3D Vlasov theory of the plasma cascade instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates \( x_1, x_2, x_3, t \). The unperturbed plasma has an average velocity

\[
v_0(x_1, x_2, x_3, t) = \sum_{j=1}^{3} \hat{x}_j x_j \omega_j(t)
\]

This average velocity is driven by the acceleration

\[
A_0(x_1, x_2, x_3, t) = \sum_{j=1}^{3} \hat{x}_j x_j k_j(t)
\]

We wish to find a distribution with these parameters. We will take an unperturbed distribution of the form

\[
f_0(x, v, t) = f_0(H(x, v, t))
\]

where

\[
H = \sum_{j=1}^{3} \frac{\alpha_j(t)}{2} (v_j - \omega_j x_j)^2.
\]

The Vlasov equation is

\[
\frac{\partial f_0}{\partial t} + v \cdot \frac{\partial f_0}{\partial x} + A_0(x, t) \cdot \frac{\partial f_0}{\partial v} = 0,
\]

and since \( f_0 = f_0(H) \), \( H \) satisfies equation (4) as well. Insert \( H \) for \( f_0 \) in equation (4). The resulting terms proportional to \( x_j^2, x_j v_j \) and \( v_j^2 \) to zero we find equation (4) is satisfied if

\[
\dot{\alpha}_j = 2 \alpha_j \omega_j, \quad \dot{\omega}_j + \omega_j^2 = k_j
\]

for \( j = 1, 2, 3 \). If equations (6) are satisfied then any function \( f_0(H) \) will satisfy equation (4). For physical solutions we require \( f_0(H)d^3xd^3v \) to be the number of electrons in the phase space volume \( d^3xd^3v \).

We will use first order perturbation theory with \( f = f_0 + f_1 \) so that

\[
\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial x} + A_0(x, t) \cdot \frac{\partial f_1}{\partial v} + A_1(x, t) \cdot \frac{\partial f_0}{\partial v} = 0,
\]

where \( A_1 \) is the acceleration created by \( f_1 \) and by the ion seeding the instability. Using the results of [3] we introduce time dependent spatial wave numbers and assume a perturbation where the spatial density of the electrons varies as

\[
n_{\mathbf{p}}(x, t) = \hat{n}_{\mathbf{p}}(t) \exp \left( i \sum_{j=1}^{3} P_j \lambda_j(t) x_j \right) \equiv \hat{n}_{\mathbf{p}}(t) \exp(i\Psi(x, t))
\]
where \( \dot{\lambda}_j + \omega_j \lambda_j = 0 \) and the \( P_j, s \) are constant in time. The total perturbed density from all wavenumbers is

\[
n_1(x, t) = \int n_P(x, t) d^3 P.
\]

Consider an ion implanted at \( t = 0 \) and located at \( x = x_0 + v_0 t \). The acceleration it generates is given by (cgs units)

\[
A_I(x, t) = \frac{qQ}{m |x - x_0 - v_0 t|^3} \int d^3 P \lambda_1 \lambda_2 \lambda_3 \left( \frac{P_1 \lambda_1, P_2 \lambda_2, P_3 \lambda_3}{\sum_j P_j^2 \lambda_j^2} \right) \exp \left( i \sum_m P_m \lambda_m (x_m - x_{0m} - v_{0m} t) \right),
\]

where \( q = -|e| \) is the electron charge, \( m \) is its mass, and \( Q \) is the charge on the ion. The net acceleration for wavenumber \( P \) due to both the ions and electrons is \( \vec{A}_P(x, t) = \vec{A}(t) \exp(i \Psi(x, t)) \) with[5]

\[
\vec{A}(t) = -4\pi i \left( \frac{P_1 \lambda_1, P_2 \lambda_2, P_3 \lambda_3}{\sum_m P_m^2 \lambda_m^2} \right) \left\{ \frac{q^2}{m} \vec{v}_p(t) + \frac{qQ}{(2\pi)^3 m} \lambda_1 \lambda_2 \lambda_3 \exp \left( -i \sum_m P_m \lambda_m (x_{0m} + v_{0m} t) \right) \right\}.
\]

To solve the Vlasov equation we consider a single \( P \). Consider the Ansatz

\[
f_P(x, v, t) = \frac{df_0}{dH} f(v_1 - \omega_1 x_1, v_2 - \omega_2 x_2, v_3 - \omega_3 x_3, t) \exp(i \Psi(x, t)).
\]

Notice that the \( x_j \) dependence in \( g \) and \( f_0 \) only shows up as \( v_j - \omega_j x_j \) so it drops out after integrating over \( v_j \). This generates the correct spatial dependence for \( n_P(x, t) \). For convenient notation define \( u_j = v_j - \omega_j x_j \) and remember that

\[
\frac{\partial g(x, v, t)}{\partial t} = \frac{\partial g(u, t)}{\partial t} + \frac{\partial g(u, t)}{\partial u} \cdot \frac{\partial u(x, v, t)}{\partial t}.
\]

Plugging into the Vlasov eq one finds

\[
\frac{\partial g(u, t)}{\partial t} + \sum_{j=1}^3 i g P_j \lambda_j u_j - \omega_j u_j \frac{\partial g}{\partial u_j} + \alpha_j u_j \dot{A}_j = 0
\]

To proceed we multiply the last term on the right of equation (11) by \( \delta(t - t_0) \) with the intention of integrating over \( t_0 \) later.

\[
\frac{\partial g(u, t, t_0)}{\partial t} + \sum_{j=1}^3 i g P_j \lambda_j u_j - \omega_j u_j \frac{\partial g}{\partial u_j} + \alpha_j u_j \dot{A}_j \delta(t - t_0) = 0
\]

We look for solutions of the form

\[
\tilde{g}(u, t, t_0) = H(t - t_0) q(t) \cdot u \exp(i K(t) \cdot u)
\]

where \( H(t - t_0) \) is 1 for \( t \geq t_0 \) and zero otherwise. Inserting this expression into (11) and using the same sort of tricks used to solve (5) we find that (12) is satisfied if

\[
\dot{K}_j + P_j \lambda_j (t) - \omega_j(t) K_j = 0, \quad \text{with} \quad K_j(t_0) = 0
\]

\[
\dot{q}_j - \omega_j(t) q_j = 0, \quad \text{with} \quad q_j(t_0) = -\alpha_j(t_0) \dot{A}_j(t_0).
\]

To bring the pieces together define the general solution \( M_j(t) \) as the solution to equation (13) but with the boundary condition \( M_j(0) = 0 \). Also define the phases \( \Phi_j(t) \) so that \( \omega_j(t) = \dot{\Phi}_j(t) \). With these definitions \( K_m(t, t_0) = M_m(t) - M_m(t_0) \exp[\Phi_m(t) - \Phi_m(t_0)] \) and \( q_j(t, t_0) = -\alpha_j(t_0) \dot{A}_j(t_0) \exp[\Phi_j(t) - \Phi_j(t_0)] \). This yields

\[
g(u, t) = \sum_{j=1}^3 \int_0^t dt_0 q_j(t, t_0) u_j \exp \left( \sum_{j=1}^3 \frac{1}{m} u_m K_m(t, t_0) \right)
\]
To close the equations we note that the only unknown in $\tilde{A}_j(t)$ is

$$\dot{\mathbf{n}}_\mathbf{p}(t) = \int d^3u \frac{\partial f_0}{\partial H}(\mathbf{u}, t) g(\mathbf{u}, t)$$

$$= - \sum_{j=1}^{3} \int dt_0 \alpha_j(t_0) \tilde{A}_j(t_0) \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0' \left( \sum_{m=1}^{3} \alpha_m(t) u_m^2 / 2 \right) u_j \exp \left( i \sum_{k=1}^{3} u_k K_k(t, t_0) \right).$$

$$= - \sum_{j=1}^{3} \int dt_0 \tilde{A}_j(t_0) G_j(t, t_0),$$

$$= \int dt_0 \left[ \frac{q^2}{m} \dot{\mathbf{n}}_\mathbf{p}(t_0) + D_f(t_0) \right] \sum_{j=1}^{3} \left( \frac{4\pi i P_j \lambda_j(t_0)}{\sum_{m=1}^{3} \lambda_m^2(t_0) P_m^2} G_j(t, t_0) \right)$$

where

$$G_j(t, t_0) = \alpha_j(t_0) \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0' \left( \sum_{m=1}^{3} \alpha_m(t) u_m^2 / 2 \right) u_j \exp \left( i \sum_{k=1}^{3} u_k K_k(t, t_0) \right),$$

$$= - \frac{i \alpha_j(t_0) K_j(t, t_0)}{\alpha_j(t)} \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0 \left( \sum_{m=1}^{3} \alpha_m(t) u_m^2 / 2 \right) \exp \left( i \sum_{k=1}^{3} u_k K_k(t, t_0) \right).$$

and

$$D_f(t) = \frac{qQ}{(2\pi)^3 m} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp \left( -i \sum_m P_m \lambda_m(t)(x_{0m} + v_{0m} t) \right).$$

We now have a Volterra equation of the second kind for $\dot{\mathbf{n}}_\mathbf{p}$. While exact solutions look hopeless a numerical solution should be straightforward. We note that changing the integration variable in equation (21) to $z_i = \sqrt{\alpha_i} u_i$ turns it into the Fourier transform of a spherically symmetric function for which a wide range of exact solutions are available.

**CONNECTION TO PREVIOUS WORK**

If we set $\omega_i = 0$ then these results should reduce to those in [4]. To show this we set

$$d_i(t) = \frac{Z}{(2\pi)^3} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp \left( -i \sum_m P_m \lambda_m(t)(x_{0m} + v_{0m} t) \right),$$

$$g(\mathbf{K}(t, t_0)) = \frac{1}{n_0} \int d^3u f_0 \left( \sum_{m=1}^{3} \alpha_m(t) u_m^2 / 2 \right) \exp \left( i \sum_{k=1}^{3} u_k K_k(t, t_0) \right),$$

$$R(t, t_0) = \frac{1}{3 \sum_{m=1}^{3} \lambda_m^2(t_0) P_m^2} \sum_{j=1}^{3} \frac{P_j \lambda_j(t_0) \alpha_j(t_0) K_j(t, t_0)}{\alpha_j(t)} \exp(\Phi_j(t) - \Phi_j(t_0)),$$

where $Z$ is the atomic number of the ion. Now we have

$$\dot{\mathbf{n}}_\mathbf{p}(t) = \frac{4\pi q^2 n_0}{m} \int_0^t dt_0 [\dot{\mathbf{n}}_\mathbf{p}(t_0) - d_i(t_0)] R(t, t_0) g(\mathbf{K}(t, t_0)).$$

(23)
When $\omega_i = 0$, $R(t, t_0) = t_0 - t$, $\lambda_i = 1$, and $K(t, t_0) = (t_0 - t)P$. Make these substitutions, account for a difference in Fourier transform conventions, and include the fact that there is a constant velocity offset between reference frames. On finds that equation (23) here is equivalent to equation (8) in [4].

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[5] a factor of $(2\pi)^3$ is missing in [3]