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3D Vlasov Theory of the Plasma Cascade Instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

The plasma cascade instabilty (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. This note shows how one can reduce the Vlasov equation to a set of Volterra equations of the second kind, which are amenable to accurate numerical solution.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates x_1, x_2, x_3, t. The unperturbed plasma has an average velocity

$$\mathbf{v}_0(\mathbf{x}, t) = \sum_{j=1}^{3} \hat{x}_j x_j \omega_j(t) \tag{1}$$

This average velocity is driven by the acceleration

$$\mathbf{A}_0(\mathbf{x}, t) = \sum_{j=1}^{3} \hat{x}_j x_j k_j(t) \tag{2}$$

We wish to find a distribution with these parameters. We will take an unperturbed distribution of the form $f_0(\mathbf{x}, \mathbf{v}, t) = f_0(H(\mathbf{x}, \mathbf{v}, t))$ with

$$H = \sum_{j=1}^{3} \frac{\alpha_j(t)}{2} \left(v_j - \omega_j x_j\right)^2 . \tag{3}$$

The Vlasov equation is

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{4}$$

and since $f_0 = f_0(H)$, H satisfies equation (4) as well. Insert H for f_0 in equation (4). The resulting terms proportional to x_j and v_j are

$$-\alpha_j(v_j - \omega_j x_j)\dot{\omega}_j x_j + \dot{\alpha}_j(v_j - \omega_j x_j)^2/2 - v_j \alpha_j \omega_j (v_j - \omega_j x_j) + x_j k_j \alpha_j (v_j - \omega_j x_j) = 0. \tag{5}$$

Setting the coefficients of x_j^2, $x_j v_j$ and v_j^2 to zero we find equation (4) is satisfied if

$$\dot{\alpha}_j = 2\alpha_j \omega_j, \qquad \dot{\omega}_j + \omega_j^2 = k_j \tag{6}$$

for $j = 1, 2, 3$. If equations (6) are satisfied then any function $f_0(H)$ will satisfy equation (4). For physical solutions we require $f_0(H) d^3x d^3v$ to be the number of electrons in the phase space volume $d^3x d^3v$.

We will use first order perturbation theory with $f = f_0 + f_1$ so that

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_1}{\partial \mathbf{v}} + \mathbf{A}_1(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{7}$$

where $\mathbf{A}_1$ is the acceleration created by f_1 and by the ion seeding the instability. Using the results of [3] we introduce time dependent spatial wave numbers and assume a perturbation where the spatial density of the electrons varies as

$$n_{\mathbf{P}}(\mathbf{x}, t) = \hat{n}_{\mathbf{P}}(t) \exp\left(i \sum_{j=1}^{3} P_j \lambda_j(t) x_j \right) \equiv \hat{n}_{\mathbf{P}}(t) \exp(i\Psi(\mathbf{x}, t)) \tag{8}$$

where $\dot{\lambda}_j + \omega_j \lambda_j = 0$ and the P_j, s are constant in time. The total perturbed density from all wavenumbers is

$$n_1(\mathbf{x}, t) = \int n_{\mathbf{P}}(\mathbf{x}, t) d^3 P.$$

Consider an ion implanted at $t = 0$ and located at $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$. The acceleration it generates is given by (cgs units)

$$\begin{aligned}\mathbf{A}_I(\mathbf{x}, t) &= \frac{qQ}{m} \frac{\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 t}{|\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 t|^3} \\ &= -i \frac{4\pi qQ}{m(2\pi)^3} \int d^3 P \lambda_1 \lambda_2 \lambda_3 \frac{(P_1\lambda_1, P_2\lambda_2, P_3\lambda_3)}{\sum_j P_j^2 \lambda_j^2} \exp\left(i \sum_m P_m \lambda_m (x_m - x_{0m} - v_{0m} t) \right), \end{aligned} \tag{9}$$

where $q = -|e|$ is the electron charge, m is its mass, and Q is the charge on the ion. The net acceleration for wavenumber $\mathbf{P}$ due to both the ions and electrons is $\mathbf{A}_{\mathbf{P}}(x, t) = \tilde{\mathbf{A}}(t) \exp(i\Psi(\mathbf{x}, t))$ with[5]

$$\tilde{\mathbf{A}}(t) = -4\pi i \frac{(P_1\lambda_1, P_2\lambda_2, P_3\lambda_3)}{\sum_m P_m^2 \lambda_m^2} \left\{ \frac{q^2}{m} \hat{n}_{\mathbf{P}}(t) + \frac{qQ}{(2\pi)^3 m} \lambda_1 \lambda_2 \lambda_3 \exp\left(-i \sum_m P_m \lambda_m (x_{0m} + v_{0m} t) \right) \right\}. \tag{10}$$

To solve the Vlasov equation we consider a single $\mathbf{P}$. Consider the Ansantz

$$f_{\mathbf{P}}(\mathbf{x}, \mathbf{v}, t) = \frac{df_0}{dH} g(v_1 - \omega_1 x_1, v_2 - \omega_2 x_2, v_3 - \omega_3 x_3, t) \exp(i\Psi(\mathbf{x}, t)).$$

Notice that the x_j dependence in g and f_0 only shows up as $v_j - \omega_j x_j$ so it drops out after integrating over v_j. This generates the correct spatial dependence for $n_{\mathbf{P}}(\mathbf{x}, t)$. For convenient notation define $u_j = v_j - \omega_j x_j$ and remember that

$$\frac{\partial g(\mathbf{x}, \mathbf{v}, t)}{\partial t} = \frac{\partial g(\mathbf{u}, t)}{\partial t} + \frac{\partial g(\mathbf{u}, t)}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}(\mathbf{x}, \mathbf{v}, t)}{\partial t}.$$

Plugging into the Vlasov eq one finds

$$\frac{\partial g(\mathbf{u}, t)}{\partial t} + \sum_{j=1}^{3} i g P_j \lambda_j u_j - \omega_j u_j \frac{\partial g}{\partial u_j} + \alpha_j u_j \tilde{A}_j = 0 \tag{11}$$

To proceed we multiply the last term on the right of equation (11) by $\delta(t - t_0)$ with the intention of integrating over t_0 later.

$$\frac{\partial \tilde{g}(\mathbf{u}, t, t_0)}{\partial t} + \sum_{j=1}^{3} i \tilde{g} P_j \lambda_j u_j - \omega_j u_j \frac{\partial \tilde{g}}{\partial u_j} + \alpha_j u_j \tilde{A}_j \delta(t - t_0) = 0 \tag{12}$$

We look for solutions of the form

$$\tilde{g}(\mathbf{u}, t, t_0) = H(t - t_0) \mathbf{q}(t) \cdot \mathbf{u} \exp(i\mathbf{K}(t) \cdot \mathbf{u})$$

where $H(t - t_0)$ is 1 for $t \geq t_0$ and zero otherwise. Inserting this expression into (11) and using the same sort of tricks used to solve (5) we find that (12) is satisfied if

$$\dot{K}_j + P_j \lambda_j(t) - \omega_j(t) K_j = 0, \qquad \text{with } \; K_j(t_0) = 0 \tag{13}$$

$$\dot{q}_j - \omega_j(t) q_j = 0, \qquad \text{with } \; q_j(t_0) = -\alpha_j(t_0) \tilde{A}_j(t_0). \tag{14}$$

To bring the pieces together define the general solution $M_j(t)$ as the solution to equation (13) but with the boundary condition $M_j(0) = 0$. Also define the phases $\Phi_j(t)$ so that $\omega_j(t) = \dot{\Phi}_j(t)$. With these definitions $K_m(t, t_0) = M_m(t) - M_m(t_0) \exp[\Phi_m(t) - \Phi_m(t_0)]$ and $q_j(t, t_0) = -\alpha_j(t_0) \tilde{A}_j(t_0) \exp[\Phi_j(t) - \Phi_j(t_0)]$. This yields

$$g(\mathbf{u}, t) = \sum_{j=1}^{3} \int_0^t dt_0 q_j(t, t_0) u_j \exp\left(i \sum_{m=1}^{3} u_m K_m(t, t_0) \right) \tag{15}$$

To close the equations we note that the only unknown in $\tilde{A}_j(t)$ is

$$\hat{n}_{\mathbf{P}}(t) = \int d^3u \frac{\partial f_0}{\partial H}(\mathbf{u},t) g(\mathbf{u},t) \tag{16}$$

$$= -\sum_{j=1}^{3} \int_0^t dt_0 \alpha_j(t_0) \tilde{A}_j(t_0) \exp\left(\Phi_j(t) - \Phi_j(t_0)\right) \int d^3u f_0' \left(\sum_{m=1}^{3} \alpha_m(t) u_m^2/2\right) u_j \exp\left(i \sum_{k=1}^{3} u_k K_k(t,t_0)\right), \tag{17}$$

$$= -\sum_{j=1}^{3} \int_0^t dt_0 \tilde{A}_j(t_0) G_j(t, t0), \tag{18}$$

$$= \int_0^t dt_0 \left[\frac{q^2}{m} \hat{n}_{\mathbf{P}}(t_0) + D_I(t_0)\right] \sum_{j=1}^{3} \left(\frac{4\pi i P_j \lambda_j(t_0)}{\sum_{m=1}^{3} \lambda_m^2(t_0) P_m^2} G_j(t,t_0) \right) \tag{19}$$

where

$$G_j(t,t_0) = \alpha_j(t_0) \exp\left(\Phi_j(t) - \Phi_j(t_0)\right) \int d^3u f_0' \left(\sum_{m=1}^{3} \alpha_m(t) u_m^2/2\right) u_j \exp\left(i \sum_{k=1}^{3} u_k K_k(t,t_0)\right), \tag{20}$$

$$= -\frac{i\alpha_j(t_0) K_j(t,t_0)}{\alpha_j(t)} \exp\left(\Phi_j(t) - \Phi_j(t_0)\right) \int d^3u f_0 \left(\sum_{m=1}^{3} \alpha_m(t) u_m^2/2\right) \exp\left(i \sum_{k=1}^{3} u_k K_k(t,t_0)\right). \tag{21}$$

and

$$D_I(t) = \frac{qQ}{(2\pi)^3 m} \lambda_1(t)\lambda_2(t)\lambda_3(t) \exp\left(-i \sum_m P_m \lambda_m(t)(x_{0m} + v_{0m} t)\right). \tag{22}$$

We now have a Volterra equation of the second kind for $\hat{n}_{\mathbf{P}}$. While exact solutions look hopeless a numerical solution should be straightforward. We note that changing the integration variable in equation (21) to $z_i = \sqrt{\alpha_i} u_i$ turns it into the Fourier transform of a spherically symmetric function for which a wide range of exact solutions are available.

CONNECTION TO PREVIOUS WORK

If we set $\omega_i = 0$ then these results should reduce to those in [4]. To show this we set

$$d_i(t) = \frac{Z}{(2\pi)^3} \lambda_1(t)\lambda_2(t)\lambda_3(t) \exp\left(-i \sum_m P_m \lambda_m(t)(x_{0m} + v_{0m} t)\right),$$

$$g(\mathbf{K}(t,t_0)) = \frac{1}{n_0} \int d^3u f_0 \left(\sum_{m=1}^{3} \alpha_m(t) u_m^2/2\right) \exp\left(i \sum_{k=1}^{3} u_k K_k(t,t_0)\right),$$

$$R(t,t_0) = \frac{1}{\sum_{m=1}^{3} \lambda_m^2(t_0) P_m^2} \sum_{j=1}^{3} \frac{P_j \lambda_j(t_0) \alpha_j(t_0) K_j(t,t_0)}{\alpha_j(t)} \exp\left(\Phi_j(t) - \Phi_j(t_0)\right),$$

where Z is the atomic number of the ion. Now we have

$$\hat{n}_{\mathbf{P}}(t) = \frac{4\pi q^2 n_0}{m} \int_0^t dt_0 \left[\hat{n}_{\mathbf{P}}(t_0) - d_i(t_0)\right] R(t,t_0) g(\mathbf{K}(t,t_0)). \tag{23}$$

When $\omega_i = 0$, $R(t, t_0) = t_0 - t$, $\lambda_i = 1$, and $\mathbf{K}(t, t_0) = (t_0 - t)\mathbf{P}$. Make these substitutions, account for a difference in Fourier transform conventions, and include the fact that there is a constant velocity offset between reference frames. On finds that equation (23) here is equivalent to equation (8) in [4].

[1] V. N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, arxiv:1802.08677, February 2018
[2] V. N. Litvinenko, G. Wang, Y. Jing, D. Kayran, J. Ma, I. Petrushina,I. Pinayev, K. Shih, arxiv:1902.10846, February 2019
[3] M. Blaskiewicz, BNL-211960-2019-TECH, (2019)
[4] Gang Wang and Michael Blaskiewicz, *Phys. Rev. E* **78**, 026413 (2008).
[5] a factor of $(2\pi)^3$ is missing in [3]