# 3D Vlasov theory of the plasma cascade instability 

M. Blaskiewicz

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## Collider Accelerator Department

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# 3D Vlasov Theory of the Plasma Cascade Instability 

M. Blaskiewicz<br>BNL 911B, Upton, NY 11973, USA


#### Abstract

The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.


The plasma cascade instabilty (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. This note shows how one can reduce the Vlasov equation to a set of Volterra equations of the second kind, which are amenable to accurate numerical solution.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates $x_{1}, x_{2}, x_{3}, t$. The unperturbed plasma has an average velocity

$$
\begin{equation*}
\mathbf{v}_{0}(\mathbf{x}, t)=\sum_{j=1}^{3} \hat{x}_{j} x_{j} \omega_{j}(t) \tag{1}
\end{equation*}
$$

This average velocity is driven by the acceleration

$$
\begin{equation*}
\mathbf{A}_{0}(\mathbf{x}, t)=\sum_{j=1}^{3} \hat{x}_{j} x_{j} k_{j}(t) \tag{2}
\end{equation*}
$$

We wish to find a distribution with these parameters. We will take an unperturbed distribution of the form $f_{0}(\mathbf{x}, \mathbf{v}, t)=$ $f_{0}(H(\mathbf{x}, \mathbf{v}, t))$ with

$$
\begin{equation*}
H=\sum_{j=1}^{3} \frac{\alpha_{j}(t)}{2}\left(v_{j}-\omega_{j} x_{j}\right)^{2} \tag{3}
\end{equation*}
$$

The Vlasov equation is

$$
\begin{equation*}
\frac{\partial f_{0}}{\partial t}+\mathbf{v} \cdot \frac{\partial f_{0}}{\partial \mathbf{x}}+\mathbf{A}_{0}(\mathbf{x}, t) \cdot \frac{\partial f_{0}}{\partial \mathbf{v}}=0 \tag{4}
\end{equation*}
$$

and since $f_{0}=f_{0}(H)$, $H$ satisfies equation (4) as well. Insert $H$ for $f_{0}$ in equation (4). The resulting terms proportional to $x_{j}$ and $v_{j}$ are

$$
\begin{equation*}
-\alpha_{j}\left(v_{j}-\omega_{j} x_{j}\right) \dot{\omega}_{j} x_{j}+\dot{\alpha}_{j}\left(v_{j}-\omega_{j} x_{j}\right)^{2} / 2-v_{j} \alpha_{j} \omega_{j}\left(v_{j}-\omega_{j} x_{j}\right)+x_{j} k_{j} \alpha_{j}\left(v_{j}-\omega_{j} x_{j}\right)=0 \tag{5}
\end{equation*}
$$

Setting the coefficients of $x_{j}^{2}, x_{j} v_{j}$ and $v_{j}^{2}$ to zero we find equation (4) is satisfied if

$$
\begin{equation*}
\dot{\alpha}_{j}=2 \alpha_{j} \omega_{j}, \quad \dot{\omega}_{j}+\omega_{j}^{2}=k_{j} \tag{6}
\end{equation*}
$$

for $j=1,2,3$. If equations (6) are satisfied then any function $f_{0}(H)$ will satisfy equation (4). For physical solutions we require $f_{0}(H) d^{3} x d^{3} v$ to be the number of electrons in the phase space volume $d^{3} x d^{3} v$.

We will use first order perturbation theory with $f=f_{0}+f_{1}$ so that

$$
\begin{equation*}
\frac{\partial f_{1}}{\partial t}+\mathbf{v} \cdot \frac{\partial f_{1}}{\partial \mathbf{x}}+\mathbf{A}_{0}(\mathbf{x}, t) \cdot \frac{\partial f_{1}}{\partial \mathbf{v}}+\mathbf{A}_{1}(\mathbf{x}, t) \cdot \frac{\partial f_{0}}{\partial \mathbf{v}}=0 \tag{7}
\end{equation*}
$$

where $\mathbf{A}_{1}$ is the acceleration created by $f_{1}$ and by the ion seeding the instability. Using the results of [3] we introduce time dependent spatial wave numbers and assume a perturbation where the spatial density of the electrons varies as

$$
\begin{equation*}
n_{\mathbf{P}}(\mathbf{x}, t)=\hat{n}_{\mathbf{P}}(t) \exp \left(i \sum_{j=1}^{3} P_{j} \lambda_{j}(t) x_{j}\right) \equiv \hat{n}_{\mathbf{P}}(t) \exp (i \Psi(\mathbf{x}, t)) \tag{8}
\end{equation*}
$$

where $\dot{\lambda}_{j}+\omega_{j} \lambda_{j}=0$ and the $P_{j}, s$ are constant in time. The total perturbed density from all wavenumbers is

$$
n_{1}(\mathbf{x}, t)=\int n_{\mathbf{P}}(\mathbf{x}, t) d^{3} P
$$

Consider an ion implanted at $t=0$ and located at $\mathbf{x}=\mathbf{x}_{0}+\mathbf{v}_{0} t$. The acceleration it generates is given by (cgs units)

$$
\begin{align*}
& \mathbf{A}_{I}(\mathbf{x}, t)=\frac{q Q}{m} \frac{\mathbf{x}-\mathbf{x}_{0}-\mathbf{v}_{0} t}{\left|\mathbf{x}-\mathbf{x}_{0}-\mathbf{v}_{0} t\right|^{3}} \\
& =-i \frac{4 \pi q Q}{m(2 \pi)^{3}} \int d^{3} P \lambda_{1} \lambda_{2} \lambda_{3} \frac{\left(P_{1} \lambda_{1}, P_{2} \lambda_{2}, P_{3} \lambda_{3}\right)}{\sum_{j} P_{j}^{2} \lambda_{j}^{2}} \exp \left(i \sum_{m} P_{m} \lambda_{m}\left(x_{m}-x_{0 m}-v_{0 m} t\right)\right) \tag{9}
\end{align*}
$$

where $q=-|e|$ is the electron charge, $m$ is its mass, and $Q$ is the charge on the ion. The net acceleration for wavenumber $\mathbf{P}$ due to both the ions and electrons is $\mathbf{A}_{\mathbf{P}}(x, t)=\tilde{\mathbf{A}}(t) \exp (i \Psi(\mathbf{x}, t))$ with[5]

$$
\begin{equation*}
\tilde{\mathbf{A}}(t)=-4 \pi i \frac{\left(P_{1} \lambda_{1}, P_{2} \lambda_{2}, P_{3} \lambda_{3}\right)}{\sum_{m} P_{m}^{2} \lambda_{m}^{2}}\left\{\frac{q^{2}}{m} \hat{n}_{\mathbf{P}}(t)+\frac{q Q}{(2 \pi)^{3} m} \lambda_{1} \lambda_{2} \lambda_{3} \exp \left(-i \sum_{m} P_{m} \lambda_{m}\left(x_{0 m}+v_{0 m} t\right)\right)\right\} \tag{10}
\end{equation*}
$$

To solve the Vlasov equation we consider a single P. Consider the Ansantz

$$
f_{\mathbf{P}}(\mathbf{x}, \mathbf{v}, t)=\frac{d f_{0}}{d H} g\left(v_{1}-\omega_{1} x_{1}, v_{2}-\omega_{2} x_{2}, v_{3}-\omega_{3} x_{3}, t\right) \exp (i \Psi(\mathbf{x}, t))
$$

Notice that the $x_{j}$ dependence in $g$ and $f_{0}$ only shows up as $v_{j}-\omega_{j} x_{j}$ so it drops out after integrating over $v_{j}$. This generates the correct spatial dependence for $n_{\mathbf{P}}(\mathbf{x}, t)$. For convenient notation define $u_{j}=v_{j}-\omega_{j} x_{j}$ and remember that

$$
\frac{\partial g(\mathbf{x}, \mathbf{v}, t)}{\partial t}=\frac{\partial g(\mathbf{u}, t)}{\partial t}+\frac{\partial g(\mathbf{u}, t)}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}(\mathbf{x}, \mathbf{v}, t)}{\partial t}
$$

Plugging into the Vlasov eq one finds

$$
\begin{equation*}
\frac{\partial g(\mathbf{u}, t)}{\partial t}+\sum_{j=1}^{3} i g P_{j} \lambda_{j} u_{j}-\omega_{j} u_{j} \frac{\partial g}{\partial u_{j}}+\alpha_{j} u_{j} \tilde{A}_{j}=0 \tag{11}
\end{equation*}
$$

To proceed we multiply the last term on the right of equation (11) by $\delta\left(t-t_{0}\right)$ with the intention of integrating over $t_{0}$ later.

$$
\begin{equation*}
\frac{\partial \tilde{g}\left(\mathbf{u}, t, t_{0}\right)}{\partial t}+\sum_{j=1}^{3} i \tilde{g} P_{j} \lambda_{j} u_{j}-\omega_{j} u_{j} \frac{\partial \tilde{g}}{\partial u_{j}}+\alpha_{j} u_{j} \tilde{A}_{j} \delta\left(t-t_{0}\right)=0 \tag{12}
\end{equation*}
$$

We look for solutions of the form

$$
\tilde{g}\left(\mathbf{u}, t, t_{0}\right)=H\left(t-t_{0}\right) \mathbf{q}(t) \cdot \mathbf{u} \exp (i \mathbf{K}(t) \cdot \mathbf{u})
$$

where $H\left(t-t_{0}\right)$ is 1 for $t \geq t_{0}$ and zero otherwise. Inserting this expression into (11) and using the same sort of tricks used to solve (5) we find that (12) is satisfied if

$$
\begin{array}{ll}
\dot{K}_{j}+P_{j} \lambda_{j}(t)-\omega_{j}(t) K_{j}=0, & \text { with } K_{j}\left(t_{0}\right)=0 \\
\dot{q}_{j}-\omega_{j}(t) q_{j}=0, & \text { with } q_{j}\left(t_{0}\right)=-\alpha_{j}\left(t_{0}\right) \tilde{A}_{j}\left(t_{0}\right) . \tag{14}
\end{array}
$$

To bring the pieces together define the general solution $M_{j}(t)$ as the solution to equation (13) but with the boundary condition $M_{j}(0)=0$. Also define the phases $\Phi_{j}(t)$ so that $\omega_{j}(t)=\dot{\Phi}_{j}(t)$. With these definitions $K_{m}\left(t, t_{0}\right)=$ $M_{m}(t)-M_{m}\left(t_{0}\right) \exp \left[\Phi_{m}(t)-\Phi_{m}\left(t_{0}\right)\right]$ and $q_{j}\left(t, t_{0}\right)=-\alpha_{j}\left(t_{0}\right) \tilde{A}_{j}\left(t_{0}\right) \exp \left[\Phi_{j}(t)-\Phi_{j}\left(t_{0}\right)\right]$. This yields

$$
\begin{equation*}
g(\mathbf{u}, t)=\sum_{j=1}^{3} \int_{0}^{t} d t_{0} q_{j}\left(t, t_{0}\right) u_{j} \exp \left(i \sum_{m=1}^{3} u_{m} K_{m}\left(t, t_{0}\right)\right) \tag{15}
\end{equation*}
$$

To close the equations we note that the only unknown in $\tilde{A}_{j}(t)$ is

$$
\begin{align*}
& \hat{n}_{\mathbf{P}}(t)=\int d^{3} u \frac{\partial f_{0}}{\partial H}(\mathbf{u}, t) g(\mathbf{u}, t)  \tag{16}\\
& =-\sum_{j=1}^{3} \int_{0}^{t} d t_{0} \alpha_{j}\left(t_{0}\right) \tilde{A}_{j}\left(t_{0}\right) \exp \left(\Phi_{j}(t)-\Phi_{j}\left(t_{0}\right)\right) \int d^{3} u f_{0}^{\prime}\left(\sum_{m=1}^{3} \alpha_{m}(t) u_{m}^{2} / 2\right) u_{j} \exp \left(i \sum_{k=1}^{3} u_{k} K_{k}\left(t, t_{0}\right)\right)  \tag{17}\\
& =-\sum_{j=1}^{3} \int_{0}^{t} d t_{0} \tilde{A}_{j}\left(t_{0}\right) G_{j}(t, t 0)  \tag{18}\\
& =\int_{0}^{t} d t_{0}\left[\frac{q^{2}}{m} \hat{n}_{\mathbf{P}}\left(t_{0}\right)+D_{I}\left(t_{0}\right)\right] \sum_{j=1}^{3}\left(\frac{4 \pi i P_{j} \lambda_{j}\left(t_{0}\right)}{\sum_{m=1}^{3} \lambda_{m}^{2}\left(t_{0}\right) P_{m}^{2}} G_{j}\left(t, t_{0}\right)\right) \tag{19}
\end{align*}
$$

where

$$
\begin{align*}
& G_{j}\left(t, t_{0}\right)=\alpha_{j}\left(t_{0}\right) \exp \left(\Phi_{j}(t)-\Phi_{j}\left(t_{0}\right)\right) \int d^{3} u f_{0}^{\prime}\left(\sum_{m=1}^{3} \alpha_{m}(t) u_{m}^{2} / 2\right) u_{j} \exp \left(i \sum_{k=1}^{3} u_{k} K_{k}\left(t, t_{0}\right)\right)  \tag{20}\\
& =-\frac{i \alpha_{j}\left(t_{0}\right) K_{j}\left(t, t_{0}\right)}{\alpha_{j}(t)} \exp \left(\Phi_{j}(t)-\Phi_{j}\left(t_{0}\right)\right) \int d^{3} u f_{0}\left(\sum_{m=1}^{3} \alpha_{m}(t) u_{m}^{2} / 2\right) \exp \left(i \sum_{k=1}^{3} u_{k} K_{k}\left(t, t_{0}\right)\right) \tag{21}
\end{align*}
$$

and

$$
\begin{equation*}
D_{I}(t)=\frac{q Q}{(2 \pi)^{3} m} \lambda_{1}(t) \lambda_{2}(t) \lambda_{3}(t) \exp \left(-i \sum_{m} P_{m} \lambda_{m}(t)\left(x_{0 m}+v_{0 m} t\right)\right) \tag{22}
\end{equation*}
$$

We now have a Volterra equation of the second kind for $\hat{n}_{\mathbf{P}}$. While exact solutions look hopeless a numerical solution should be straightforward. We note that changing the integration variable in equation (21) to $z_{i}=\sqrt{\alpha_{i}} u_{i}$ turns it into the Fourier transform of a spherically symmetric function for which a wide range of exact solutions are available.

## CONNECTION TO PREVIOUS WORK

If we set $\omega_{i}=0$ then these results should reduce to those in [4]. To show this we set

$$
\begin{gathered}
d_{i}(t)=\frac{Z}{(2 \pi)^{3}} \lambda_{1}(t) \lambda_{2}(t) \lambda_{3}(t) \exp \left(-i \sum_{m} P_{m} \lambda_{m}(t)\left(x_{0 m}+v_{0 m} t\right)\right) \\
g\left(\mathbf{K}\left(t, t_{0}\right)\right)=\frac{1}{n_{0}} \int d^{3} u f_{0}\left(\sum_{m=1}^{3} \alpha_{m}(t) u_{m}^{2} / 2\right) \exp \left(i \sum_{k=1}^{3} u_{k} K_{k}\left(t, t_{0}\right)\right) \\
R\left(t, t_{0}\right)=\frac{1}{\sum_{m=1}^{3} \lambda_{m}^{2}\left(t_{0}\right) P_{m}^{2}} \sum_{j=1}^{3} \frac{P_{j} \lambda_{j}\left(t_{0}\right) \alpha_{j}\left(t_{0}\right) K_{j}\left(t, t_{0}\right)}{\alpha_{j}(t)} \exp \left(\Phi_{j}(t)-\Phi_{j}\left(t_{0}\right)\right)
\end{gathered}
$$

where $Z$ is the atomic number of the ion. Now we have

$$
\begin{equation*}
\hat{n}_{\mathbf{P}}(t)=\frac{4 \pi q^{2} n_{0}}{m} \int_{0}^{t} d t_{0}\left[\hat{n}_{\mathbf{P}}\left(t_{0}\right)-d_{i}\left(t_{0}\right)\right] R\left(t, t_{0}\right) g\left(\mathbf{K}\left(t, t_{0}\right)\right) \tag{23}
\end{equation*}
$$

When $\omega_{i}=0, R\left(t, t_{0}\right)=t_{0}-t, \lambda_{i}=1$, and $\mathbf{K}\left(t, t_{0}\right)=\left(t_{0}-t\right) \mathbf{P}$. Make these substitutions, account for a difference in Fourier transform conventions, and include the fact that there is a constant velocity offset between reference frames. On finds that equation (23) here is equivalent to equation (8) in [4].
[1] V. N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, arxiv:1802.08677, February 2018
[2] V. N. Litvinenko, G. Wang, Y. Jing, D. Kayran, J. Ma, I. Petrushina,I. Pinayev, K. Shih, arxiv:1902.10846, February 2019
[3] M. Blaskiewicz, BNL-211960-2019-TECH, (2019)
[4] Gang Wang and Michael Blaskiewicz, Phys. Rev. E 78, 026413 (2008).
[5] a factor of $(2 \pi)^{3}$ is missing in [3]


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