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Beam dynamics in non-magnetized electron cooler with strong hadron-electron focusing

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Abstract

In this note we develop an approach to description of beam dynamics of electrons experiencing both strong non-linear focusing from the space charge of a hadron beam and the self-field effects.

1 Introduction

In non-magnetized bunched electron coolers the dynamics of electrons is dominated by the strong non-linear focusing from the cooled hadron beam. The effects of electrons own space charge must be taken into account as well.

These effects were critical for the angular spread of electrons in LEReC [1] cooling section (CS).

The goal of this paper is to present our approach to fast calculation of electron angular spread throughout the CS of the non-magnetized cooler in the presence of strong non-linear space-charge forces (both external and self-induced).

We start with a reminder of how a standard text-book envelope equation is derived and of when its various parameterizasions are applicable. Then we move to a detailed description of the algorithm for computing the envelope and angles of electrons in the CS of non-magnetized bunched beam coolers.

2 Space charge force

Let us consider an electron beam co-traveling in the cooling section with a hadron beam. We assume that the center of masses of both beams coincide. We also assume that both beams have a circularly symmetric transverse distribution. These assumptions are relevant to the LEReC case and they simplify the derived formulas.

From Gauss's and Ampere's laws the electron displaced by radius r ($r^2 = x^2 + y^2$) from the common center of the electron and proton bunches experiences the transverse force:

$$F = \frac{e\Lambda(r)}{2\pi\varepsilon_0\gamma^2 r} \tag{1}$$

Here $\Lambda(r) = \Lambda_e(r) - \Lambda_i(r)$, $\Lambda_e(r)$ and $\Lambda_i(r)$ are linear charges of considered longitudinal slice of respectively the ion and the electron beams encircled by radius r.

From Newton's law:

$$r'' = \frac{F(r)}{\gamma\beta^2 m_e c^2} \tag{2}$$

Then the effect of the space-charge is simply:

$$r'' = \frac{2c}{I_a\gamma^3\beta^2}\frac{\Lambda(r)}{r} \tag{3}$$

where Alfven current $I_a = \frac{4\pi\varepsilon_0 m_e c^3}{e}$.

In the presence of additional linear focusing (3) becomes:

$$r'' = \frac{2c}{I_a\gamma^3\beta^2}\frac{\Lambda(r)}{r} - kr \tag{4}$$

3 Emittance and envelope

3.1 1D emittance

Let's consider an ensemble of electrons in (x, x') phase space. We will assume that the phase space occupied by these particles can be well represented by and ellipse. We parameterize the ellipse as:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon \tag{5}$$

where $\varepsilon = A/\pi$, with A being an ellipse area, and $\beta\gamma - \alpha^2 = 1$. It follows right away from (5) that the maximum x:

$$x_{\max} = \sqrt{\varepsilon\beta} \tag{6}$$

If our ensemble of particles is subject to transformations described by matrix M with $\det M = 1$ then both $\varepsilon = \text{const}$ (it's a special case of Liouville's theorem) and the "ellipticity" is preserved. Then,

$$\beta_1 = M_{11}^2\beta_0 - 2M_{11}M_{12}\alpha_0 + M_{12}^2\gamma_0 \tag{7}$$

which gives for a drift of length s:

$$\beta(s) = \beta_0 - 2s\alpha_0 + s^2\gamma_0 \tag{8}$$

Substituting (8) into (6) and taking second derivative of x_{max} with respect to s we get:

$$x''_{\text{max}} = \frac{\varepsilon^2}{x^3_{\text{max}}} \tag{9}$$

Equation (9) describes how the border of an ensemble of particles evolves in the physical space if in the phase space the ensemble can be approximated by an ellipse with area $A = \pi\varepsilon$.

Let us derive here a few other emittance-related formulas, which will be useful in the following considerations.

First of all, from (6),for two similar ensembles with border radii x_1 and x_2:

$$\varepsilon_2 = \varepsilon_1 \frac{x_2^2}{x_1^2} \tag{10}$$

Second, the angles of particles can be split into correlated and uncorrelated parts. It follows directly from (5) and (6) that the correlated angles:

$$\theta_c = -\alpha\sqrt{\frac{\varepsilon}{\beta}} \tag{11}$$

And the uncorrelated angles are given by:

$$\theta_{uc} = \sqrt{\frac{\varepsilon}{\beta}} \tag{12}$$

Of course, for the total angular spread we obtain:

$$\sqrt{\theta_c^2 + \theta_{uc}^2} = \sqrt{\varepsilon\gamma} \tag{13}$$

Figure 1 visualizes the meaning of the discussed concept.

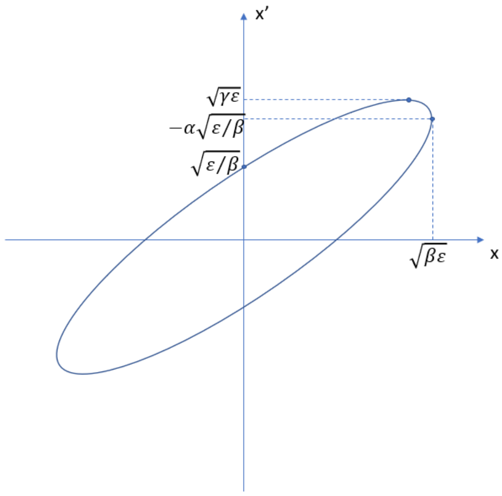

Figure 1: Uncorrelated, correlated and total angles for the particles encircled by an ellipse with area $\pi\varepsilon$.

3.2 r-emittance

Let us consider an arbitrary particle in the 4D phase space with coordinates (x, x', y, y') (see Fig. 2).

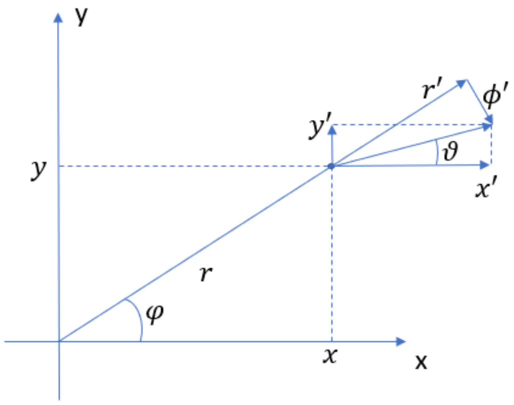

Figure 2: A particle with (x, x', y, y') coordinates.

It is not hard to see that (in notations of Fig. 2):

$$\frac{r'}{\sqrt{x'^2 + y'^2}} = \cos(\varphi - \vartheta) \tag{14}$$

Substituting $\cos(\varphi) = x/r$, $\sin(\varphi) = y/r$, $\cos(\vartheta) = x'/\sqrt{x'^2 + y'^2}$ and $\sin(\vartheta) = y/\sqrt{x'^2 + y'^2}$ into (14) we obtain:

$$rr' = xx' + yy' \tag{15}$$

In a similar fashion, noticing that $\phi'/\sqrt{x'^2 + y'^2} = \sin(\varphi - \vartheta)$ and that $\varphi' = \phi'/r$ we obtain:

$$r^2\varphi' = yx' - xy' \tag{16}$$

Taking a second derivative of $r^2 = x^2 + y^2$ we get:

$$r \cdot r'' + r'^2 = xx'' + yy'' + x'^2 + y'^2 \tag{17}$$

Combining (18) with (15) and (16) we get:

$$r^3 \cdot r'' = (xx'' + yy'')r^2 + (r^2\varphi')^2 \tag{18}$$

Let's consider the evolution of the boundary a of the transverse circularly symmetric uniform distribution over the drift ds. For such a distribution $a^2 = 2 < r^2 >$, $a^2 = 4 < x^2 >= 4 < y^2 >$ and $\varepsilon_x = \varepsilon_y$. Then, using (9) and (10) we get from (18):

$$a'' = \frac{16\varepsilon_x^2 + 4 < r^2\varphi' >^2}{a^3} \tag{19}$$

We can define r-emittance ε_a as:

$$\varepsilon_a^2 = 16\varepsilon_x^2 + 4 < r^2\varphi' >^2 \tag{20}$$

The physical meaning of two parts of ε_a is clear from (10) and (16). The first part is just a 1D emittance scaled up by a factor of 4 due to $a^2 = 4 < x^2 >$. The second part is the "rotational emittance" representing the input of correlated rotational angles in the presence of solenoidal field (it is probably the only x-y coupling preserving the circular symmetry). In the absence of continuous solenoidal field in the cooling section, which is the LEReC case, $\varepsilon_a^2 = (4\varepsilon_x)^2$.

4 r-layer equation

We can write an equation of r-layer as superposition of two special cases (4) and (20):

$$r'' = \frac{2c}{I_a\gamma^3\beta^2}\frac{\Lambda(r)}{r} - kr + \frac{\varepsilon_r^2}{r^3} \tag{21}$$

Equation (21) is mathematically robust for the case of linear space charge force. It is an approximation (but a good one for properly done numerical simulations) for the nonlinear case.

5 Uniform transverse distribution

For the uniform transverse distribution of electrons and ions in the CS we get:

$$\Lambda_e = \frac{I_e}{\beta c}\frac{r^2}{a_e^2(s)} \tag{22}$$

$$\Lambda_e = \frac{I_i}{\beta c}\frac{r^2}{a_i^2} \tag{23}$$

Here a_e and a_i are total radii of respectively electron and ion beams. In (23) we assumed that $a_i = \text{const}$ and that $r \leq a_i \ \forall\ s$ in the CS. In (22) we used $a_e(s)$ to stress that the beam radius changes as the beam travels through the CS.

From (21-23):

$$r'' = K_e\left(\frac{r}{a_e^2} - \frac{I_i}{I_e}\frac{r}{a_i^2}\right) - kr + \frac{\varepsilon_r^2}{r^3} \tag{24}$$

where, generalized perviance $K_e = \frac{2I_e}{I_a\beta^3\gamma^3}$.

The envelope equation ($r = a_e$) is:

$$a_e'' = K_e\left(\frac{1}{a_e} - \frac{I_i}{I_e}\frac{a_e}{a_i^2}\right) - ka_e + \frac{\varepsilon_a^2}{a_e^3} \tag{25}$$

Equation (25) with $I_i = 0$ is exactly the equation 4.85 in [2], which is a K-V equation for round beam with equal horizontal and vertical emittances (see Chapter 5.3.2 of [2]).

If we are interested in equation for the root mean square (rms) radius $\sigma_r = \sqrt{2}a_r$, then noticing that the linear focusing preserves the uniformity of distribution and using (10) we get from (24):

$$\sigma_r'' = K_e\left(\frac{1}{2\sigma_r} - \frac{I_i}{I_e}\frac{\sigma_r}{a_i^2}\right) - k\sigma_r + \frac{\varepsilon_\sigma^2}{\sigma_r^3} \tag{26}$$

Here $\varepsilon_\sigma = \varepsilon_a\frac{\sigma_r^2}{a_e^2}$.

Finally, if we are interested in "one dimensional" equation for $\sigma_{xy} \equiv \sigma_x = \sigma_y$, $(\sigma_x^2 + \sigma_y^2 = \sigma_r^2)$ then we get:

$$\sigma_{xy}'' = K_e \left(\frac{1}{4\sigma_{xy}} - \frac{I_i}{I_e} \frac{\sigma_{xy}}{a_i^2} \right) - k\sigma_{xy} + \frac{\varepsilon_{xy}^2}{\sigma_{xy}^3} \tag{27}$$

where $\varepsilon_{xy} = \varepsilon_a/4$.

With $I_i = 0$ (27) is the "Sacherer equation" [3].

Therefore, when writing the envelope equation for the beam with uniform distribution use (25) if you are interested in the full envelope (usually it is the best idea for the beam, which transverse profile is literally a circle), use (26) if you are interested in rms envelope, and if you want to work with "1D" parameters use (27). Of course, you must not forget about proper emittance scaling provided by (10).

6 Gaussian transverse distribution

Let us assume that at the entrance to the CS both the electron and the ion beams have a circularly symmetric Gaussian transverse distributions with linear charge density function:

$$f_{e,i}(r,\phi) = \frac{I_{e,i}}{\beta c} \frac{1}{2\pi\sigma_{e,i}^2} e^{-\frac{r^2}{2\sigma_{e,i}^2}} \tag{28}$$

It is important to point out that in the notations used in (28) $\sigma \equiv \sigma_x = \sigma_y$ and $r^2 = x^2 + y^2$.

Then the respective linear charges encircled by radius r:

$$\Lambda_e(r) = \frac{I_e}{\beta c} \left(1 - e^{-\frac{r^2}{2\sigma_e(s)^2}} \right) \tag{29}$$

$$\Lambda_i(r) = \frac{I_i}{\beta c} \left(1 - e^{-\frac{r^2}{2\sigma_i^2}} \right) \tag{30}$$

Then, from (21) we get:

$$r'' = K_e \frac{1}{r} \left[\left(1 - e^{-\frac{r^2}{2\sigma_e(s)^2}} \right) - \frac{I_i}{I_e} \left(1 - e^{-\frac{r^2}{2\sigma_i^2}} \right) \right] - kr + \frac{\varepsilon_r^2}{r^3} \tag{31}$$

The space charge-induced force in (31) is essentially nonlinear. This means that the beam distribution changes along the CS. This also means that while Liouville's theorem holds true (because the forces acting on the beam are still Hamiltonian and because the flow of the ensemble of particles in the phase space is still "smooth") and the emittance $\varepsilon = \text{const}$, the ellipticity of the phase space area characterized by ε is not preserved. Therefore,

(31) is correct only at the entrance of the CS, the envelope equation can not be obtained in the explicit form and, generally speaking, the rms correlated angular spread of the beam $\sigma_{\theta c}$ at any given location s along the CS is not equal to the "envelope angle" $d\sigma_r/ds$.

Of course, at each point along the cooling section for step ds small enough to preserve the ellipticity of the chosen phase space area (21) is still correct. So, the brutal force approach to numerical simulations of beam dynamics is to split initial distribution into N_r layers, start solving (21) (initially written in form (31)) and on each step of the simulation recalculate the transverse distribution, split new distribution into N_r layers again and repeat the integration step.

We suggest a simplified approach resulting in a much faster and cleaner numerical calculations. We assume that the linear charge of the electron beam encircled by $r(s)$ stays constant $\forall s$ along the CS. That is, we assume that while the transverse distribution is changing freely and the radius $r(s)$ of each r-layer is evolving along the CS, the charge encircled by each layer:

$$\Lambda_e(r(s)) = \text{const} = \frac{I_e}{\beta c}\left(1 - e^{-\frac{r_0^2}{2\sigma_{e0}^2}}\right) \tag{32}$$

Then, with assumption (32) and taking (10) into account (21) becomes:

$$r'' = K_e \frac{1}{r}\left[\left(1 - e^{-\frac{r_0^2}{2\sigma_{e0}^2}}\right) - \frac{I_i}{I_e}\left(1 - e^{-\frac{r^2}{2\sigma_i^2}}\right)\right] - kr + \frac{\varepsilon_{x,y}^2 r_0^4}{\sigma_{e0}^4}\frac{1}{r^3} \tag{33}$$

We solve (33) numerically with an explicit, exactly simplectic, second order method [4]. Explicitly written, the numerical solution for layer m of N_r layers is given by:

$$r_{n+1,m} = r_{n,m} + ds\theta_{n,m} + \frac{ds^2}{2}\Phi(r_{n,m}, r_{0,m}) \tag{34}$$

$$\theta_{n+1,m} = \theta_{n,m} + ds\Phi(r_{n,m} + \frac{ds}{2}\theta_{n,m}, r_{0,m}) \tag{35}$$

$$\Phi(r_n, r_0) = K_e \frac{1}{r_n}\left[\left(1 - e^{-\frac{r_0^2}{2\sigma_{e0}^2}}\right) - \frac{I_i}{I_e}\left(1 - e^{-\frac{r_n^2}{2\sigma_i^2}}\right)\right] - kr_n + \frac{\varepsilon_{x,y}^2 r_0^4}{\sigma_{e0}^4}\frac{1}{r_n^3} \tag{36}$$

On each step of simulations the correlated angular spread $\sigma_{\theta c}$ can be calculated from obtained N_r layer angles $\theta_{n,m}$ and the rms beam size σ_r must be calculated from N_r $r_{n,m}$. Remember that you do not need to re-calculate the distribution since you already know the charge within each layer. Finally, one shall apply (12) to obtain uncorrelated energy spread as:

$$\sigma_{\theta uc} = \frac{\varepsilon_r}{\sigma_r} = \frac{2\varepsilon_{x,y}}{\sigma_r} \tag{37}$$

Of course, the total angular spread on each step is $\sqrt{\sigma_{\theta c}^2 + \sigma_{\theta uc}^2}$. When providing the final result one has to remember that we calculated $\sigma_r = \sqrt{2}\sigma_{x,y}$ and $\sigma_\theta = \sqrt{2}\sigma_{\theta x,y}$.

7 Conclusion

In this note we described the electron beam dynamics in non-magnetized electron cooler with strong hadron-electron focusing. We derived a set of explicit equations (34-37) for fast numerical simulations of such a beam.

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