad (16)$$

Using eq. (16), the changes of single particle emittance and momentum spread are

$$\Delta \epsilon_x = -C_x n_e \beta_x x'^2 - C_p n_e \frac{k D_i x_\beta^2}{\beta_x} + C_p n_e \frac{D_i (1 - k D_i) \delta x_\beta}{\beta_x} \quad (17)$$

$$\Delta \delta^2 = -2 C_p n_e [(1 - k D_i) \delta^2 - k \delta x_\beta] \quad (18)$$

With the same method, the final cooling rates are

$$\frac{\langle \Delta \epsilon_x \rangle}{\epsilon_{x,rms}} = - \frac{M}{\sqrt{\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2}} \left[ C_x + \frac{C_p (D_i^2 \delta_{ip}^2 + D_i D_e \delta_{ep}^2)}{\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2} \right] \quad (19)$$

$$\frac{\langle \Delta \delta \rangle}{\delta_p} = - \frac{M}{\sqrt{\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2}} \left[ C_p - \frac{C_p (D_i^2 \delta_{ip}^2 + D_i D_e \delta_{ep}^2)}{\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2} \right] \quad (20)$$

$$M = \frac{e N_{e0}}{(2\pi)^{3/2} \sqrt{(\sigma_{ey}^2 + \sigma_{iy}^2)(\sigma_{es}^2 + \sigma_{is}^2)}}$$

and the final gain factors as a function of  $D_i$  and  $D_e$  is given by

$$k_x = \frac{\sigma_{ex}^2 + \sigma_{ix}^2 + (1 + C_p/C_x) D_i^2 \delta_{ip}^2 + (1 + D_i C_p/D_e C_x) D_e^2 \delta_{ep}^2}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2)^{3/2}} \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2} \quad (21)$$

$$k_p = \frac{\sigma_{ex}^2 + \sigma_{ix}^2 + (1 - D_i/D_e) D_e^2 \delta_{ep}^2}{(\sigma_{ex}^2 + \sigma_{ix}^2 + D_i^2 \delta_{ip}^2 + D_e^2 \delta_{ep}^2)^{3/2}} \sqrt{\sigma_{ex}^2 + \sigma_{ix}^2} \quad (22)$$

The dependency of the gain factors on electron dispersion and ion dispersion is shown in Fig. 3, in which the arbitrary parameters are used ( $\sigma_{ex} = \sigma_{ix} = \delta_{ep} = \delta_{ip} = 1, C_p/C_x = 3$ ). It shows that the dispersion on electron beam also contributes to this redistribution effect. With the electron dispersion, the maximum increase of the horizontal cooling rate can reach to 100%. However, the coefficients  $C_x$  and  $C_p$  in eq. (16) are correlated because of the electron dispersion. When there is a large electron dispersion, the ratio between  $C_x$  and  $C_p$  will change a lot. Therefore, this result is just suitable for small electron dispersion.

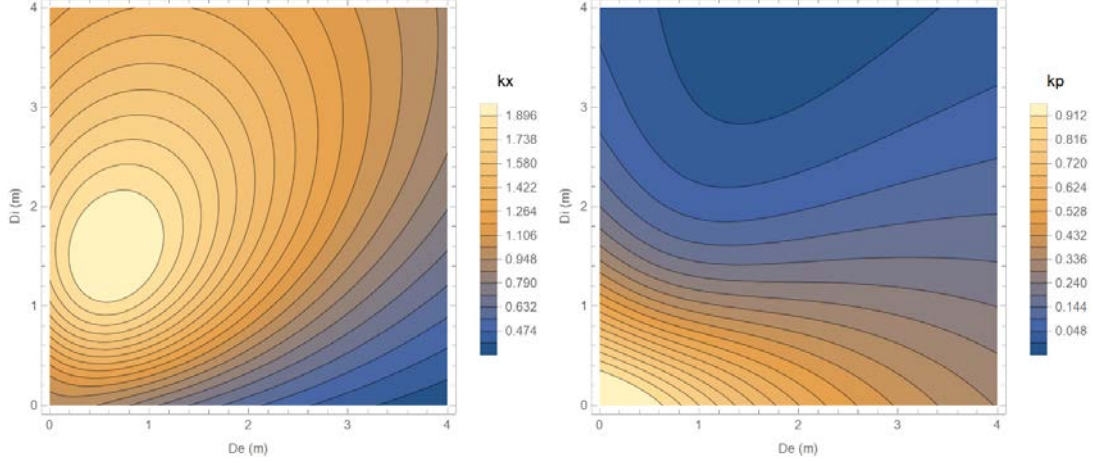


Fig. 3. Dependency of the gain factors on the electron dispersion and ion dispersion.

$$(\sigma_{ex} = \sigma_{ix} = \delta_{ep} = \delta_{ip} = 1, C_p/C_x = 3)$$

#### IV. Simulation with real beam conditions

In section II and III, we just give a simple analytical model, in which the nonlinear cooling force and beam evolution along the cooling section are not considered. In this section, we simulated the cooling process using real beam conditions and studied the dependency of the cooling rate on dispersions. In the simulation, the beam parameters are listed in Table 1. The dependency of the cooling rates in all 3 dimensions on electron dispersion and ion dispersion is given in Fig. 4. We can see that the horizontal cooling rate can be strengthened by electron and ion dispersions at the expense of the longitudinal and vertical cooling rates, which is consistent with our analytical model.

Table 1. Beam parameters in simulation

	Electron	Proton
Relativistic Factor $\gamma$	293.2	293.2
Density	3e11	6.9e10
RMS Emittance x/y (nm)	5/5	9.6/9.6
RMS dp/p	1e-3	6.8e-3
RMS Bunch length (m)	0.1	0.06
$\sigma_x/\sigma_y$ (mm)	1.4	1.4
$\sigma_x/\sigma_y$	1e-3	2e-3
L_cool (m)	170	170



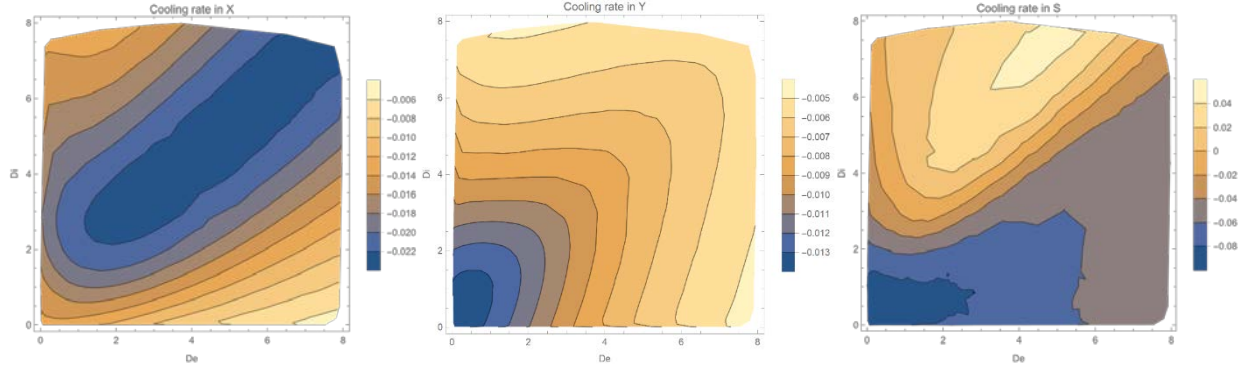


Fig. 3. Simulation result of the cooling rates in three dimensions. (Unit of the dispersion: m)

## V. Conclusion

We present a simple analytical model to explain and estimate the dispersion effect on the horizontal and longitudinal cooling rates. It shows that the horizontal dispersion on ions can enhance horizontal cooling at the expense of longitudinal cooling rate. The electron dispersion is also beneficial to this effect. The simulation under real beam conditions gives consistent conclusion.

## Acknowledgement

Thanks to A. Fedotov, J. Kewisch and D. Kayran for many useful suggestions and discussions.

## References

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