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3D Vlasov Theory of the Plasma Cascade Instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. This note shows how one can reduce the Vlasov equation to a set of Volterra equations of the second kind, which are amenable to accurate numerical solution.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates x_1, x_2, x_3, t . The unperturbed plasma has an average velocity

$$\mathbf{v}_0(\mathbf{x}, t) = \sum_{j=1}^3 \hat{x}_j x_j \omega_j(t) \quad (1)$$

This average velocity is driven by the acceleration

$$\mathbf{A}_0(\mathbf{x}, t) = \sum_{j=1}^3 \hat{x}_j x_j k_j(t) \quad (2)$$

We wish to find a distribution with these parameters. We will take an unperturbed distribution of the form $f_0(\mathbf{x}, \mathbf{v}, t) = f_0(H(\mathbf{x}, \mathbf{v}, t))$ with

$$H = \sum_{j=1}^3 \frac{\alpha_j(t)}{2} (v_j - \omega_j x_j)^2. \quad (3)$$

The Vlasov equation is

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \quad (4)$$

and since $f_0 = f_0(H)$, H satisfies equation (4) as well. Insert H for f_0 in equation (4). The resulting terms proportional to x_j and v_j are

$$-\alpha_j (v_j - \omega_j x_j) \dot{\omega}_j x_j + \dot{\alpha}_j (v_j - \omega_j x_j)^2 / 2 - v_j \alpha_j \omega_j (v_j - \omega_j x_j) + x_j k_j \alpha_j (v_j - \omega_j x_j) = 0. \quad (5)$$

Setting the coefficients of x_j^2 , $x_j v_j$ and v_j^2 to zero we find equation (4) is satisfied if

$$\dot{\alpha}_j = 2\alpha_j \omega_j, \quad \dot{\omega}_j + \omega_j^2 = k_j \quad (6)$$

for $j = 1, 2, 3$. If equations (6) are satisfied then any function $f_0(H)$ will satisfy equation (4). For physical solutions we require $f_0(H) d^3 x d^3 v$ to be the number of electrons in the phase space volume $d^3 x d^3 v$.

We will use first order perturbation theory with $f = f_0 + f_1$ so that

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_1}{\partial \mathbf{v}} + \mathbf{A}_1(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \quad (7)$$

where \mathbf{A}_1 is the acceleration created by f_1 and by the ion seeding the instability. Using the results of [3] we introduce time dependent spatial wave numbers and assume a perturbation where the spatial density of the electrons varies as

$$n_{\mathbf{P}}(\mathbf{x}, t) = \hat{n}_{\mathbf{P}}(t) \exp \left(i \sum_{j=1}^3 P_j \lambda_j(t) x_j \right) \equiv \hat{n}_{\mathbf{P}}(t) \exp(i\Psi(\mathbf{x}, t)) \quad (8)$$

where $\dot{\lambda}_j + \omega_j \lambda_j = 0$ and the P_j, s are constant in time. The total perturbed density from all wavenumbers is

$$n_1(\mathbf{x}, t) = \int n_{\mathbf{P}}(\mathbf{x}, t) d^3 P.$$

Consider an ion implanted at $t = 0$ and located at $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$. The acceleration it generates is given by (cgs units)

$$\begin{aligned} \mathbf{A}_I(\mathbf{x}, t) &= \frac{qQ}{m} \frac{\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 t}{|\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 t|^3} \\ &= -i \frac{4\pi qQ}{m(2\pi)^3} \int d^3 P \lambda_1 \lambda_2 \lambda_3 \frac{(P_1 \lambda_1, P_2 \lambda_2, P_3 \lambda_3)}{\sum_j P_j^2 \lambda_j^2} \exp\left(i \sum_m P_m \lambda_m (x_m - x_{0m} - v_{0m} t)\right), \end{aligned} \quad (9)$$

where $q = -|e|$ is the electron charge, m is its mass, and Q is the charge on the ion. The net acceleration for wavenumber \mathbf{P} due to both the ions and electrons is $\mathbf{A}_{\mathbf{P}}(x, t) = \tilde{\mathbf{A}}(t) \exp(i\Psi(\mathbf{x}, t))$ with[5]

$$\tilde{\mathbf{A}}(t) = -4\pi i \frac{(P_1 \lambda_1, P_2 \lambda_2, P_3 \lambda_3)}{\sum_m P_m^2 \lambda_m^2} \left\{ \frac{q^2}{m} \hat{n}_{\mathbf{P}}(t) + \frac{qQ}{(2\pi)^3 m} \lambda_1 \lambda_2 \lambda_3 \exp\left(-i \sum_m P_m \lambda_m (x_{0m} + v_{0m} t)\right) \right\}. \quad (10)$$

To solve the Vlasov equation we consider a single \mathbf{P} . Consider the Ansatz

$$f_{\mathbf{P}}(\mathbf{x}, \mathbf{v}, t) = \frac{df_0}{dH} g(v_1 - \omega_1 x_1, v_2 - \omega_2 x_2, v_3 - \omega_3 x_3, t) \exp(i\Psi(\mathbf{x}, t)).$$

Notice that the x_j dependence in g and f_0 only shows up as $v_j - \omega_j x_j$ so it drops out after integrating over v_j . This generates the correct spatial dependence for $n_{\mathbf{P}}(\mathbf{x}, t)$. For convenient notation define $u_j = v_j - \omega_j x_j$ and remember that

$$\frac{\partial g(\mathbf{x}, \mathbf{v}, t)}{\partial t} = \frac{\partial g(\mathbf{u}, t)}{\partial t} + \frac{\partial g(\mathbf{u}, t)}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}(\mathbf{x}, \mathbf{v}, t)}{\partial t}.$$

Plugging into the Vlasov eq one finds

$$\frac{\partial g(\mathbf{u}, t)}{\partial t} + \sum_{j=1}^3 i g P_j \lambda_j u_j - \omega_j u_j \frac{\partial g}{\partial u_j} + \alpha_j u_j \tilde{A}_j = 0 \quad (11)$$

To proceed we multiply the last term on the right of equation (11) by $\delta(t - t_0)$ with the intention of integrating over t_0 later.

$$\frac{\partial \tilde{g}(\mathbf{u}, t, t_0)}{\partial t} + \sum_{j=1}^3 i \tilde{g} P_j \lambda_j u_j - \omega_j u_j \frac{\partial \tilde{g}}{\partial u_j} + \alpha_j u_j \tilde{A}_j \delta(t - t_0) = 0 \quad (12)$$

We look for solutions of the form

$$\tilde{g}(\mathbf{u}, t, t_0) = H(t - t_0) \mathbf{q}(t) \cdot \mathbf{u} \exp(i\mathbf{K}(t) \cdot \mathbf{u})$$

where $H(t - t_0)$ is 1 for $t \geq t_0$ and zero otherwise. Inserting this expression into (11) and using the same sort of tricks used to solve (5) we find that (12) is satisfied if

$$\dot{K}_j + P_j \lambda_j(t) - \omega_j(t) K_j = 0, \quad \text{with } K_j(t_0) = 0 \quad (13)$$

$$\dot{q}_j - \omega_j(t) q_j = 0, \quad \text{with } q_j(t_0) = -\alpha_j(t_0) \tilde{A}_j(t_0). \quad (14)$$

To bring the pieces together define the general solution $M_j(t)$ as the solution to equation (13) but with the boundary condition $M_j(0) = 0$. Also define the phases $\Phi_j(t)$ so that $\omega_j(t) = \dot{\Phi}_j(t)$. With these definitions $K_m(t, t_0) = M_m(t) - M_m(t_0) \exp[\Phi_m(t) - \Phi_m(t_0)]$ and $q_j(t, t_0) = -\alpha_j(t_0) \tilde{A}_j(t_0) \exp[\Phi_j(t) - \Phi_j(t_0)]$. This yields

$$g(\mathbf{u}, t) = \sum_{j=1}^3 \int_0^t dt_0 q_j(t, t_0) u_j \exp\left(i \sum_{m=1}^3 u_m K_m(t, t_0)\right) \quad (15)$$

To close the equations we note that the only unknown in $\tilde{A}_j(t)$ is

$$\hat{n}_{\mathbf{P}}(t) = \int d^3u \frac{\partial f_0}{\partial H}(\mathbf{u}, t) g(\mathbf{u}, t) \quad (16)$$

$$= - \sum_{j=1}^3 \int_0^t dt_0 \alpha_j(t_0) \tilde{A}_j(t_0) \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0' \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2 \right) u_j \exp \left(i \sum_{k=1}^3 u_k K_k(t, t_0) \right), \quad (17)$$

$$= - \sum_{j=1}^3 \int_0^t dt_0 \tilde{A}_j(t_0) G_j(t, t_0), \quad (18)$$

$$= \int_0^t dt_0 \left[\frac{q^2}{m} \hat{n}_{\mathbf{P}}(t_0) + D_I(t_0) \right] \sum_{j=1}^3 \left(\frac{4\pi i P_j \lambda_j(t_0)}{\sum_{m=1}^3 \lambda_m^2(t_0) P_m^2} G_j(t, t_0) \right) \quad (19)$$

where

$$G_j(t, t_0) = \alpha_j(t_0) \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0' \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2 \right) u_j \exp \left(i \sum_{k=1}^3 u_k K_k(t, t_0) \right), \quad (20)$$

$$= - \frac{i \alpha_j(t_0) K_j(t, t_0)}{\alpha_j(t)} \exp(\Phi_j(t) - \Phi_j(t_0)) \int d^3u f_0 \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2 \right) \exp \left(i \sum_{k=1}^3 u_k K_k(t, t_0) \right). \quad (21)$$

and

$$D_I(t) = \frac{qQ}{(2\pi)^3 m} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp \left(-i \sum_m P_m \lambda_m(t) (x_{0m} + v_{0m} t) \right). \quad (22)$$

We now have a Volterra equation of the second kind for $\hat{n}_{\mathbf{P}}$. While exact solutions look hopeless a numerical solution should be straightforward. We note that changing the integration variable in equation (21) to $z_i = \sqrt{\alpha_i} u_i$ turns it into the Fourier transform of a spherically symmetric function for which a wide range of exact solutions are available.

CONNECTION TO PREVIOUS WORK

If we set $\omega_i = 0$ then these results should reduce to those in [4]. To show this we set

$$d_i(t) = \frac{Z}{(2\pi)^3} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp \left(-i \sum_m P_m \lambda_m(t) (x_{0m} + v_{0m} t) \right),$$

$$g(\mathbf{K}(t, t_0)) = \frac{1}{n_0} \int d^3u f_0 \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2 \right) \exp \left(i \sum_{k=1}^3 u_k K_k(t, t_0) \right),$$

$$R(t, t_0) = \frac{1}{\sum_{m=1}^3 \lambda_m^2(t_0) P_m^2} \sum_{j=1}^3 \frac{P_j \lambda_j(t_0) \alpha_j(t_0) K_j(t, t_0)}{\alpha_j(t)} \exp(\Phi_j(t) - \Phi_j(t_0)),$$

where Z is the atomic number of the ion. Now we have

$$\hat{n}_{\mathbf{P}}(t) = \frac{4\pi q^2 n_0}{m} \int_0^t dt_0 [\hat{n}_{\mathbf{P}}(t_0) - d_i(t_0)] R(t, t_0) g(\mathbf{K}(t, t_0)). \quad (23)$$

When $\omega_i = 0$, $R(t, t_0) = t_0 - t$, $\lambda_i = 1$, and $\mathbf{K}(t, t_0) = (t_0 - t)\mathbf{P}$. Make these substitutions, account for a difference in Fourier transform conventions, and include the fact that there is a constant velocity offset between reference frames. One finds that equation (23) here is equivalent to equation (8) in [4].

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