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3D Vlasov Theory of the Plasma Cascade Instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

The plasma cascade instabilty (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. This note shows how one can reduce the Vlasov equation to a set of Volterra equations of the second kind, which are amenable to accurate numerical solution.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates x_1, x_2, x_3, t . The unperturbed plasma has an average velocity

$$\mathbf{v}_0(\mathbf{x},t) = \sum_{j=1}^3 \hat{x}_j x_j \omega_j(t) \tag{1}$$

This average velocity is driven by the acceleration

$$\mathbf{A}_0(\mathbf{x},t) = \sum_{j=1}^3 \hat{x}_j x_j k_j(t) \tag{2}$$

We wish to find a distribution with these parameters. We will take an unperturbed distribution of the form $f_0(\mathbf{x}, \mathbf{v}, t) = f_0(H(\mathbf{x}, \mathbf{v}, t))$ with

$$H = \sum_{j=1}^{3} \frac{\alpha_j(t)}{2} (v_j - \omega_j x_j)^2.$$
 (3)

The Vlasov equation is

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{4}$$

and since $f_0 = f_0(H)$, H satisfies equation (4) as well. Insert H for f_0 in equation (4). The resulting terms proportional to x_j and v_j are

$$-\alpha_j(v_j - \omega_j x_j)\dot{\omega}_j x_j + \dot{\alpha}_j(v_j - \omega_j x_j)^2 / 2 - v_j \alpha_j \omega_j(v_j - \omega_j x_j) + x_j k_j \alpha_j(v_j - \omega_j x_j) = 0.$$
 (5)

Setting the coefficients of x_i^2 , $x_j v_j$ and v_i^2 to zero we find equation (4) is satisfied if

$$\dot{\alpha}_j = 2\alpha_j \omega_j, \qquad \dot{\omega}_j + \omega_j^2 = k_j \tag{6}$$

for j = 1, 2, 3. If equations (6) are satisfied then any function $f_0(H)$ will satisfy equation (4). For physical solutions we require $f_0(H)d^3xd^3v$ to be the number of electrons in the phase space volume d^3xd^3v .

We will use first order perturbation theory with $f = f_0 + f_1$ so that

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{A}_0(\mathbf{x}, t) \cdot \frac{\partial f_1}{\partial \mathbf{v}} + \mathbf{A}_1(\mathbf{x}, t) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{7}$$

where A_1 is the acceleration created by f_1 and by the ion seeding the instability. Using the results of [3] we introduce time dependent spatial wave numbers and assume a perturbation where the spatial density of the electrons varies as

$$n_{\mathbf{P}}(\mathbf{x},t) = \hat{n}_{\mathbf{P}}(t) \exp\left(i \sum_{j=1}^{3} P_j \lambda_j(t) x_j\right) \equiv \hat{n}_{\mathbf{P}}(t) \exp(i\Psi(\mathbf{x},t))$$
 (8)

where $\dot{\lambda}_j + \omega_j \lambda_j = 0$ and the P_j , s are constant in time. The total perturbed density from all wavenumbers is

$$n_1(\mathbf{x}, t) = \int n_{\mathbf{P}}(\mathbf{x}, t) d^3 P.$$

Consider an ion implanted at t = 0 and located at $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$. The acceleration it generates is given by (cgs units)

$$\mathbf{A}_{I}(\mathbf{x},t) = \frac{qQ}{m} \frac{\mathbf{x} - \mathbf{x}_{0} - \mathbf{v}_{0}t}{|\mathbf{x} - \mathbf{x}_{0} - \mathbf{v}_{0}t|^{3}}$$

$$= -i \frac{4\pi qQ}{m(2\pi)^{3}} \int d^{3}P \lambda_{1} \lambda_{2} \lambda_{3} \frac{(P_{1}\lambda_{1}, P_{2}\lambda_{2}, P_{3}\lambda_{3})}{\sum_{i} P_{j}^{2} \lambda_{j}^{2}} \exp\left(i \sum_{m} P_{m} \lambda_{m} (x_{m} - x_{0m} - v_{0m}t)\right), \tag{9}$$

where q = -|e| is the electron charge, m is its mass, and Q is the charge on the ion. The net acceleration for wavenumber \mathbf{P} due to both the ions and electrons is $\mathbf{A}_{\mathbf{P}}(x,t) = \tilde{\mathbf{A}}(t) \exp(i\Psi(\mathbf{x},t))$ with [5]

$$\tilde{\mathbf{A}}(t) = -4\pi i \frac{(P_1 \lambda_1, P_2 \lambda_2, P_3 \lambda_3)}{\sum_{m} P_m^2 \lambda_m^2} \left\{ \frac{q^2}{m} \hat{n}_{\mathbf{P}}(t) + \frac{qQ}{(2\pi)^3 m} \lambda_1 \lambda_2 \lambda_3 \exp\left(-i \sum_{m} P_m \lambda_m (x_{0m} + v_{0m} t)\right) \right\}. \tag{10}$$

To solve the Vlasov equation we consider a single P. Consider the Ansantz

$$f_{\mathbf{P}}(\mathbf{x}, \mathbf{v}, t) = \frac{df_0}{dH}g(v_1 - \omega_1 x_1, v_2 - \omega_2 x_2, v_3 - \omega_3 x_3, t) \exp(i\Psi(\mathbf{x}, t)).$$

Notice that the x_j dependence in g and f_0 only shows up as $v_j - \omega_j x_j$ so it drops out after integrating over v_j . This generates the correct spatial dependence for $n_{\mathbf{P}}(\mathbf{x},t)$. For convenient notation define $u_j = v_j - \omega_j x_j$ and remember that

$$\frac{\partial g(\mathbf{x}, \mathbf{v}, t)}{\partial t} = \frac{\partial g(\mathbf{u}, t)}{\partial t} + \frac{\partial g(\mathbf{u}, t)}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}(\mathbf{x}, \mathbf{v}, t)}{\partial t}.$$

Plugging into the Vlasov eq one finds

$$\frac{\partial g(\mathbf{u},t)}{\partial t} + \sum_{j=1}^{3} ig P_j \lambda_j u_j - \omega_j u_j \frac{\partial g}{\partial u_j} + \alpha_j u_j \tilde{A}_j = 0$$
(11)

To proceed we multiply the last term on the right of equation (11) by $\delta(t-t_0)$ with the intention of integrating over t_0 later.

$$\frac{\partial \tilde{g}(\mathbf{u}, t, t_0)}{\partial t} + \sum_{j=1}^{3} i \tilde{g} P_j \lambda_j u_j - \omega_j u_j \frac{\partial \tilde{g}}{\partial u_j} + \alpha_j u_j \tilde{A}_j \delta(t - t_0) = 0$$
(12)

We look for solutions of the form

$$\tilde{g}(\mathbf{u}, t, t_0) = H(t - t_0)\mathbf{q}(t) \cdot \mathbf{u} \exp(i\mathbf{K}(t) \cdot \mathbf{u})$$

where $H(t-t_0)$ is 1 for $t \ge t_0$ and zero otherwise. Inserting this expression into (11) and using the same sort of tricks used to solve (5) we find that (12) is satisfied if

$$\dot{K}_i + P_i \lambda_i(t) - \omega_i(t) K_i = 0, \quad \text{with } K_i(t_0) = 0$$

$$\tag{13}$$

$$\dot{q}_j - \omega_j(t)q_j = 0,$$
 with $q_j(t_0) = -\alpha_j(t_0)\tilde{A}_j(t_0).$ (14)

To bring the pieces together define the general solution $M_j(t)$ as the solution to equation (13) but with the boundary condition $M_j(0) = 0$. Also define the phases $\Phi_j(t)$ so that $\omega_j(t) = \dot{\Phi}_j(t)$. With these definitions $K_m(t,t_0) = M_m(t) - M_m(t_0) \exp[\Phi_m(t) - \Phi_m(t_0)]$ and $q_j(t,t_0) = -\alpha_j(t_0)\tilde{A}_j(t_0) \exp[\Phi_j(t) - \Phi_j(t_0)]$. This yields

$$g(\mathbf{u},t) = \sum_{j=1}^{3} \int_{0}^{t} dt_0 q_j(t,t_0) u_j \exp\left(i \sum_{m=1}^{3} u_m K_m(t,t_0)\right)$$
(15)

To close the equations we note that the only unknown in $\tilde{A}_i(t)$ is

$$\hat{n}_{\mathbf{P}}(t) = \int d^3 u \frac{\partial f_0}{\partial H}(\mathbf{u}, t) g(\mathbf{u}, t)$$
(16)

$$= -\sum_{j=1}^{3} \int_{0}^{t} dt_{0} \alpha_{j}(t_{0}) \tilde{A}_{j}(t_{0}) \exp\left(\Phi_{j}(t) - \Phi_{j}(t_{0})\right) \int d^{3}u f_{0}' \left(\sum_{m=1}^{3} \alpha_{m}(t) u_{m}^{2} / 2\right) u_{j} \exp\left(i \sum_{k=1}^{3} u_{k} K_{k}(t, t_{0})\right), \quad (17)$$

$$= -\sum_{j=1}^{3} \int_{0}^{t} dt_0 \tilde{A}_j(t_0) G_j(t, t_0), \tag{18}$$

$$= \int_{0}^{t} dt_{0} \left[\frac{q^{2}}{m} \hat{n}_{\mathbf{P}}(t_{0}) + D_{I}(t_{0}) \right] \sum_{j=1}^{3} \left(\frac{4\pi i P_{j} \lambda_{j}(t_{0})}{\sum_{m=1}^{3} \lambda_{m}^{2}(t_{0}) P_{m}^{2}} G_{j}(t, t_{0}) \right)$$

$$(19)$$

where

$$G_j(t, t_0) = \alpha_j(t_0) \exp\left(\Phi_j(t) - \Phi_j(t_0)\right) \int d^3 u f_0' \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2\right) u_j \exp\left(i \sum_{k=1}^3 u_k K_k(t, t_0)\right), \tag{20}$$

$$= -\frac{i\alpha_j(t_0)K_j(t, t_0)}{\alpha_j(t)} \exp\left(\Phi_j(t) - \Phi_j(t_0)\right) \int d^3u f_0\left(\sum_{m=1}^3 \alpha_m(t)u_m^2/2\right) \exp\left(i\sum_{k=1}^3 u_k K_k(t, t_0)\right). \tag{21}$$

and

$$D_I(t) = \frac{qQ}{(2\pi)^3 m} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp\left(-i \sum_m P_m \lambda_m(t) (x_{0m} + v_{0m}t)\right). \tag{22}$$

We now have a Volterra equation of the second kind for $\hat{n}_{\mathbf{P}}$. While exact solutions look hopeless a numerical solution should be straightforward. We note that changing the integration variable in equation (21) to $z_i = \sqrt{\alpha_i} u_i$ turns it into the Fourier transform of a spherically symmetric function for which a wide range of exact solutions are available.

CONNECTION TO PREVIOUS WORK

If we set $\omega_i = 0$ then these results should reduce to those in [4]. To show this we set

$$d_i(t) = \frac{Z}{(2\pi)^3} \lambda_1(t) \lambda_2(t) \lambda_3(t) \exp\left(-i \sum_m P_m \lambda_m(t) (x_{0m} + v_{0m}t)\right),$$

$$g(\mathbf{K}(t,t_0)) = \frac{1}{n_0} \int d^3 u f_0 \left(\sum_{m=1}^3 \alpha_m(t) u_m^2 / 2 \right) \exp\left(i \sum_{k=1}^3 u_k K_k(t,t_0) \right),$$

$$R(t,t_0) = \frac{1}{\sum_{m=1}^{3} \lambda_m^2(t_0) P_m^2} \sum_{j=1}^{3} \frac{P_j \lambda_j(t_0) \alpha_j(t_0) K_j(t,t_0)}{\alpha_j(t)} \exp\left(\Phi_j(t) - \Phi_j(t_0)\right),$$

where Z is the atomic number of the ion. Now we have

$$\hat{n}_{\mathbf{P}}(t) = \frac{4\pi q^2 n_0}{m} \int_0^t dt_0 \left[\hat{n}_{\mathbf{P}}(t_0) - d_i(t_0) \right] R(t, t_0) g(\mathbf{K}(t, t_0)). \tag{23}$$

When $\omega_i = 0$, $R(t, t_0) = t_0 - t$, $\lambda_i = 1$, and $\mathbf{K}(t, t_0) = (t_0 - t)\mathbf{P}$. Make these substitutions, account for a difference in Fourier transform conventions, and include the fact that there is a constant velocity offset between reference frames. On finds that equation (23) here is equivalent to equation (8) in [4].

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- [3] M. Blaskiewicz, BNL-211960-2019-TECH, (2019)
- [4] Gang Wang and Michael Blaskiewicz, Phys. Rev. E 78, 026413 (2008).
- [5] a factor of $(2\pi)^3$ is missing in [3]