

Preservation of the distribution of beam particles with respect to longitudinal oscillation amplitude in a 3 to 1 bunch merge

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September 2019

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U.S. Department of Energy

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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Preservation of the distribution of beam particles with respect to longitudinal oscillation amplitude in a 3 to 1 bunch merge

C.J. Gardner

September 3, 2019

In the work presented here we show in detail how the 3 to 1 bunch merge can be set up so that it preserves the distribution of beam particles with respect to longitudinal oscillation amplitude. This is an elaboration based on the work presented in [1]. The desired merge is achieved by slowly reducing the area inside the three-lobed separatrix that contains the beam while keeping the areas of the three lobes equal to one another. A sequence of RF voltages that does the job is found by employing a numerical integration routine and a search algorithm. The integration routine calculates the areas of the three lobes for given voltages. This is used by the search algorithm to find voltages that give three lobes of equal area. The resulting voltages are listed in **Section 12** in order of decreasing lobe area. These may be multiplied by a common scaling factor to make the scheme work for any given longitudinal emittance in any given machine. The analytic basis of the scheme is presented in **Sections 2** through **13**. (These sections may be treated as appendices.) Demonstration of the merge is done by simulation. The results are shown in **Figures 1** through **39** and are discussed in **Section 1**.

References [1] through [10] and those cited therein provide a history of bunch merging (and splitting) relevant to the present work. Refs. [1] and [2] record the early work done at CERN. Refs. [3] through [10] record the setup and study of the 2 to 1 and 3 to 1 merges used in Booster and AGS. In particular, Refs. [9] and [10] record the recent work of K. Zeno who proposed and developed the 3 to 1 bunch merge in Booster. It is this merge that is studied in the present work.

1 Simulation setup and results

The simulation is carried out following the equations and method presented in [6]. The relevant machine parameters are the radius

$$R = 201.780/(2\pi) \text{ m} \quad (1)$$

and transition gamma

$$\gamma_t = 4.832. \quad (2)$$

The simulation is done with Au31+ ions that have revolution frequency

$$f = 400.000 \text{ KHz} \quad (3)$$

at the nominal radius. This gives

$$\gamma = 1.0383386216559 \quad (4)$$

and

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} = -0.884687414860. \quad (5)$$

The ion mass-energy equivalent is

$$mc^2 = 183.457368352 \text{ GeV}. \quad (6)$$

In practice the merge done in Booster is a 6 to 1 merge. It is carried out by doing a 6 to 3 followed by a 3 to 1 merge. This is also done in the simulation where the 6 to 3 merge provides the three bunches to be used in the 3 to 1 merge.

The RF voltage program used in the simulation is shown in **Figure 1**. The tabulated and plotted voltages come from Section 12. The six initial bunches are shown in **Figure 2**. Here each particle in each bunch is assigned a color according to its longitudinal oscillation amplitude. This assignment stays with the particle throughout the simulation. The total longitudinal emittance of the six bunches to be merged is 0.10 eV-s per nucleon. The horizontal axis gives the particle phase in degrees.

Figures 2 through **4** show the 6 to 3 merge. They show clearly the preservation of particle distribution with respect to longitudinal oscillation amplitude. Although not shown explicitly, there are inner two-lobed separatrices centered on unstable fixed points at -120 , 0 , and 120 degrees. These all have the same area, and this guarantees that as the lobe area is

reduced, the distribution of particles (with respect to longitudinal oscillation amplitude) will be preserved if the merge proceeds sufficiently slowly.

Figures 4 through **14** show the essential 3 to 1 merge. They show explicitly the progressive shrinking of the area enclosed by the three-lobed separatrix while keeping the area of each outer lobe equal to that of the central lobe. This maintains the distribution of particles with respect to longitudinal oscillation amplitude as shown. The time required is $480 - 160 = 320$ ms, which is much longer than the time available in practice (some tens of ms).

In **Figures 14** through **16** the harmonic 3 and 2 voltages are slowly reduced to zero while the harmonic 1 voltage is held constant. The result is the final merged bunch sitting in a harmonic 1 bucket. **Figure 17** shows a black curve that is matched to the bucket and encloses the merged bunch. Its area is 1.017 times that of the initial six bunches, which shows that there is very little growth of the gross emittance.

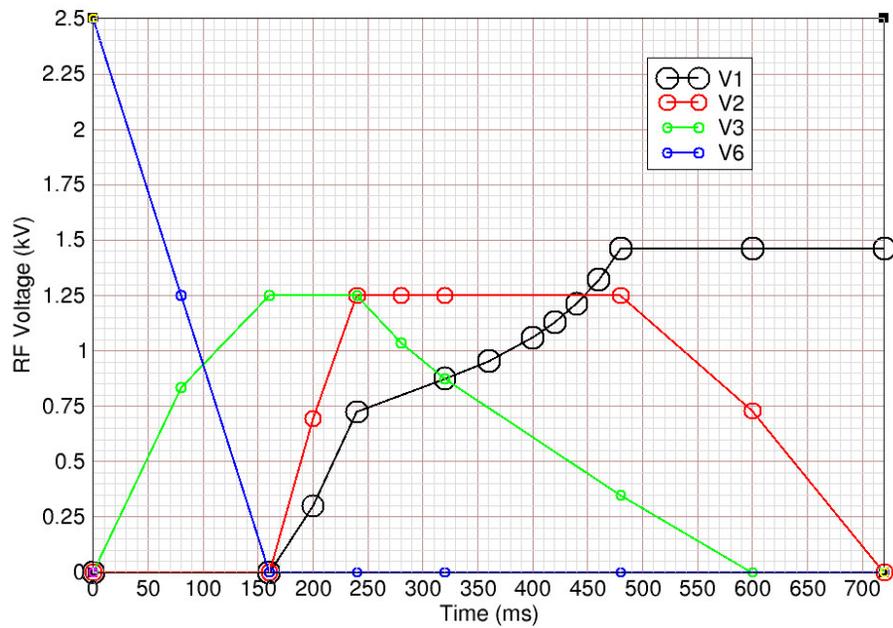
Figure 18 shows the evolution of the three-lobed separatrices during the 3 to 1 merge. **Figures 19** through **22** show the corresponding evolution of the separatrix fixed points, enclosed area, and potential wells.

In **Figures 23** through **37** the total 6 to 1 merge time has been reduced by a factor of 20 to just 36 ms. This is much closer to what is done in practice. The figures show vividly the consequences of merging too quickly. There is significant filamentation and mixing of the longitudinal oscillation amplitude layers. As layers with differing particle densities are mixed, the distribution develops lumps.

In **Figures 35** through **37** the harmonic 3 and 2 voltages are (as before) reduced to zero while the harmonic 1 voltage is held constant. The result is the final merged bunch sitting in a harmonic 1 bucket. **Figure 38** shows a black curve that is matched to the bucket and encloses the merged bunch. Its area is 1.361 times that of the initial six bunches. This is to be compared with **Figure 39** which is a copy of **Figure 17** and shows the bunch resulting from the much longer 720 ms merge. This again illustrates the consequences of merging too quickly.

Time (ms)	V6 (kV)	V3 (kV)	V2 (kV)	V1 (kV)
0	2.50	0	0	0
80	1.25	0.8335	0	0
160	0	1.25	0	0
200	0	1.25	0.695	0.3001
240	0	1.25	1.25	0.724807
280	0	1.03455	1.25	0.799380
320	0	0.873952	1.25	0.873952
360	0	0.742964	1.25	0.953600
400	0	0.611976	1.25	1.06050
420	0	0.546482	1.25	1.12962
440	0	0.480988	1.25	1.21420
460	0	0.415494	1.25	1.32080
480	0	0.35	1.25	1.46045
600	0	0	0.730225	1.46045
720	0	0	0	1.46045

Figure 1: Voltage program for RF harmonics 6, 3, 2, 1.



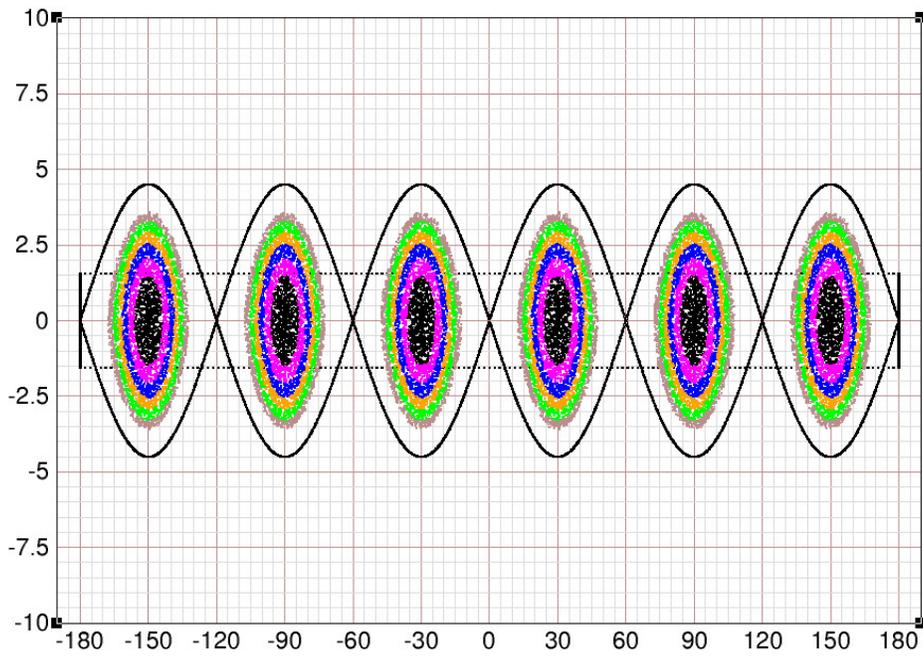
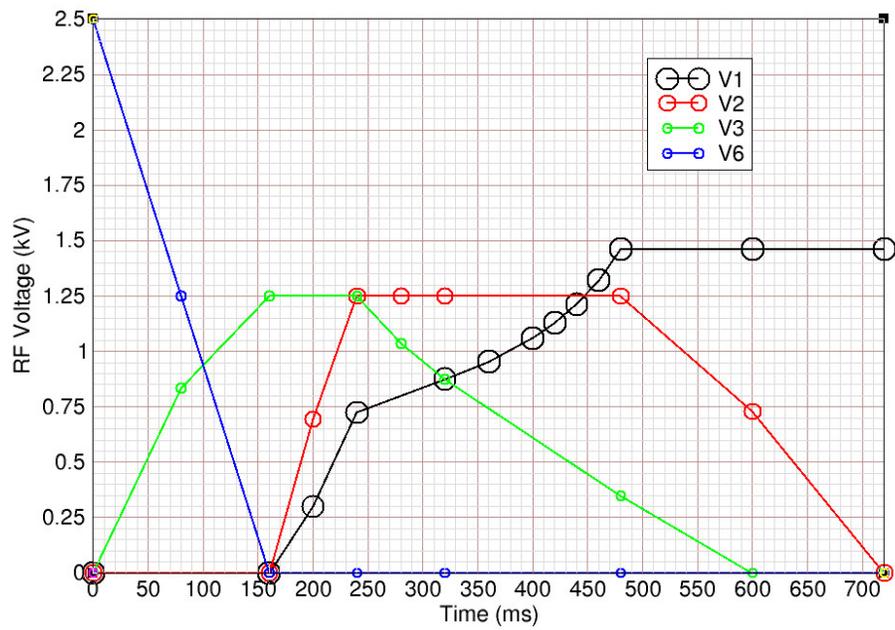


Figure 2: $(V_6, V_3) = (2.50, 0)$ kV. Time = 0 ms.



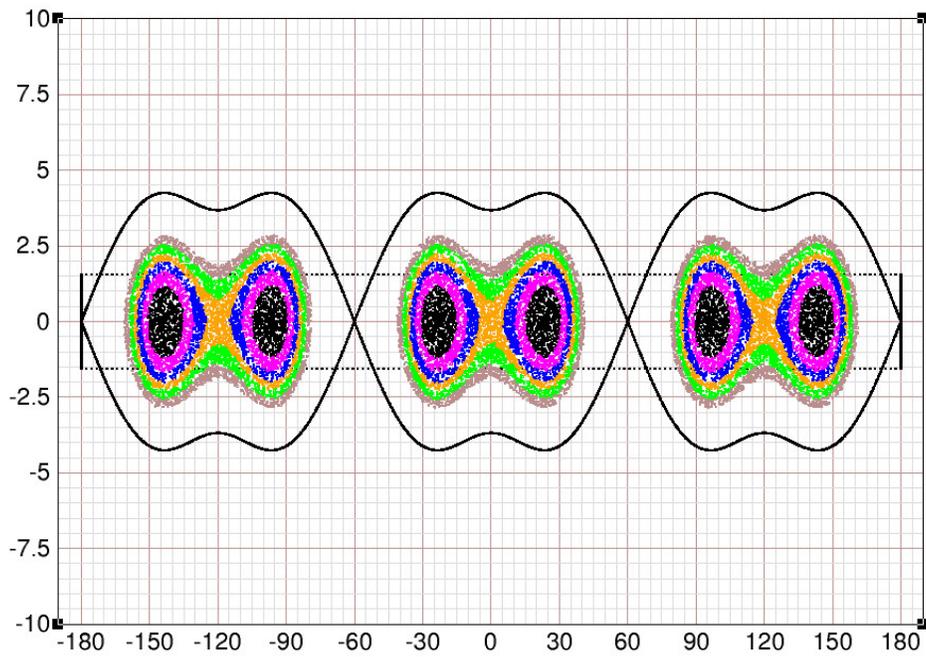
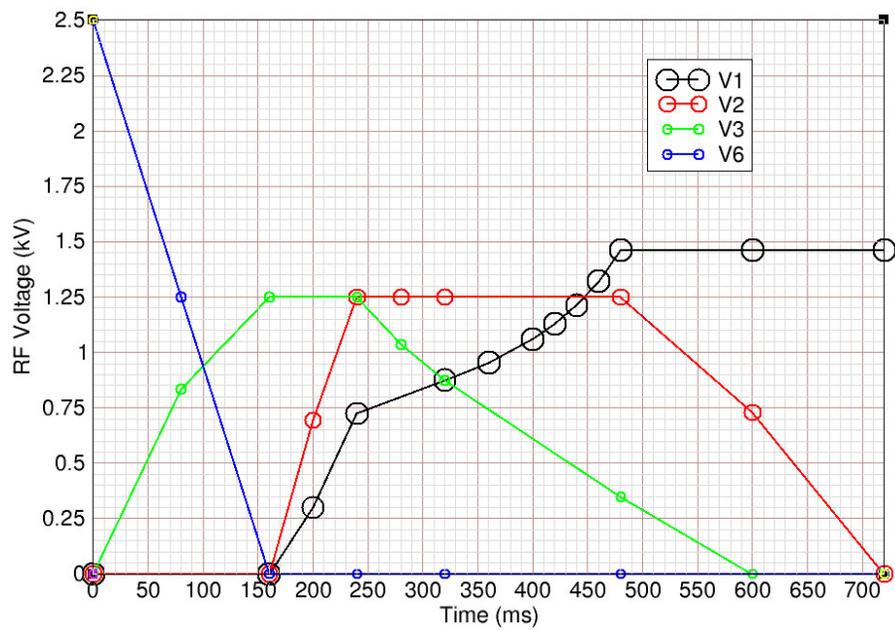


Figure 3: $(V6, V3) = (1.25, 0.8335)$ kV. Time = 80 ms.



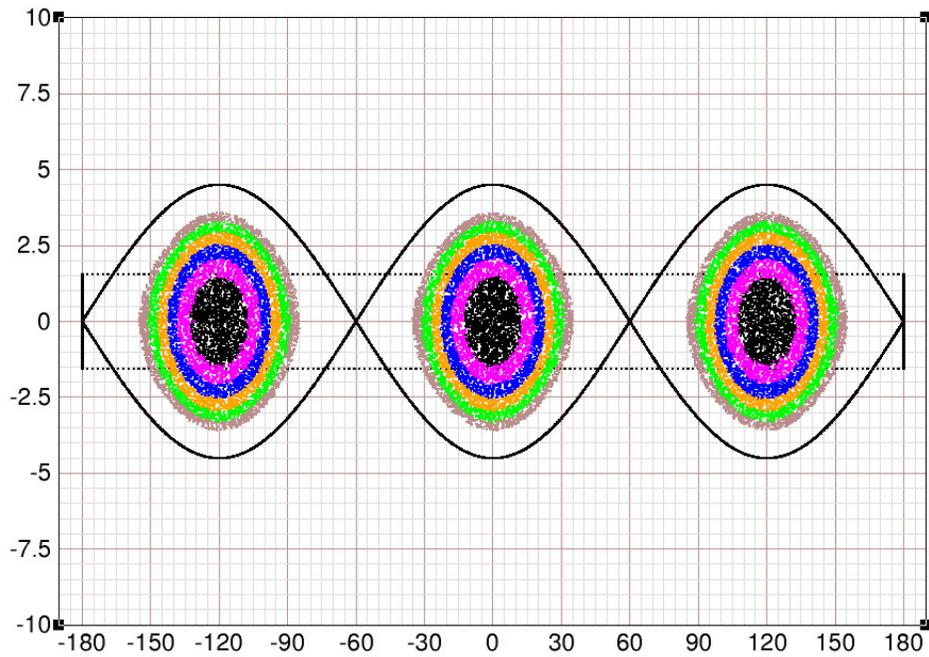
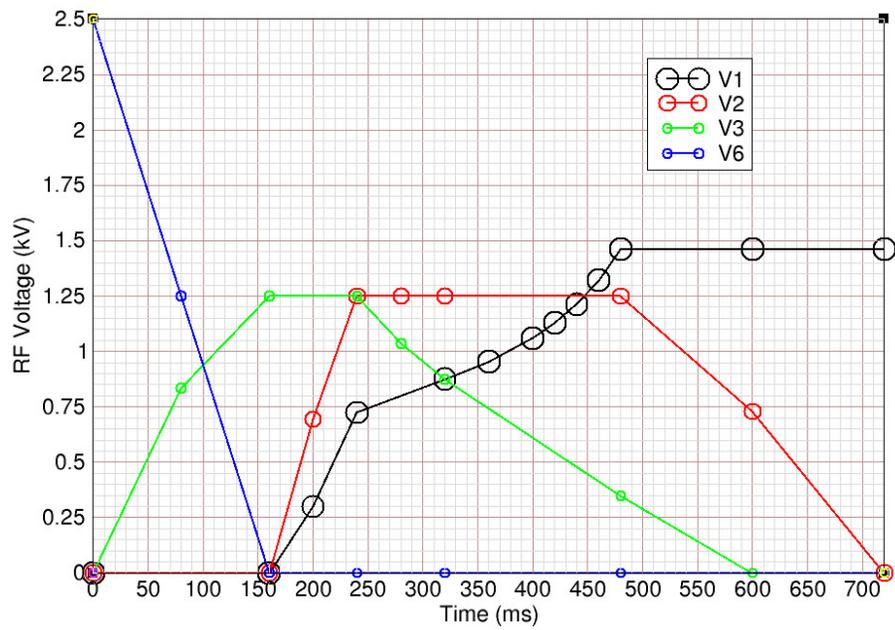


Figure 4: $(V6, V3) = (0, 1.25)$ kV. Time = 160 ms.



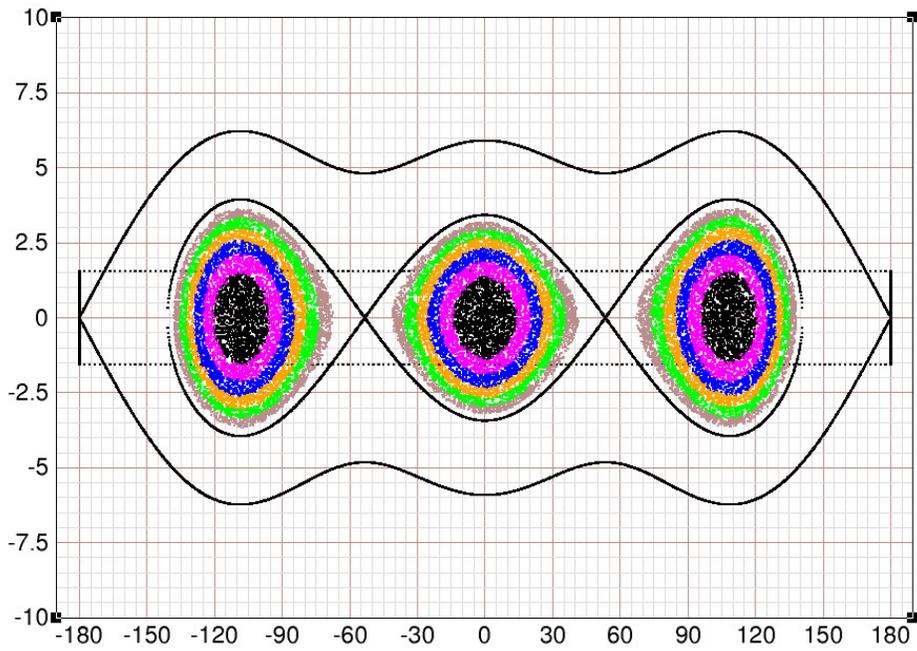
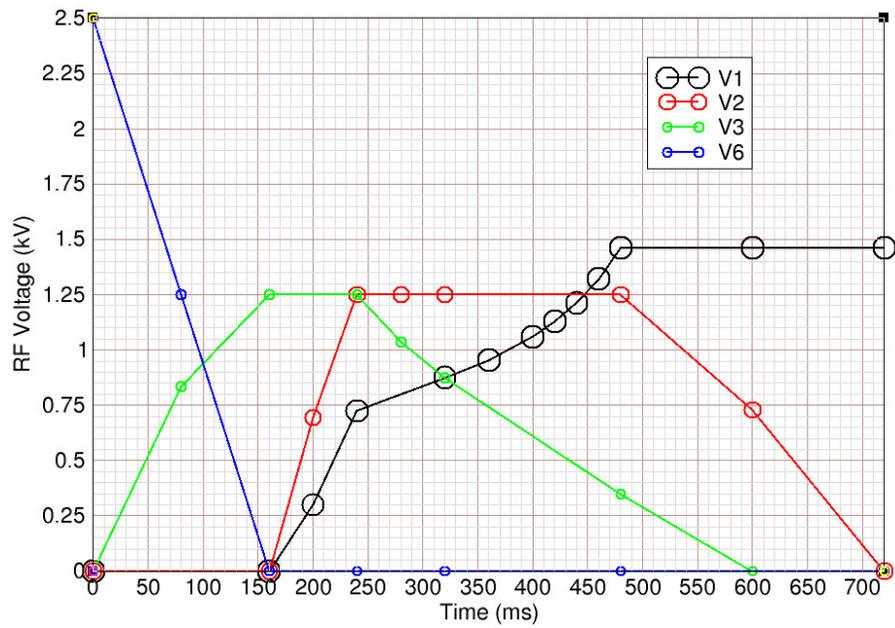


Figure 5: $(V_3, V_2, V_1) = (1.25, 0.695, 0.3001)$ kV. Time = 200 ms.



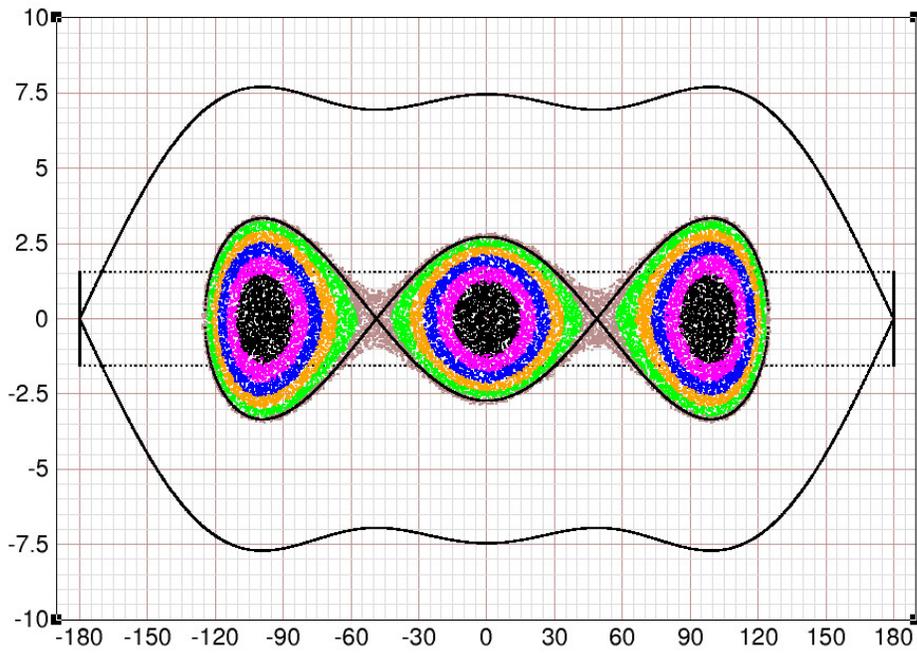
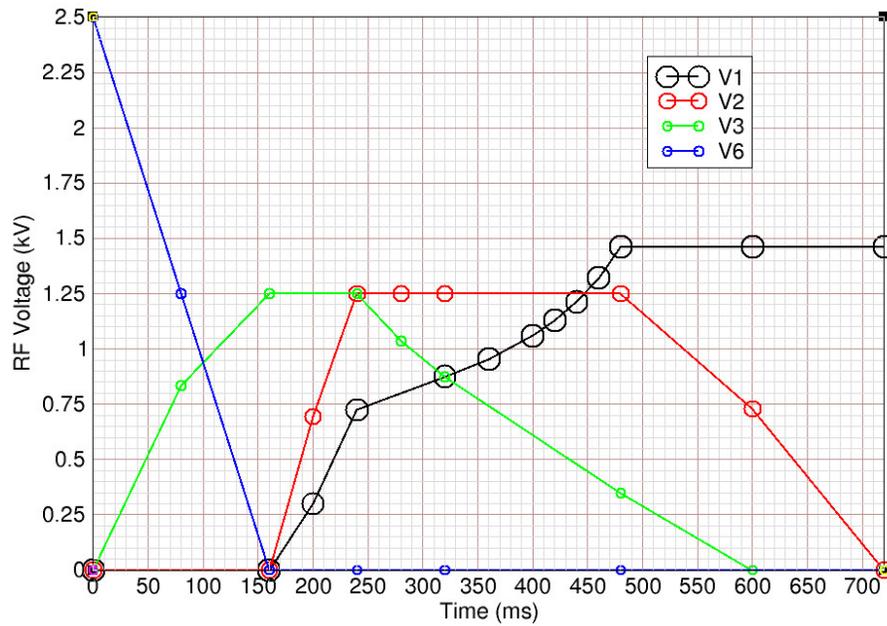


Figure 6: $(V_3, V_2, V_1) = (1.25, 1.25, 0.724807)$ kV. Time = 240 ms.



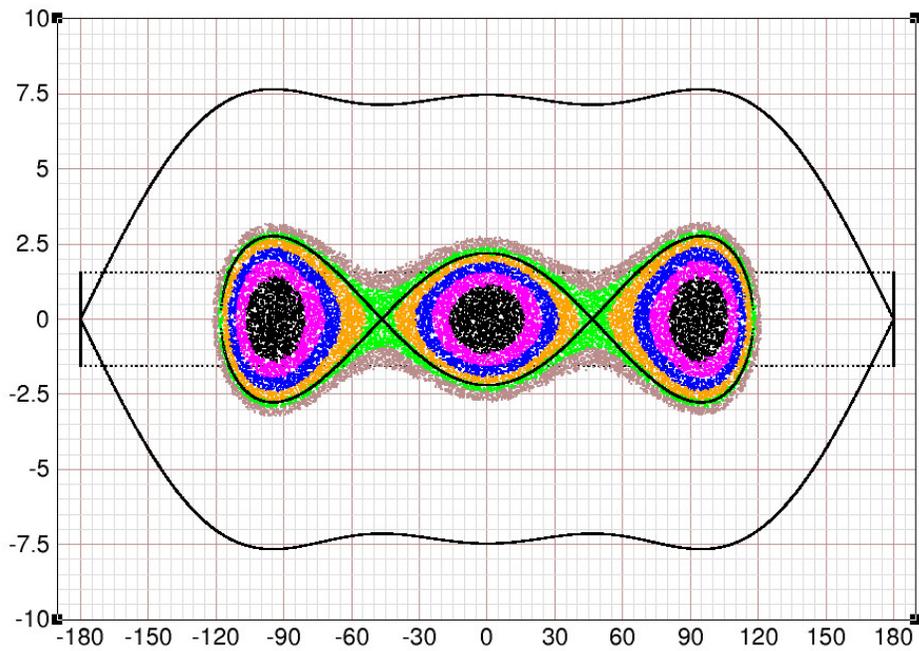
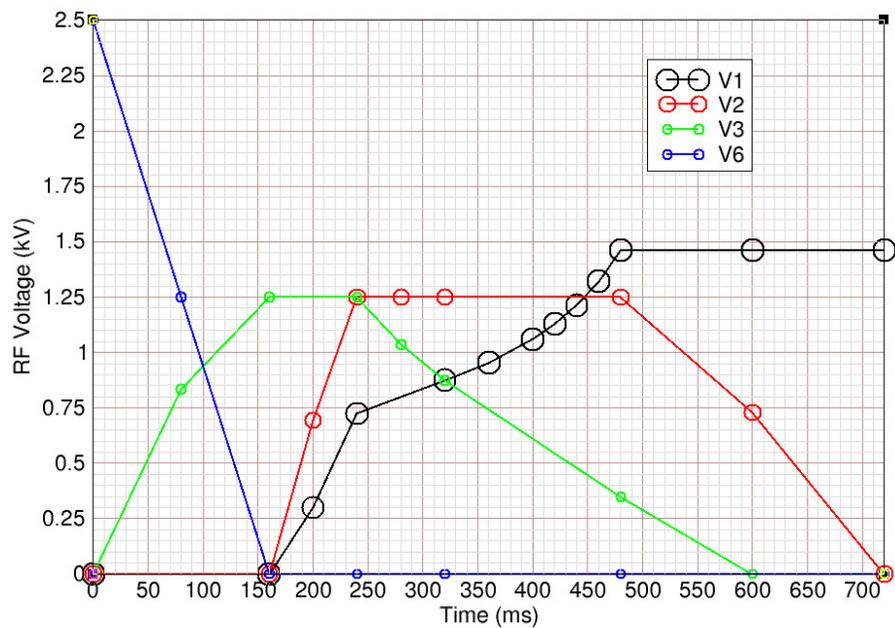


Figure 7: $(V_3, V_2, V_1) = (1.03455, 1.25, 0.799380)$ kV. Time = 280 ms.



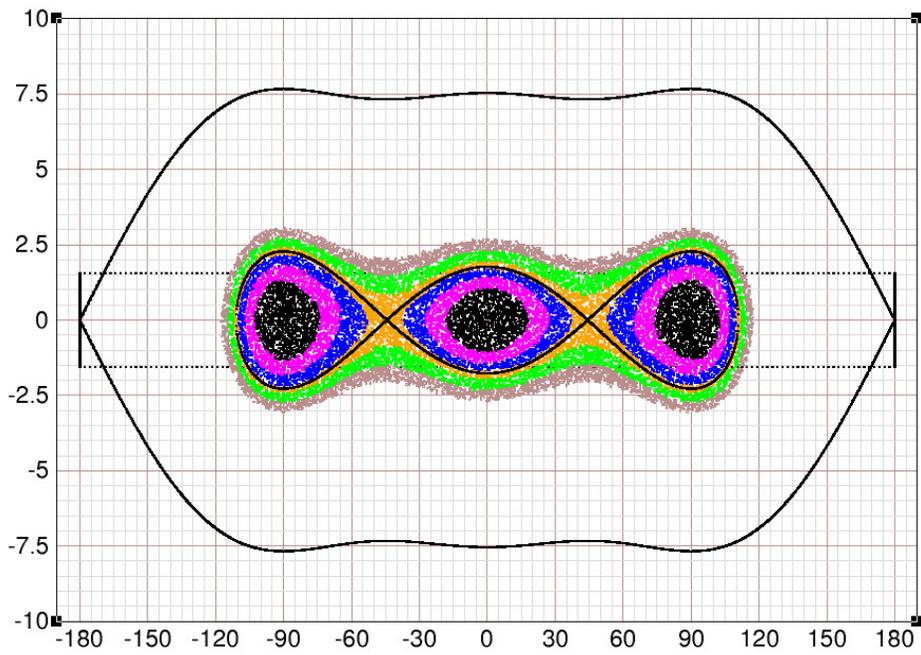
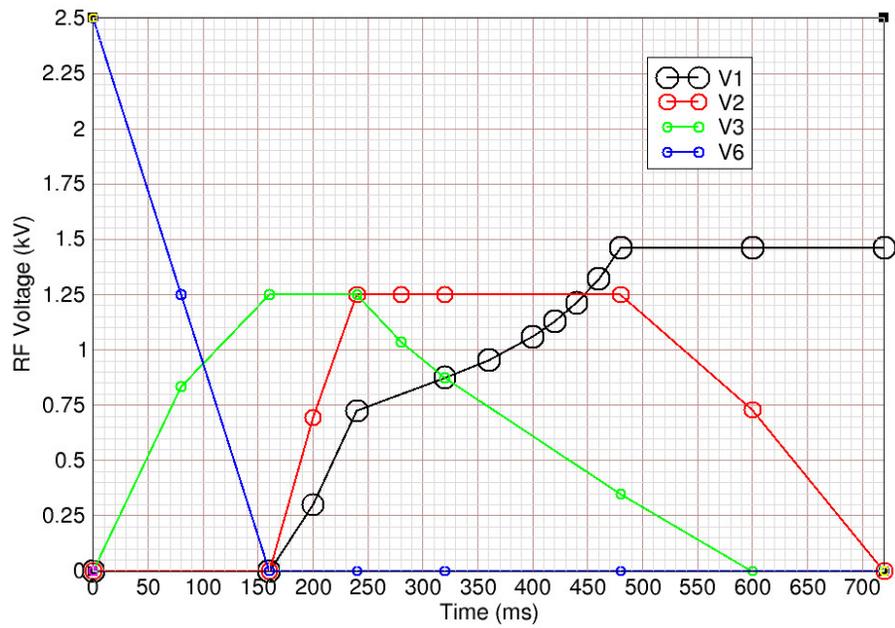


Figure 8: $(V_3, V_2, V_1) = (0.873952, 1.25, 0.873952)$ kV. Time = 320 ms.



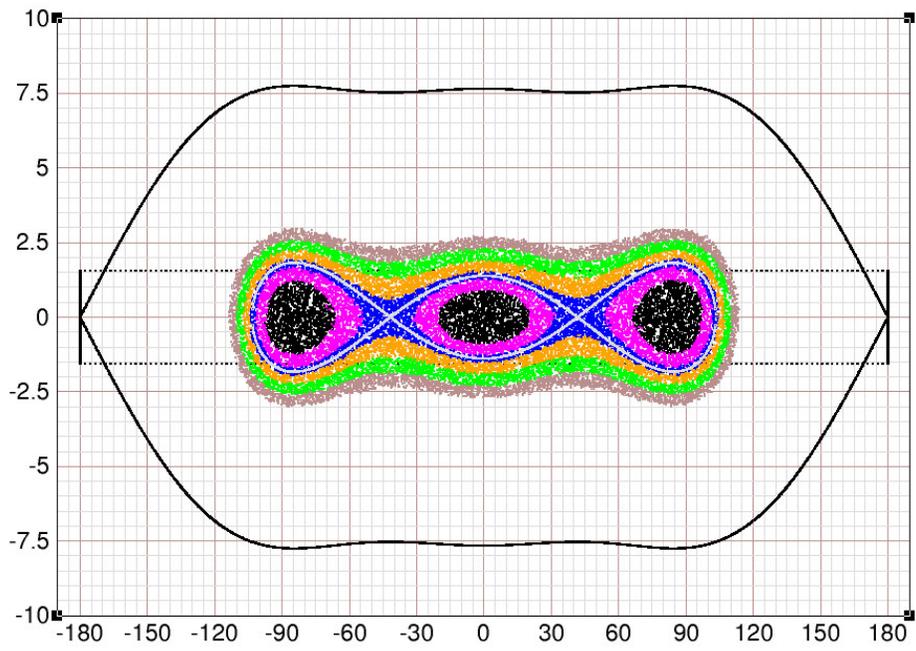
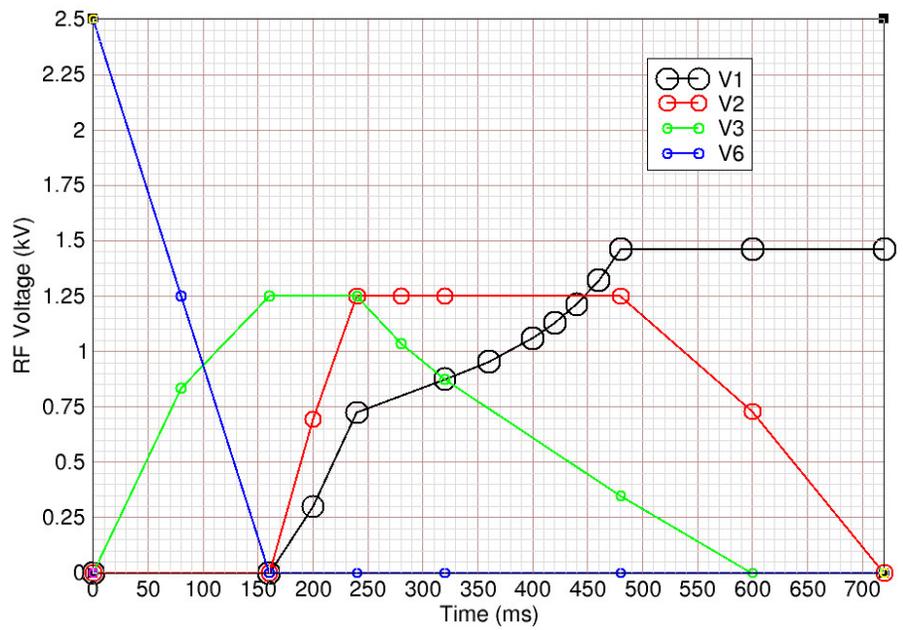


Figure 9: $(V3, V2, V1) = (0.742964, 1.25, 0.953600)$ kV. Time = 360 ms.



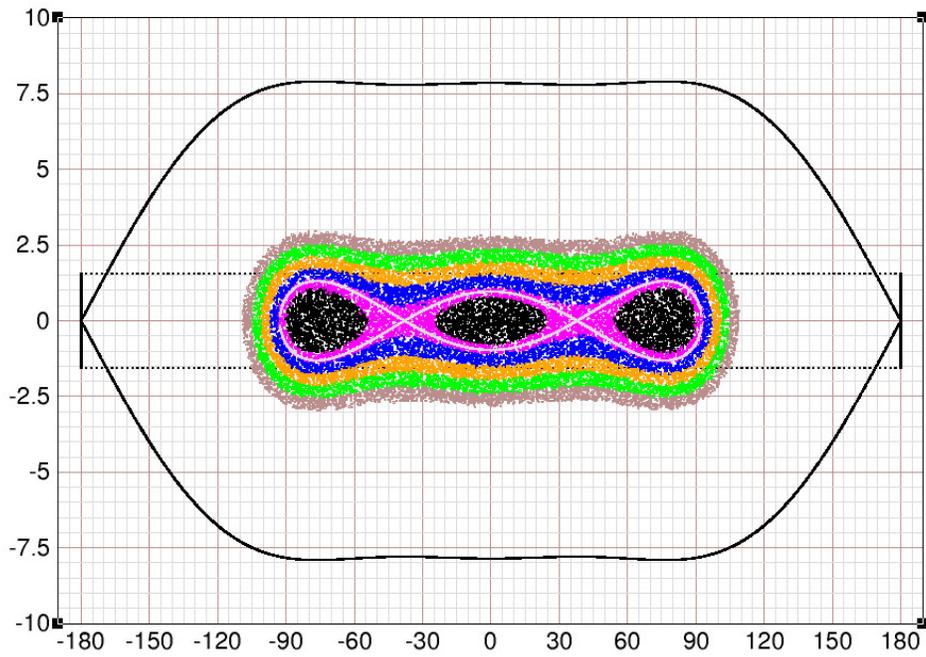
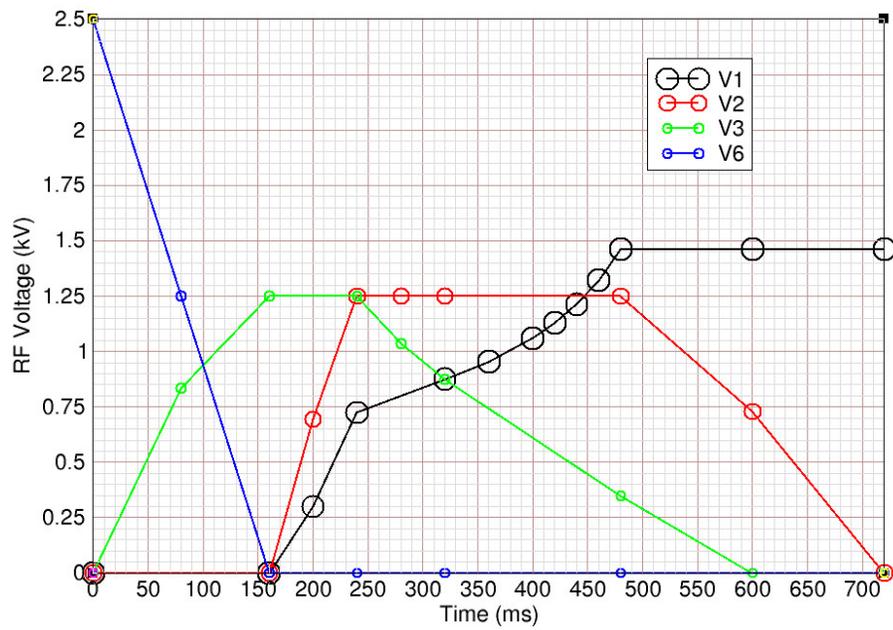


Figure 10: $(V3, V2, V1) = (0.611976, 1.25, 1.06050)$ kV. Time = 400 ms.



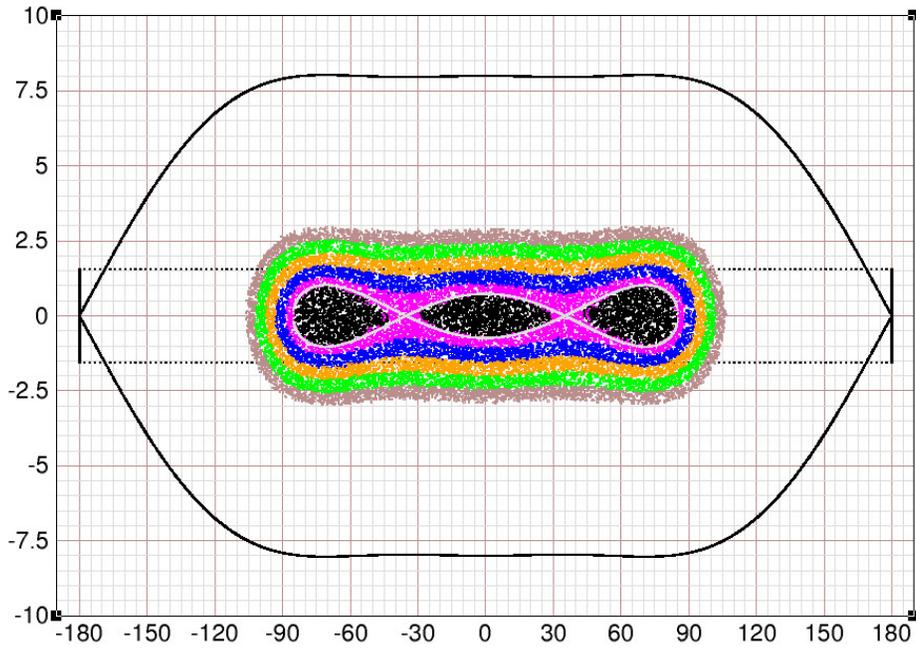
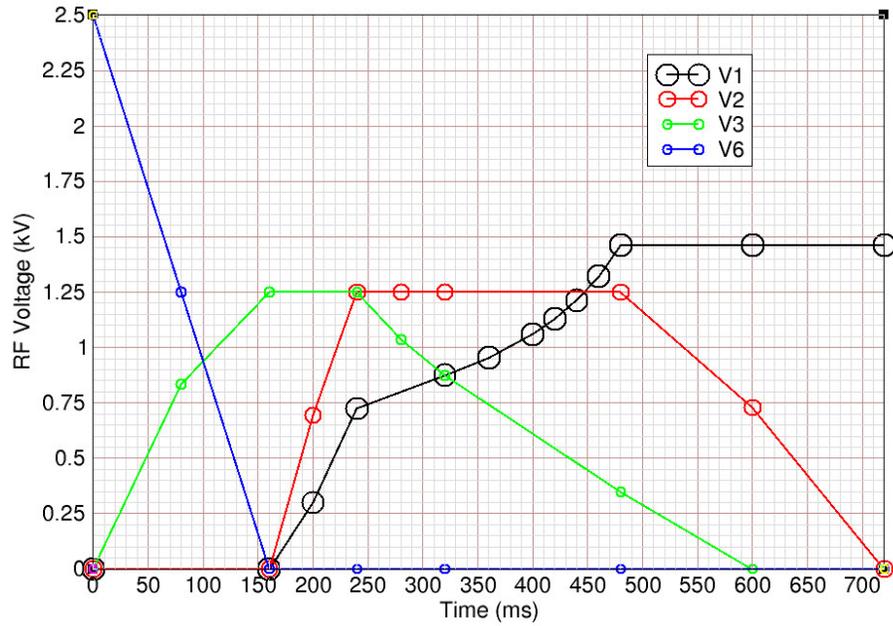


Figure 11: $(V3, V2, V1) = (0.546482, 1.25, 1.12962)$ kV. Time = 420 ms.



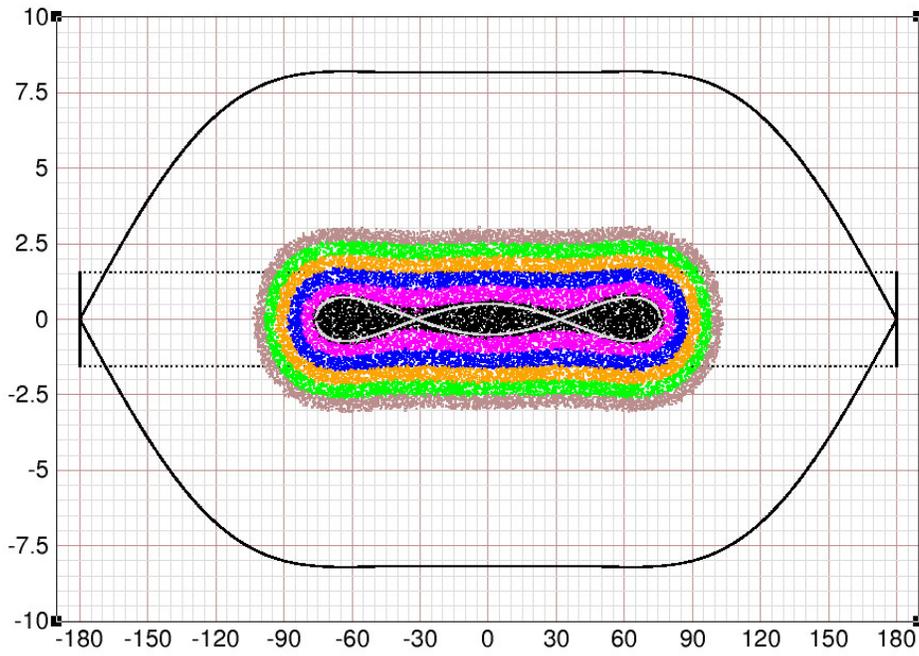
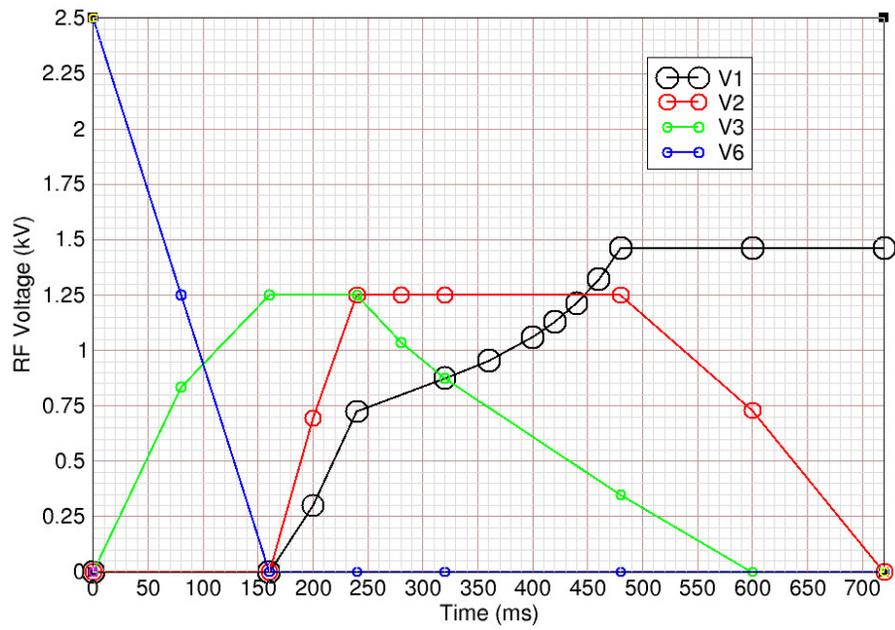


Figure 12: $(V_3, V_2, V_1) = (0.480988, 1.25, 1.21420)$ kV. Time = 440 ms.



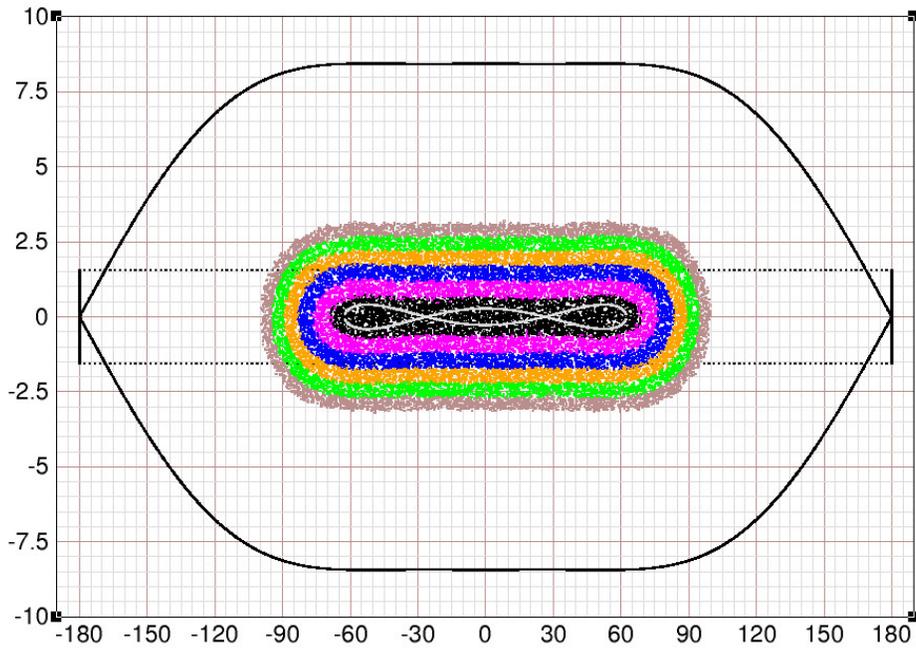
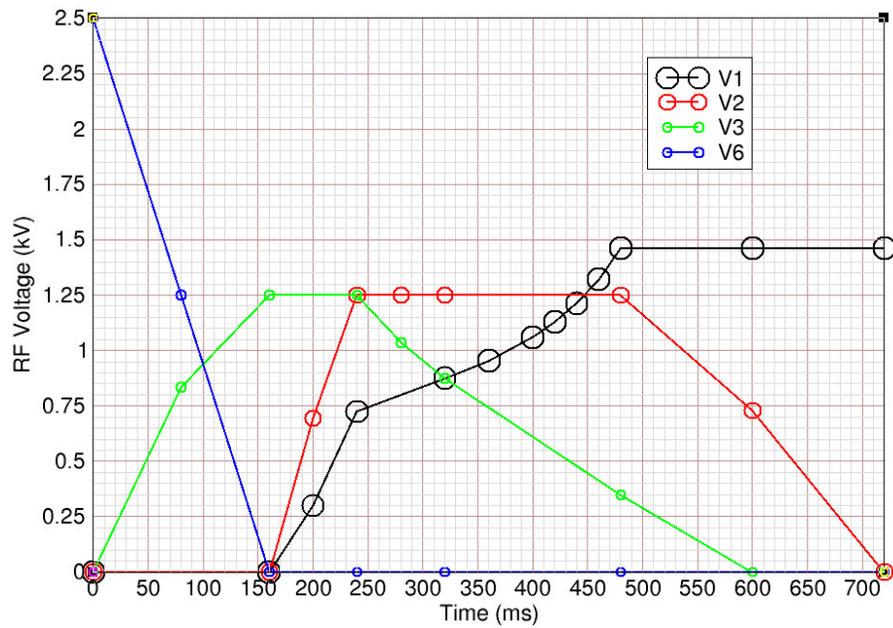


Figure 13: $(V3, V2, V1) = (0.415494, 1.25, 1.32080)$ kV. Time = 460 ms.



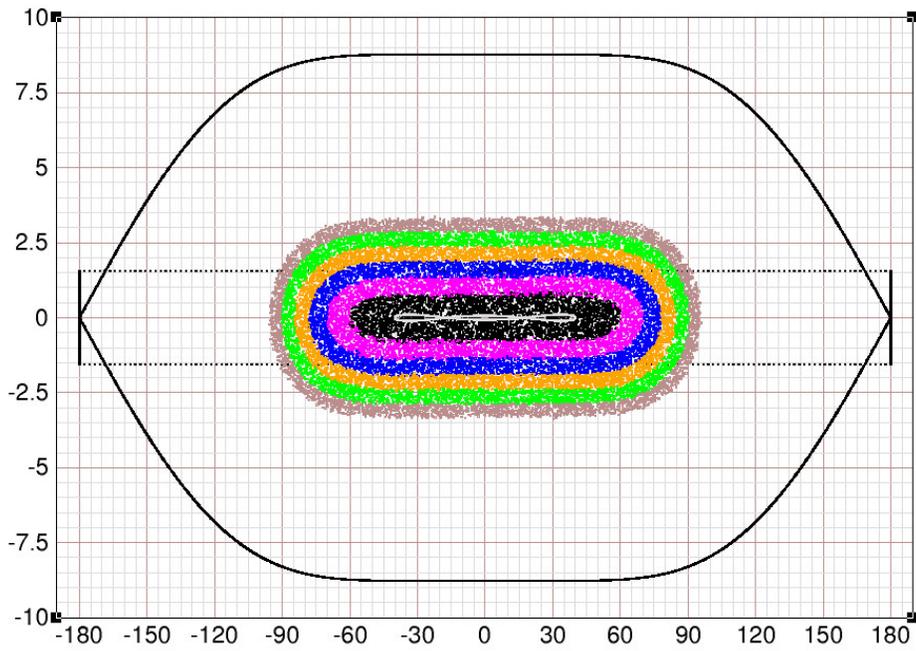
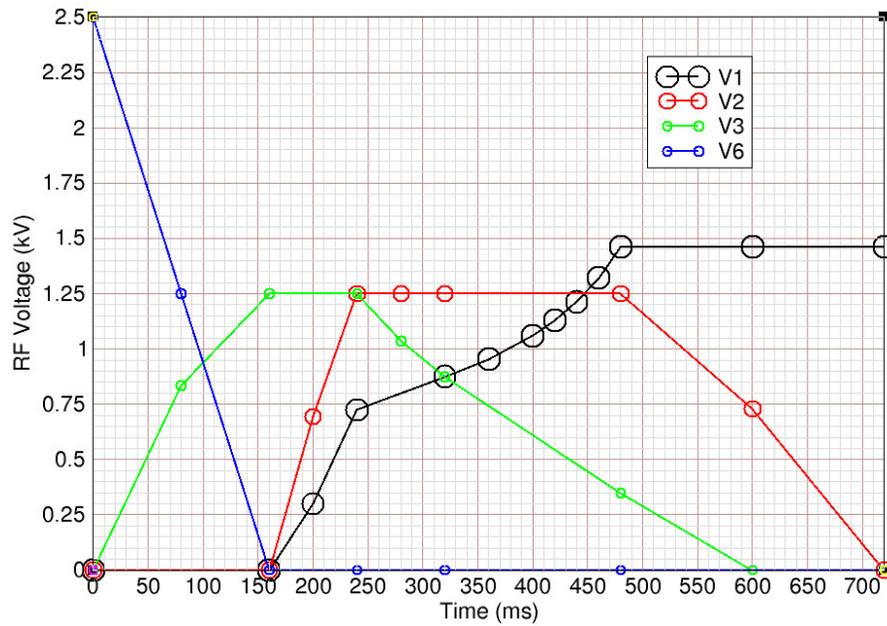


Figure 14: $(V3, V2, V1) = (0.35, 1.25, 1.46045)$ kV. Time = 480 ms.



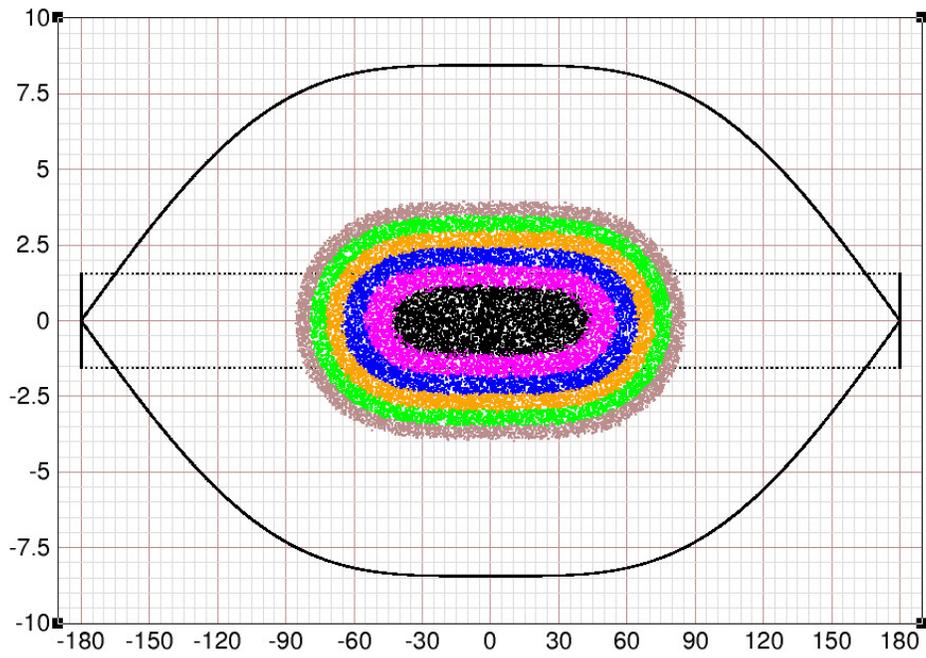
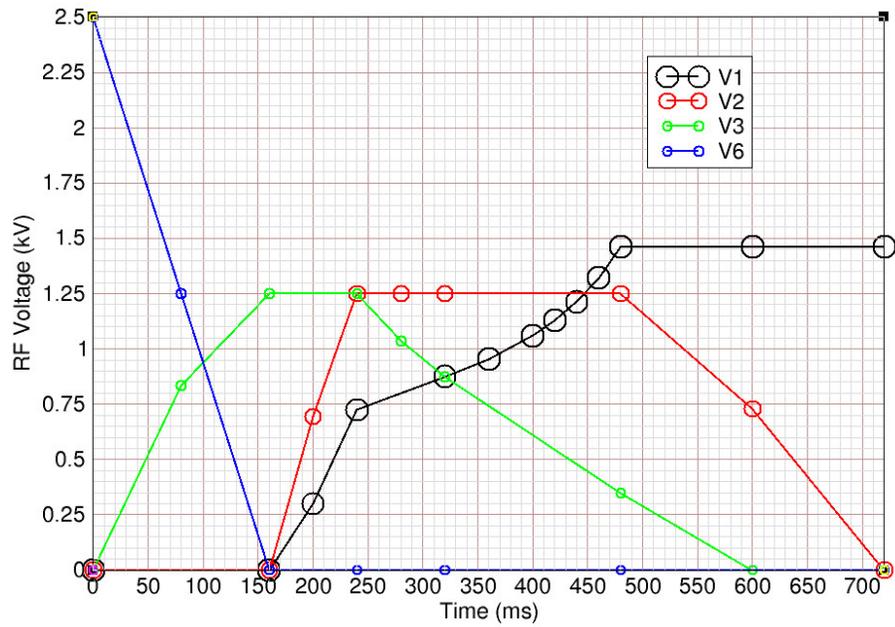


Figure 15: $(V_3, V_2, V_1) = (0, 0.730225, 1.46045)$ kV. Time = 600 ms.



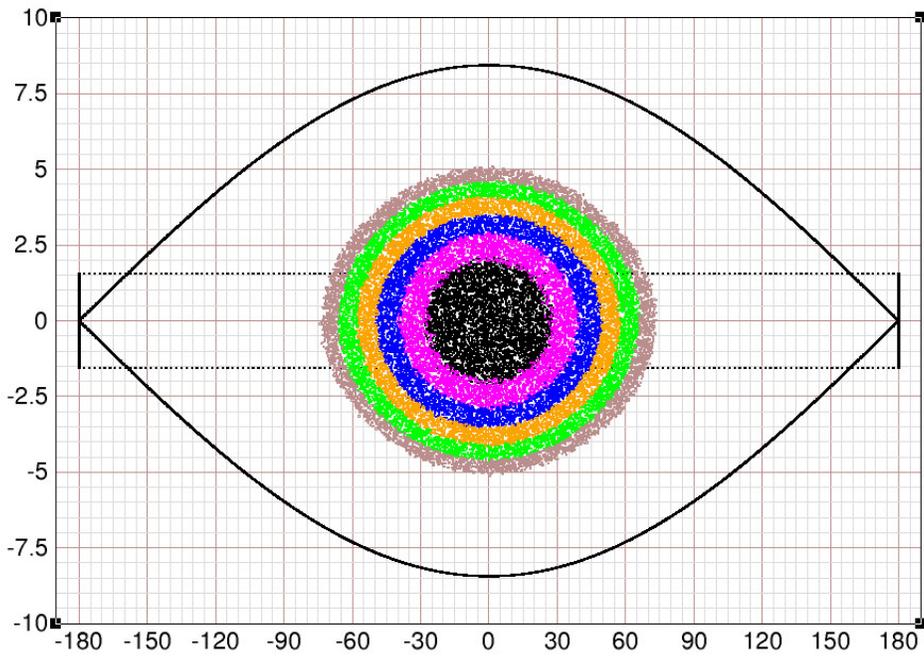
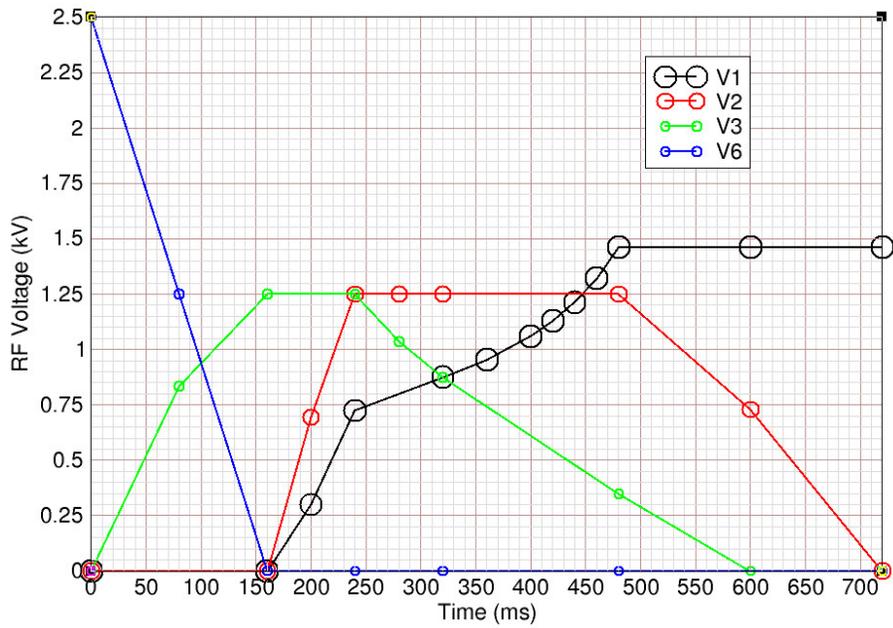


Figure 16: $(V3, V2, V1) = (0, 0, 1.46045)$ kV. Time = 720 ms.



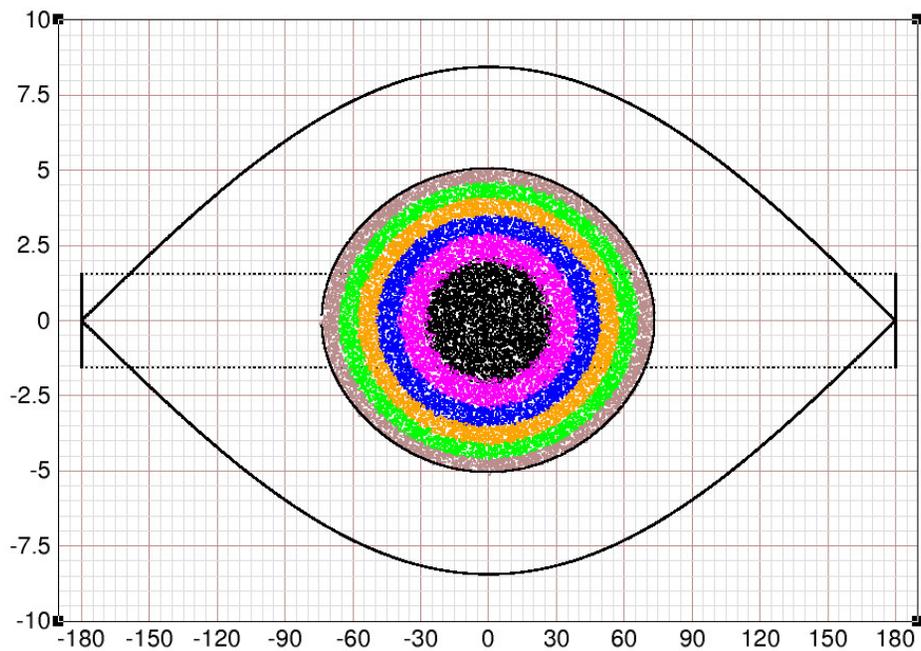
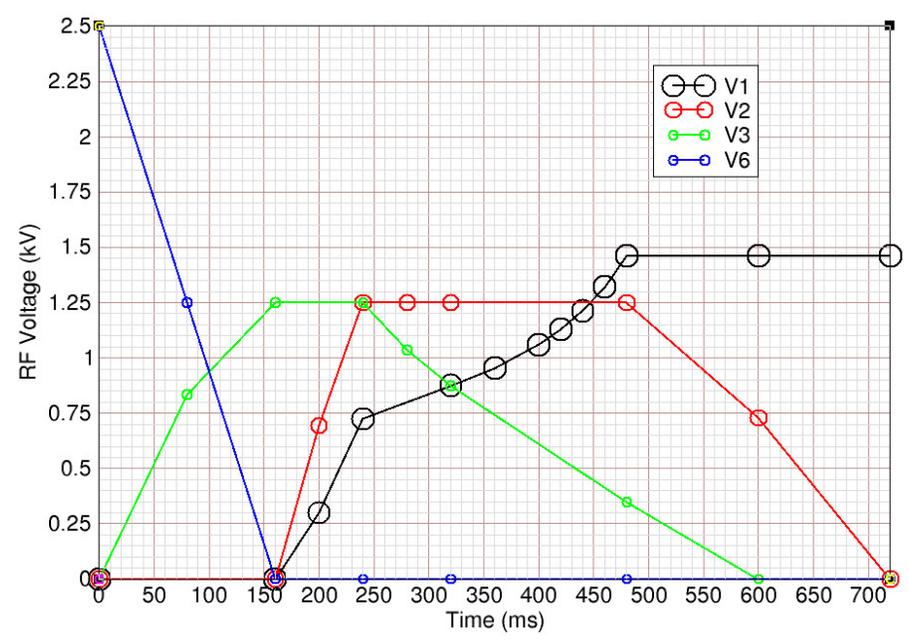


Figure 17: Final merged bunch at 720 ms. The inner black curve is matched to the RF bucket and encloses an area 1.017 times the area of the initial six bunches. This shows very little growth of the gross emittance.



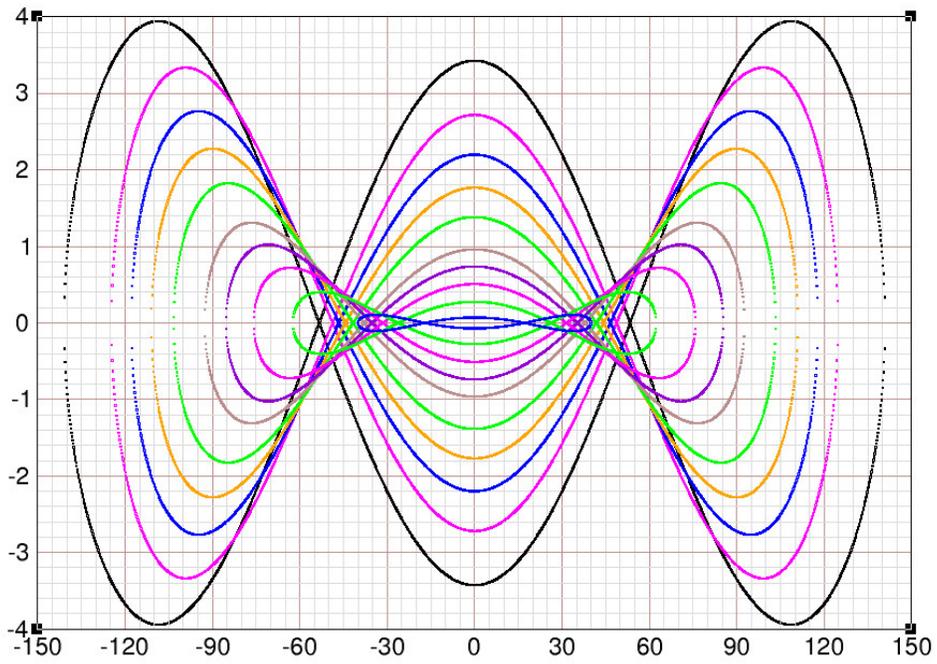
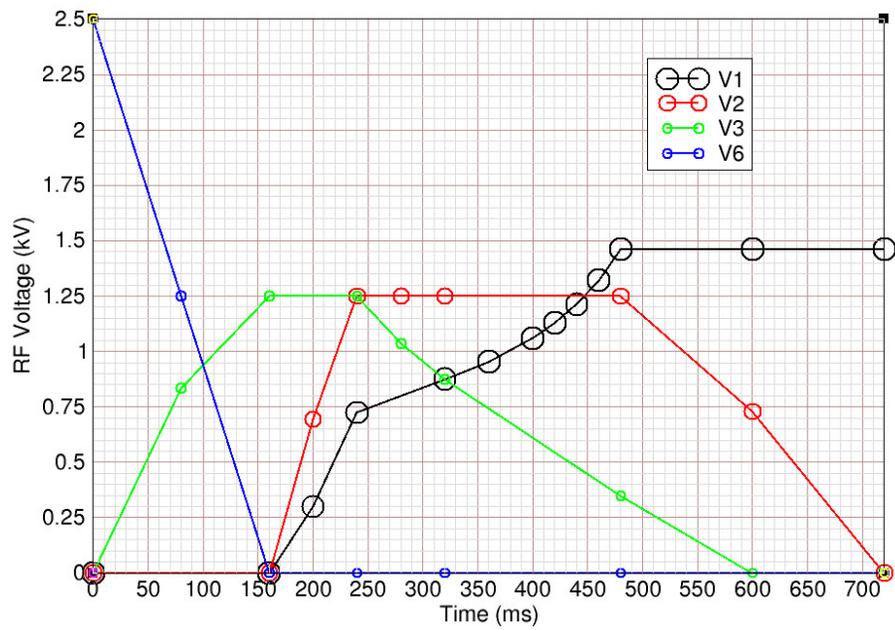


Figure 18: Evolution of separatrix lobes from 200 to 480 ms.



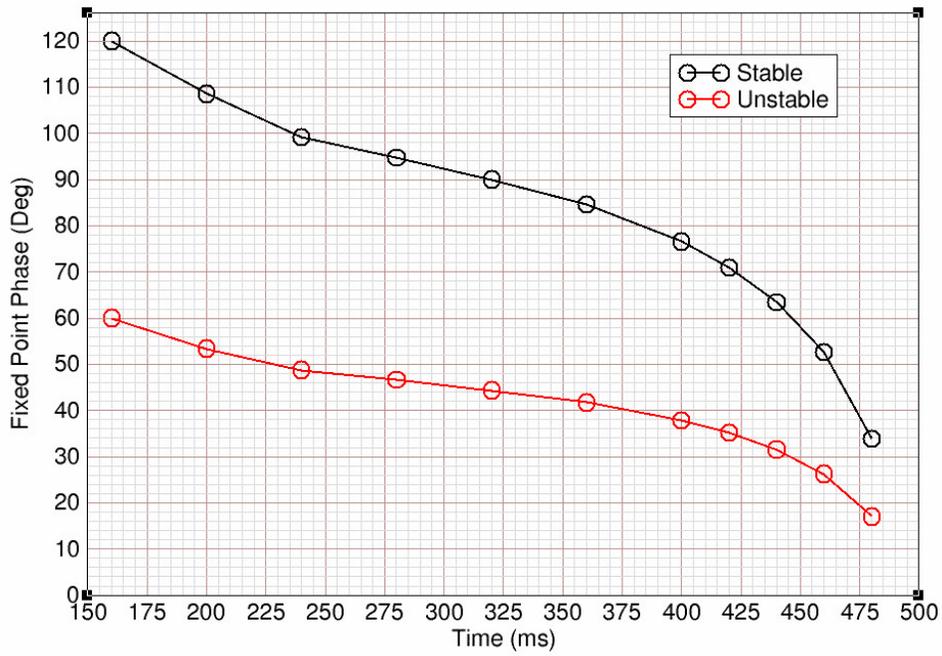
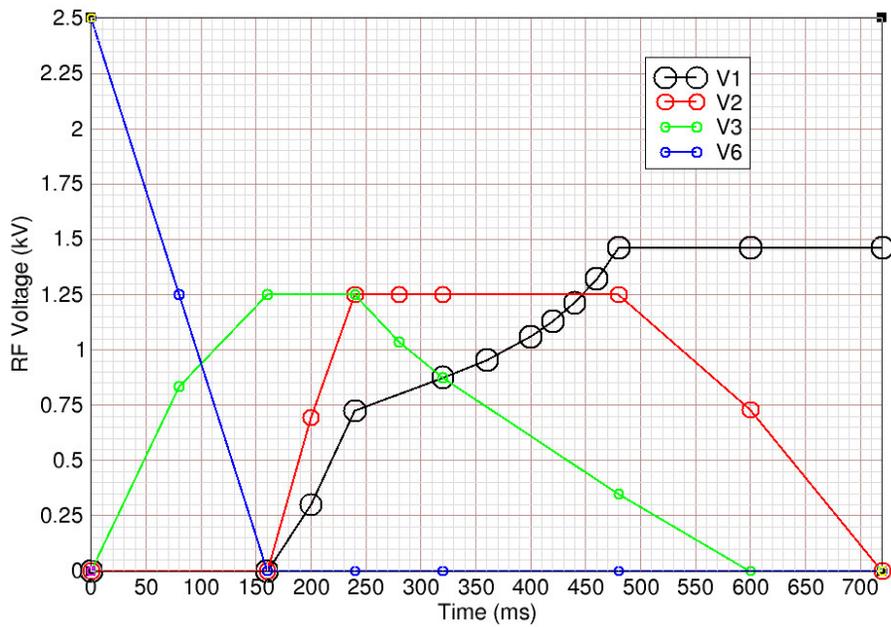
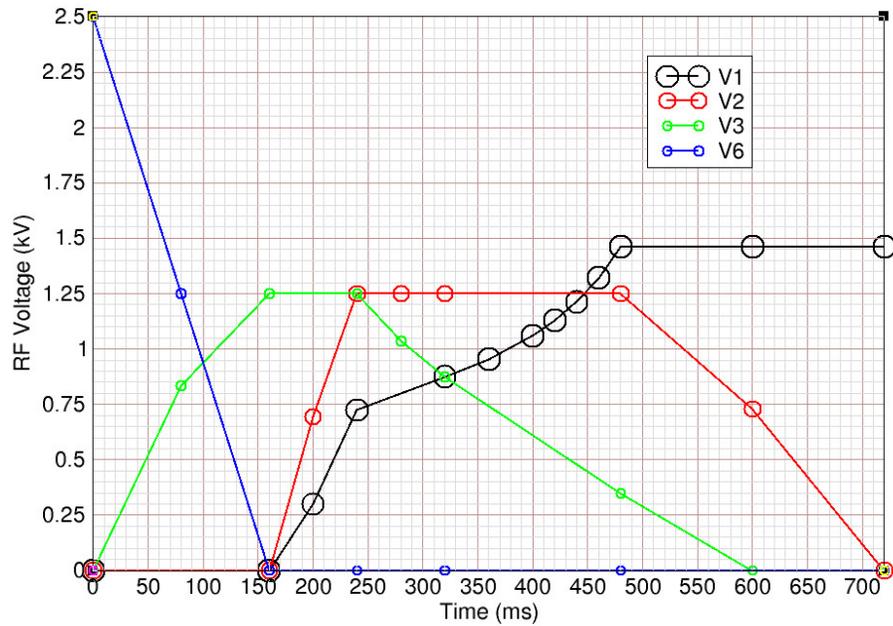


Figure 19: Evolution of stable and unstable fixed point phases.



Time (ms)	Phiu (deg)	Phis (deg)	Area
160	60	120	1.00000
200	53.3808	108.572	0.68125
240	48.7524	99.1671	0.49557
280	46.6176	94.7459	0.38400
320	44.3451	90.0000	0.29409
360	41.7336	84.5500	0.21679
400	37.9007	76.5731	0.13726
420	35.1885	70.9484	0.09791
440	31.5443	63.4375	0.06047
460	26.2461	52.6032	0.02758
480	17.0159	33.9535	0.00461

Figure 20: Table of stable and unstable fixed point phases, and area enclosed by separatrix lobes. Here the unit of area is the area enclosed by the three RF buckets at time 160 ms.



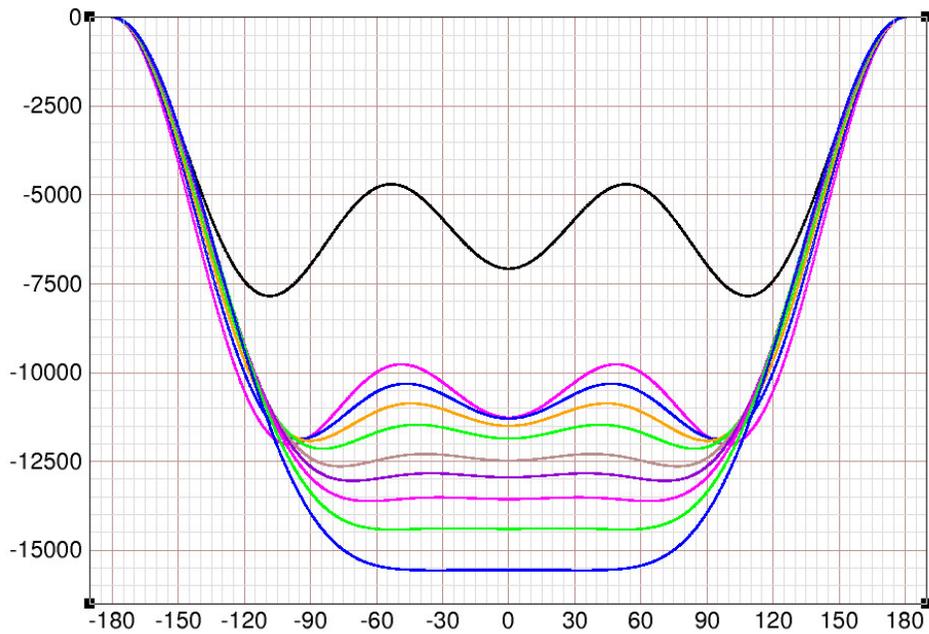
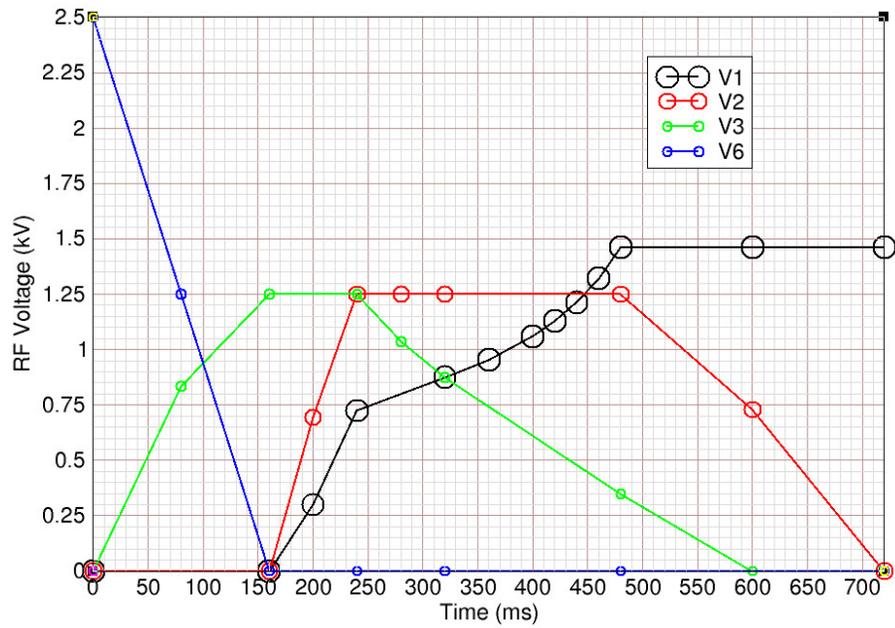


Figure 21: Evolution of separatrix potentials from 200 to 480 ms.



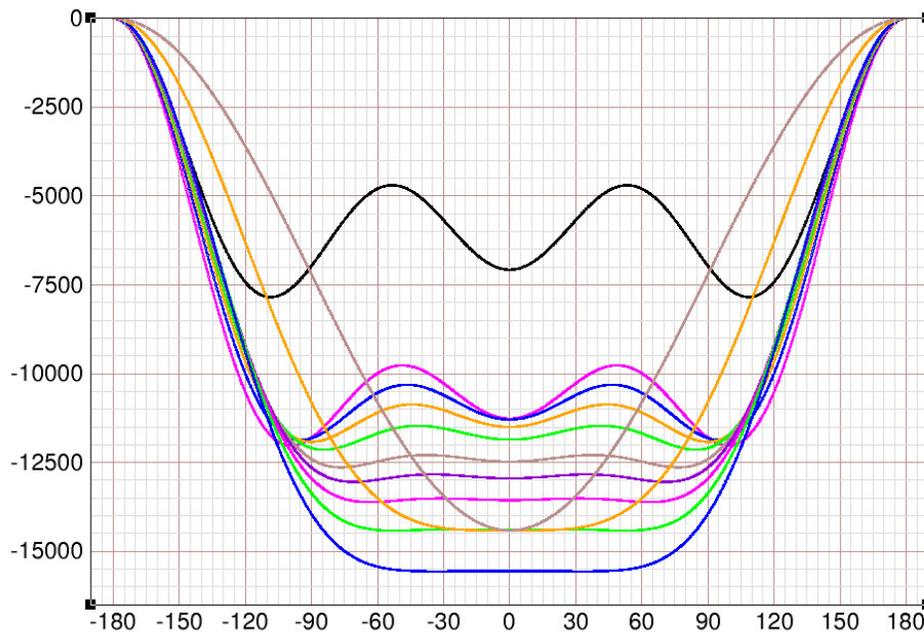
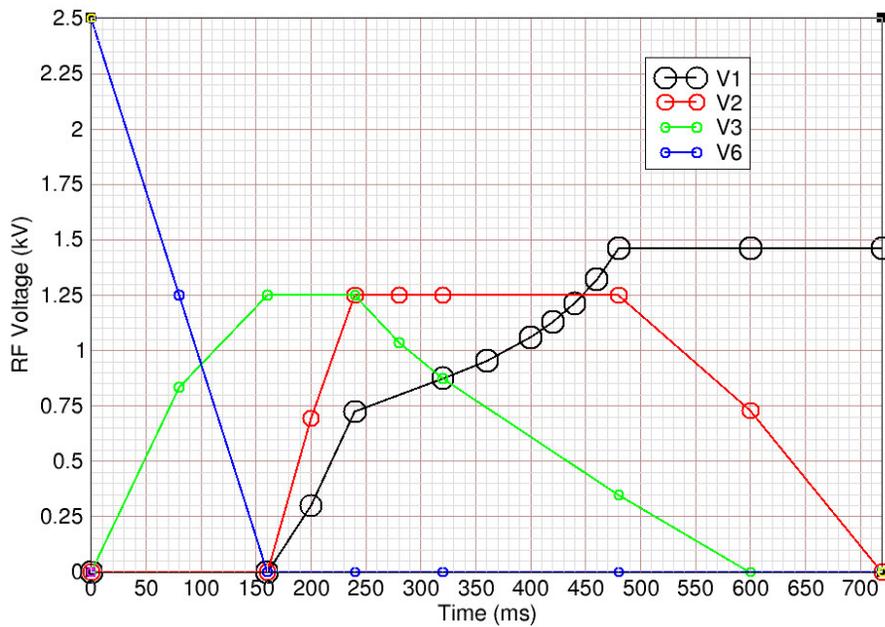


Figure 22: Here the additional curves are the potentials at 600 and 720 ms.



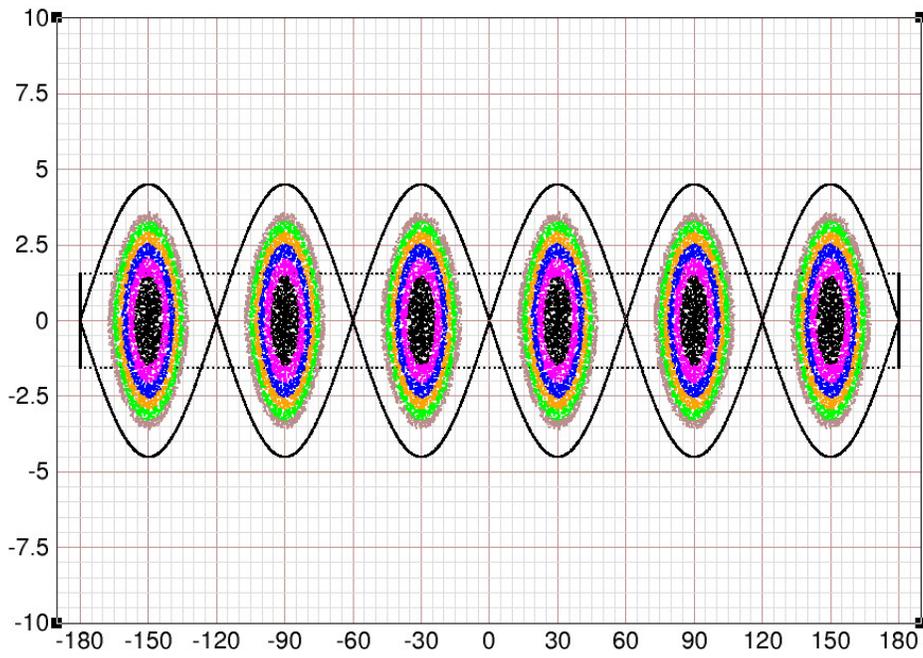
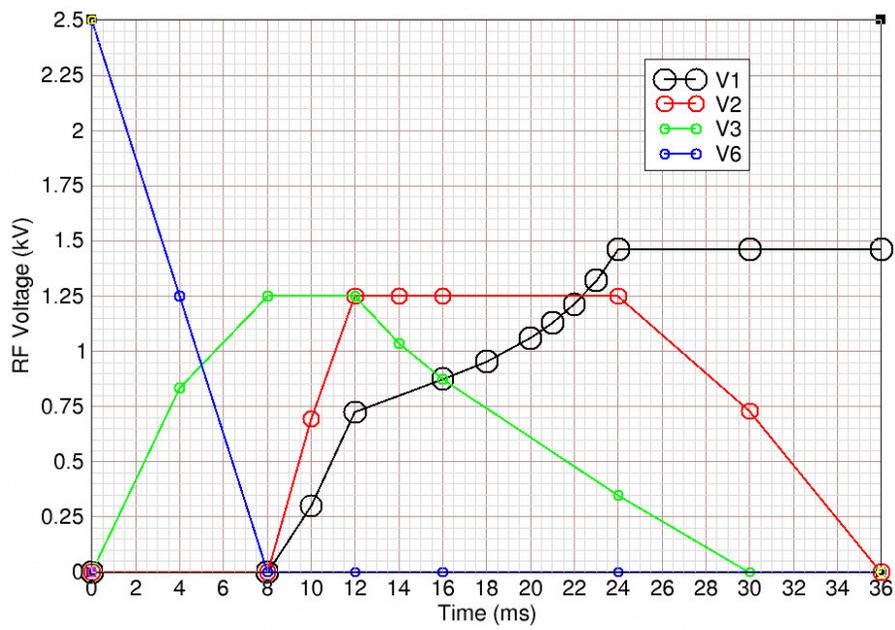


Figure 23: $(V_6, V_3) = (2.50, 0)$ kV. Time = 0 ms.



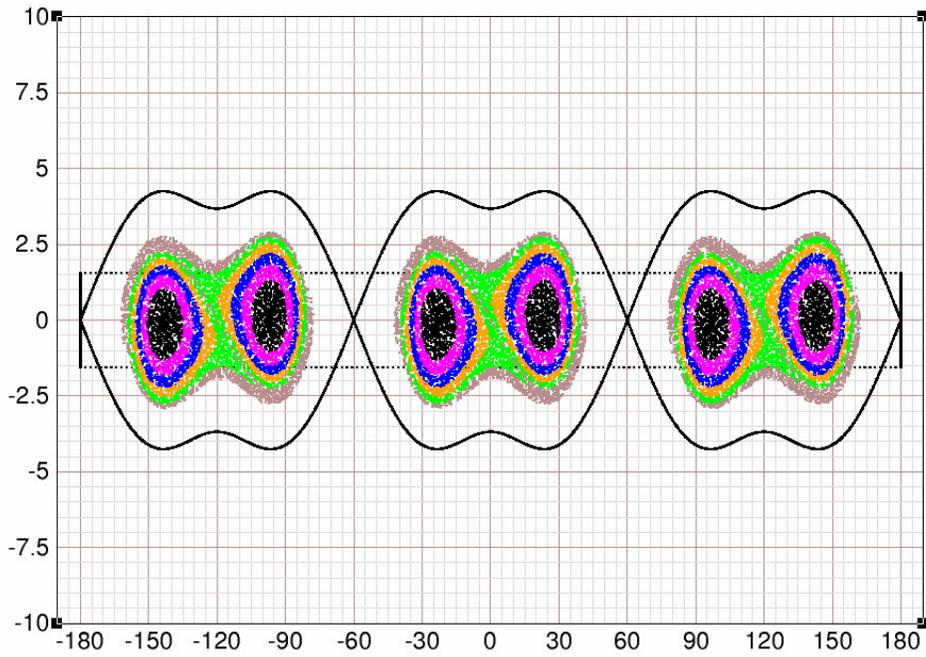
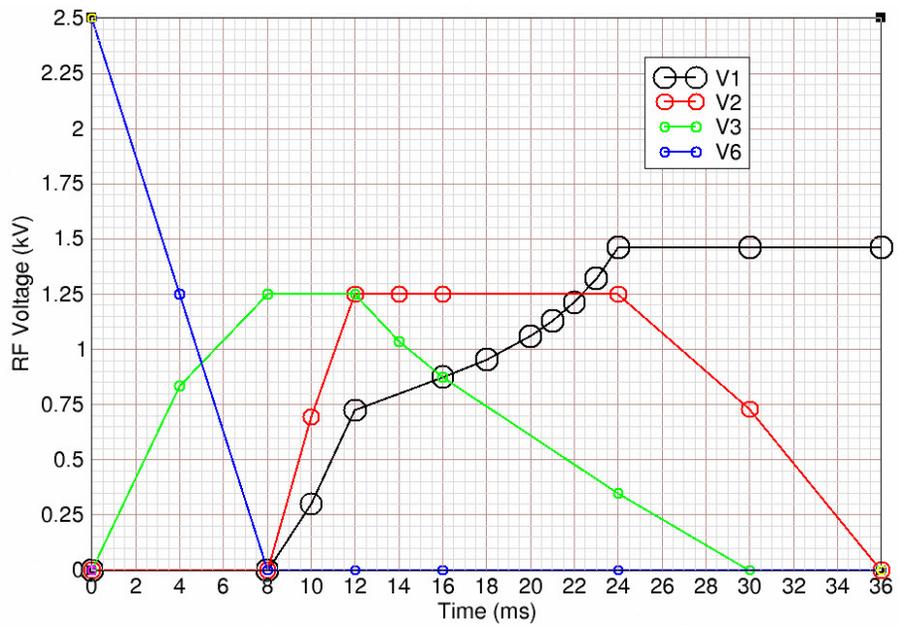


Figure 24: $(V6, V3) = (1.25, 0.8335)$ kV. Time = 4 ms.



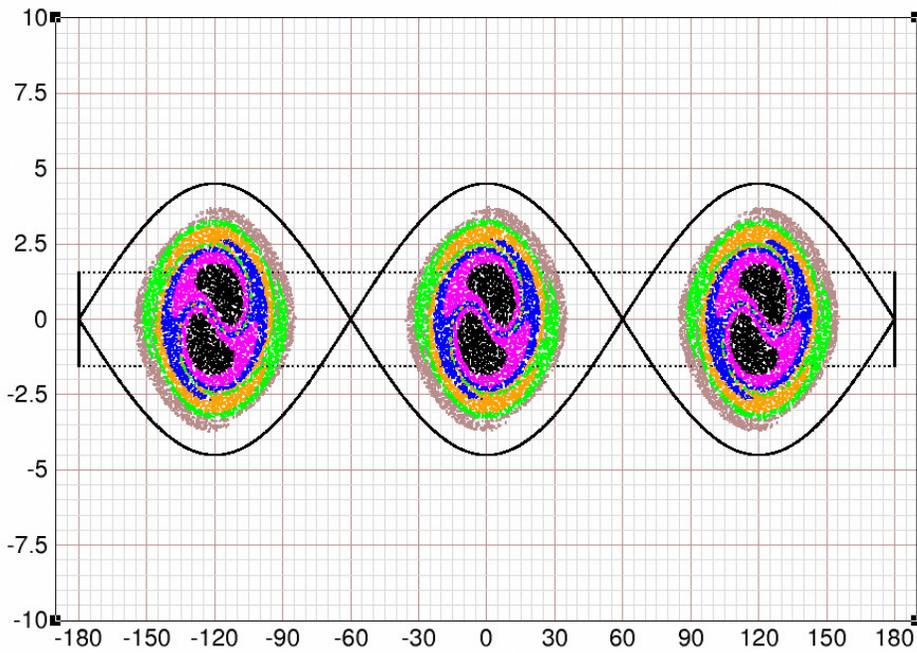
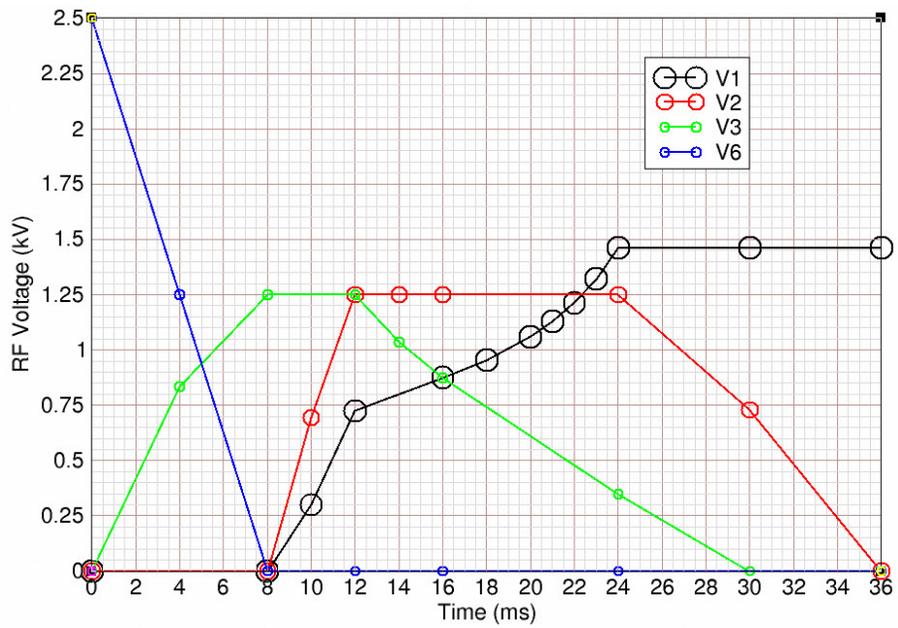


Figure 25: $(V_6, V_3) = (0, 1.25)$ kV. Time = 8 ms.



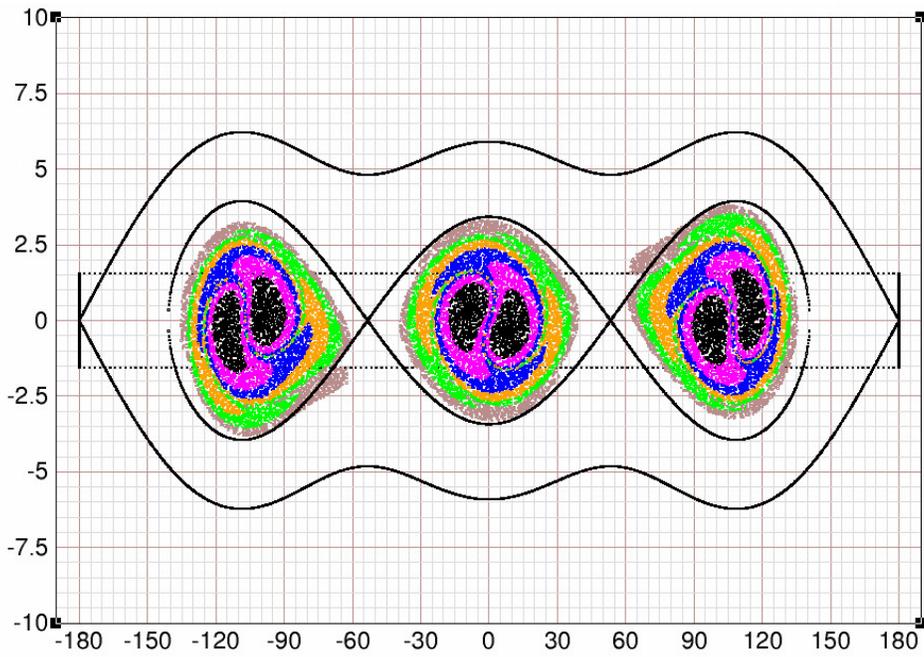
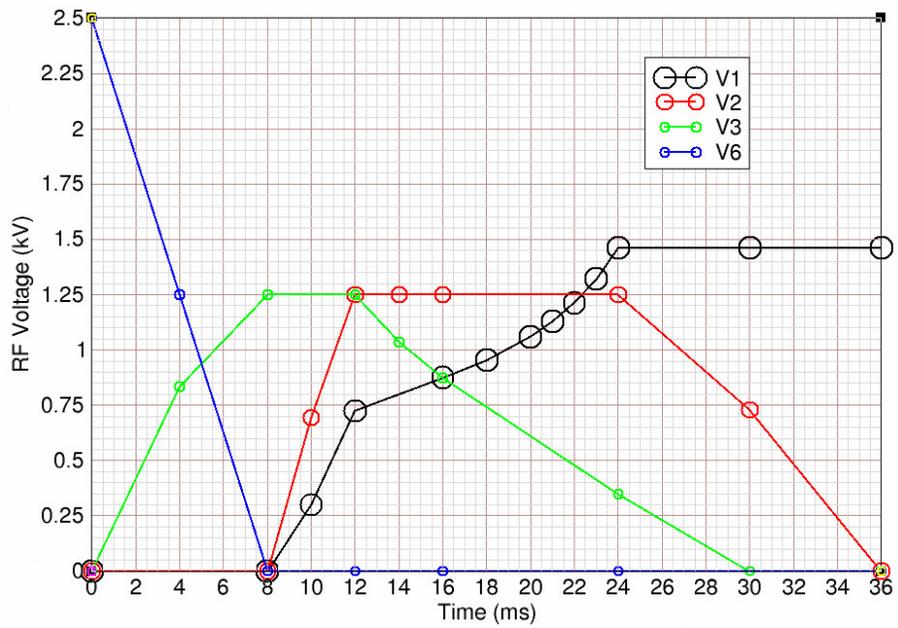


Figure 26: $(V_3, V_2, V_1) = (1.25, 0.695, 0.3001)$ kV. Time = 10 ms.



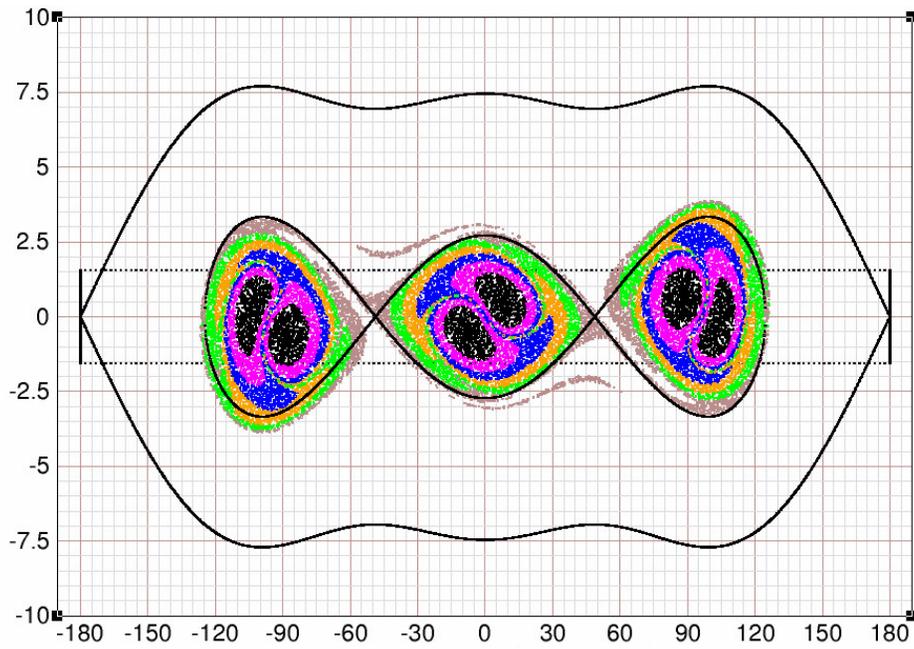
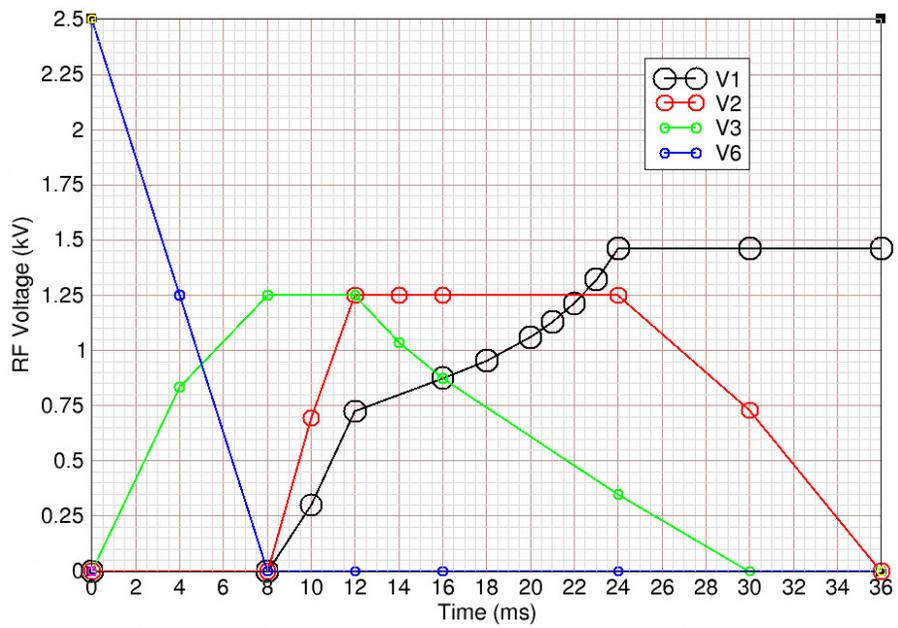


Figure 27: $(V_3, V_2, V_1) = (1.25, 1.25, 0.724807)$ kV. Time = 12 ms.



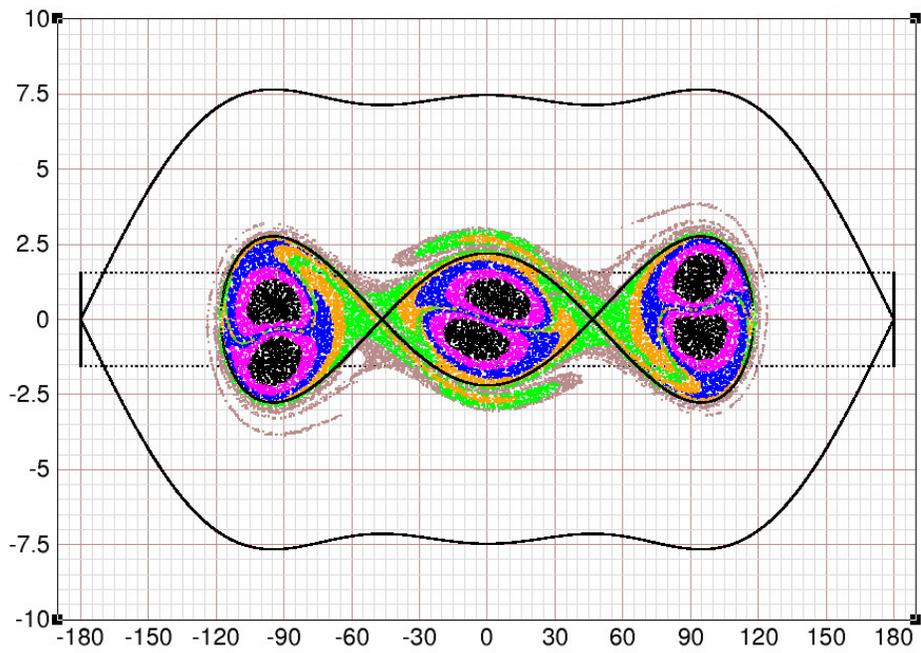
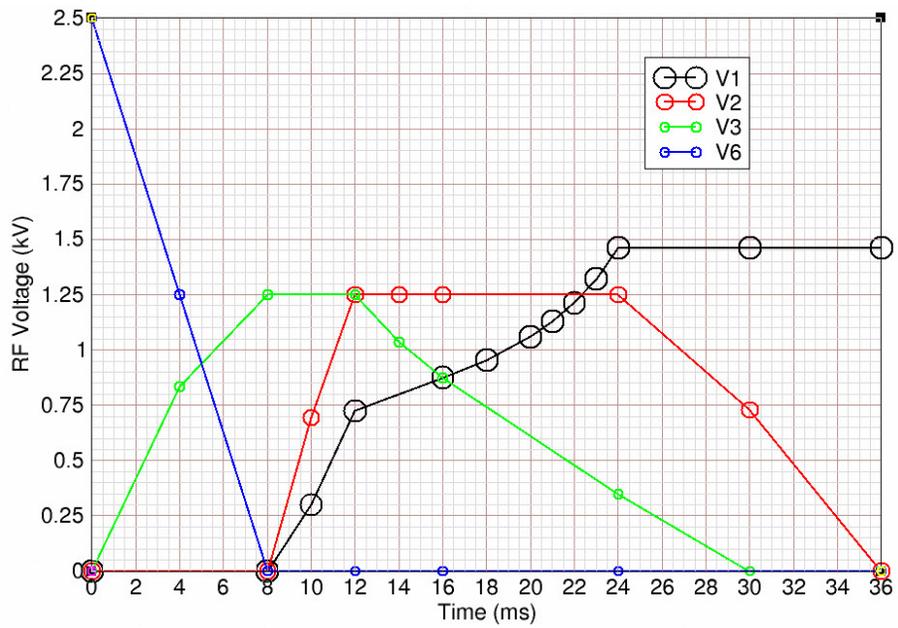


Figure 28: $(V3, V2, V1) = (1.03455, 1.25, 0.799380)$ kV. Time = 14 ms.



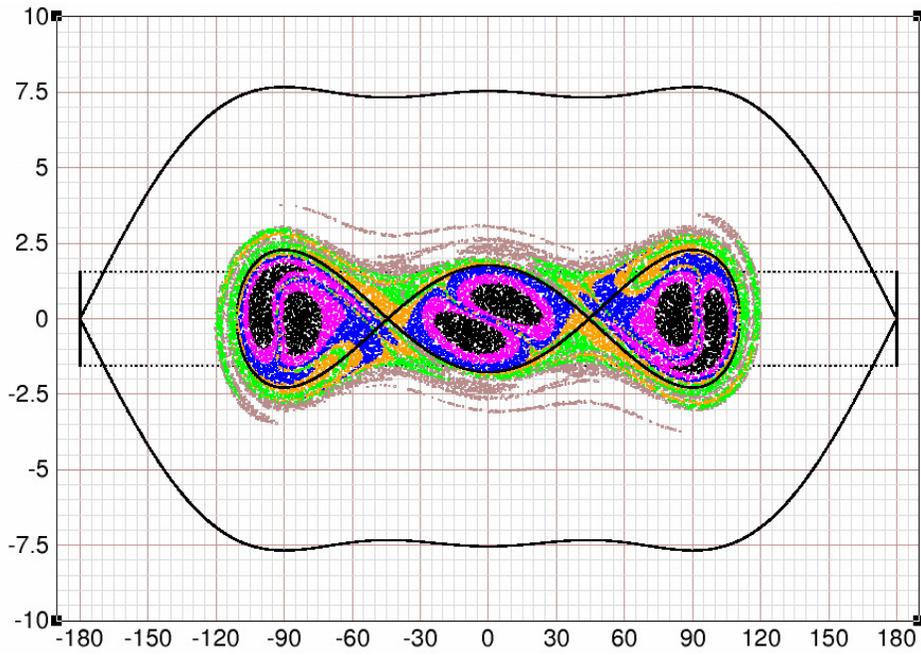
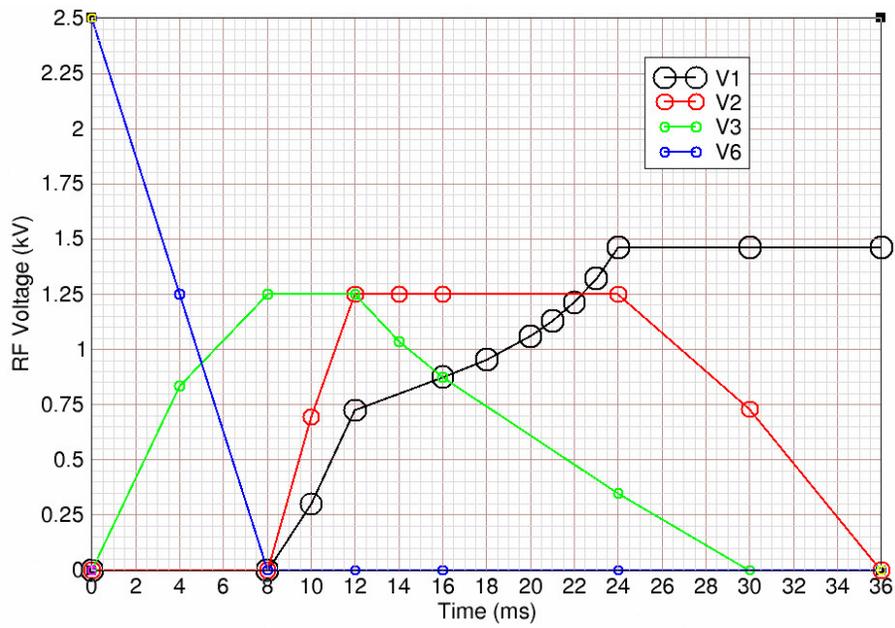


Figure 29: $(V3, V2, V1) = (0.873952, 1.25, 0.873952)$ kV. Time = 16 ms.



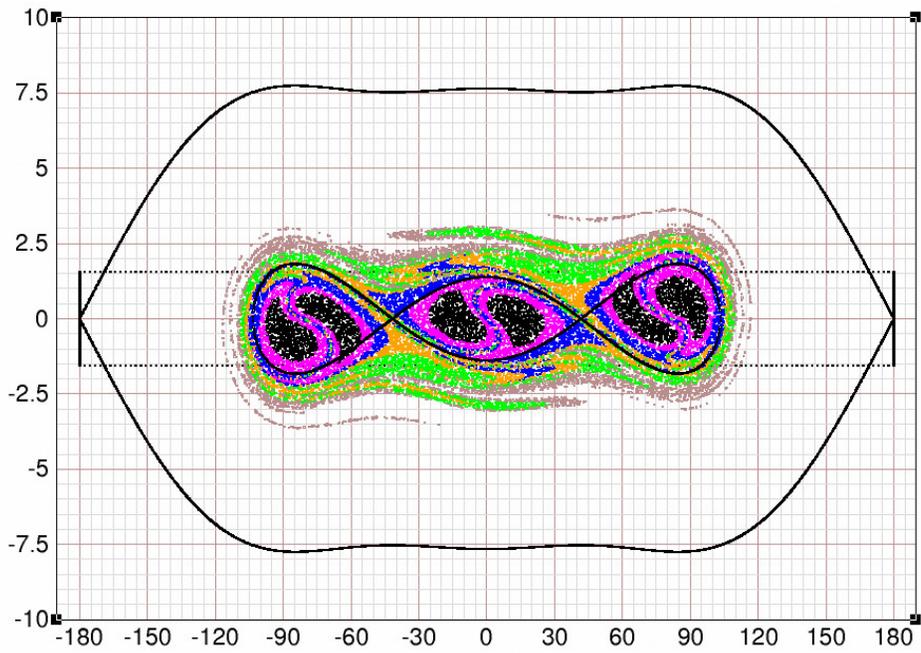
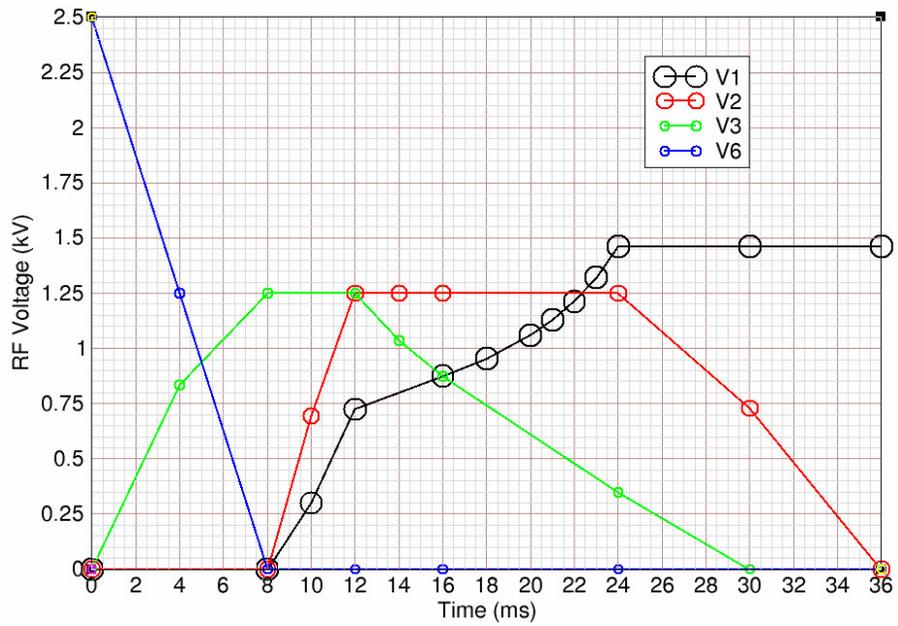


Figure 30: $(V3, V2, V1) = (0.742964, 1.25, 0.953600)$ kV. Time = 18 ms.



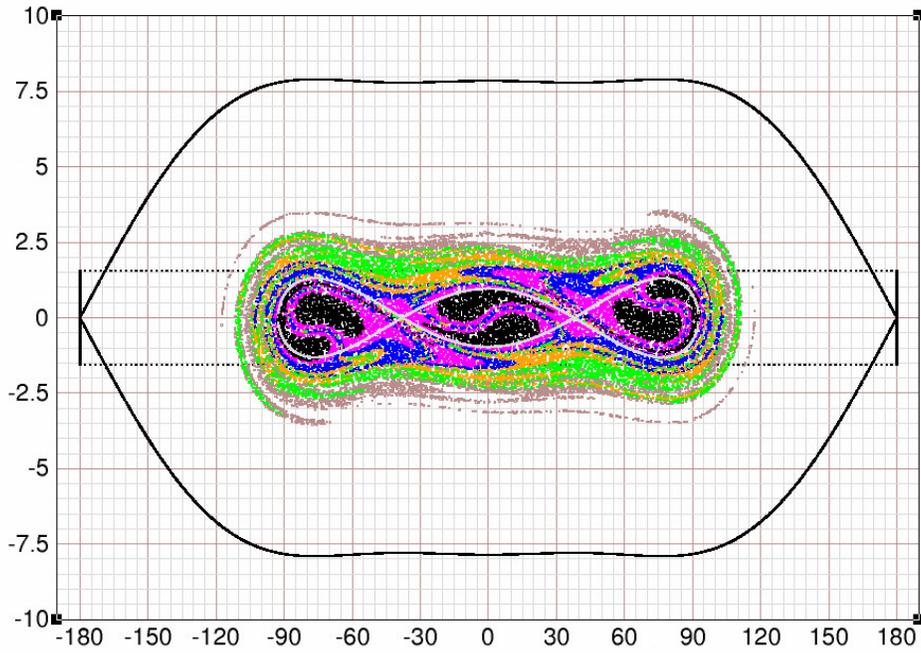
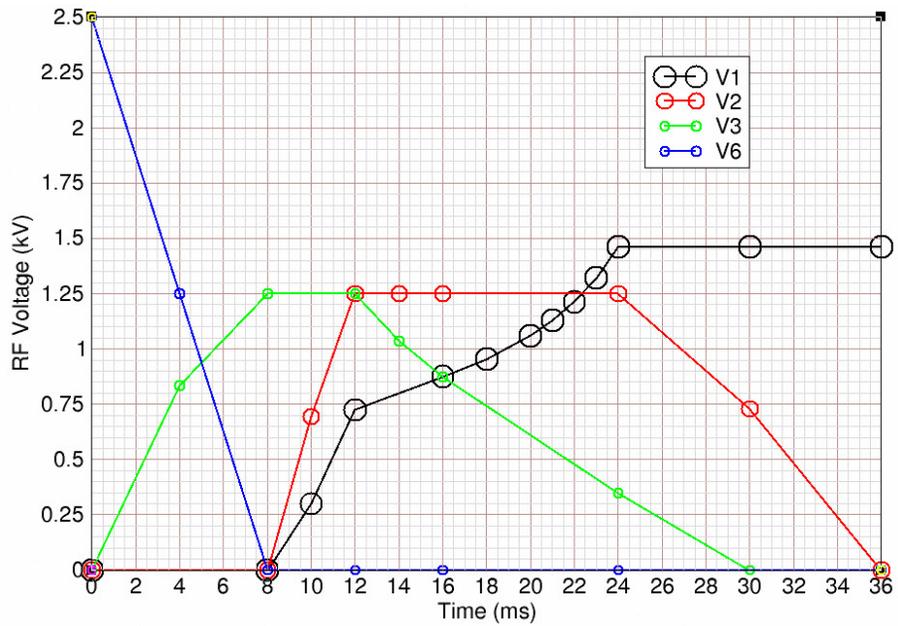


Figure 31: $(V_3, V_2, V_1) = (0.611976, 1.25, 1.06050)$ kV. Time = 20 ms.



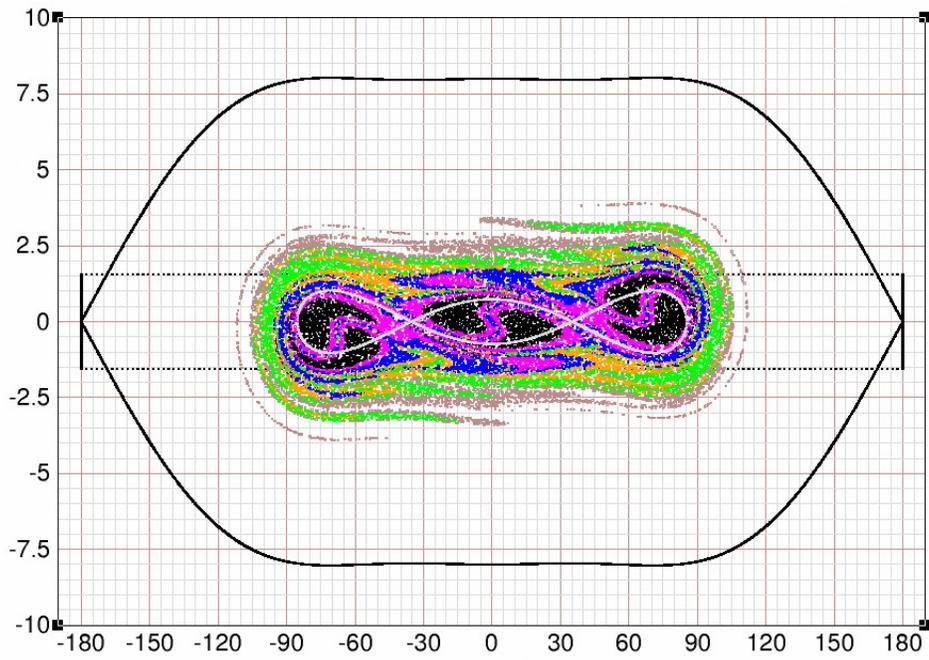
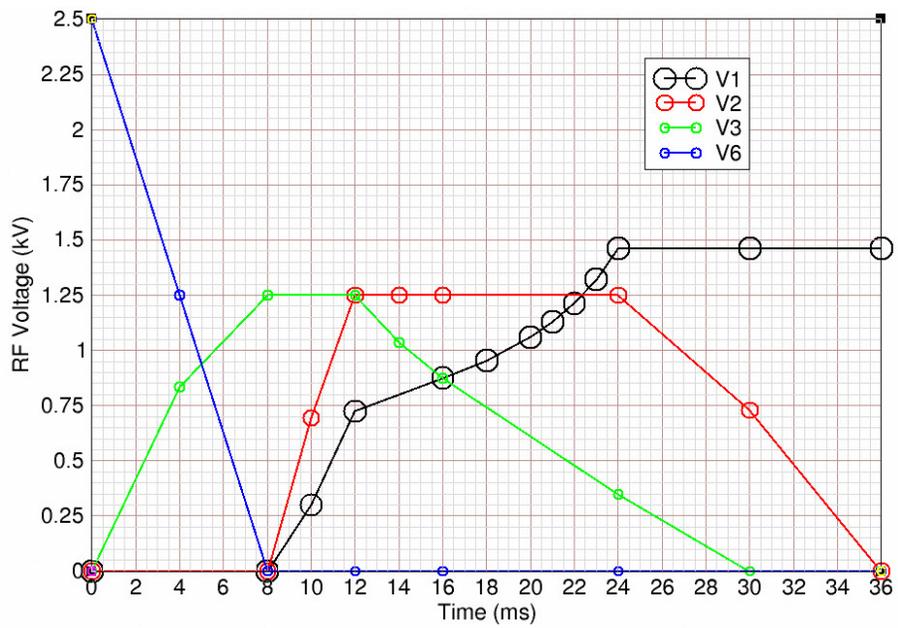


Figure 32: $(V3, V2, V1) = (0.546482, 1.25, 1.12962)$ kV. Time = 21 ms.



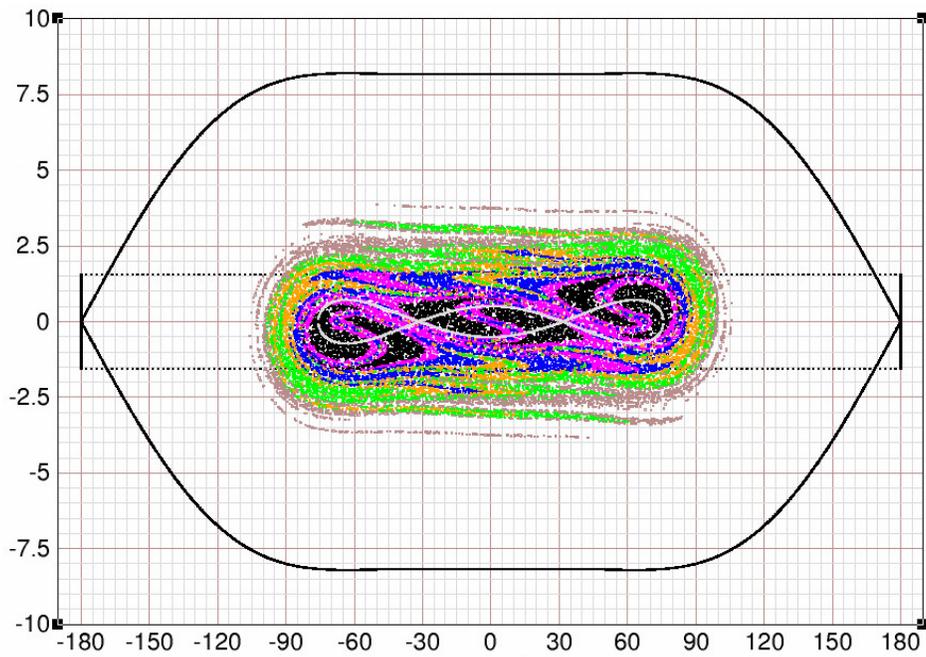
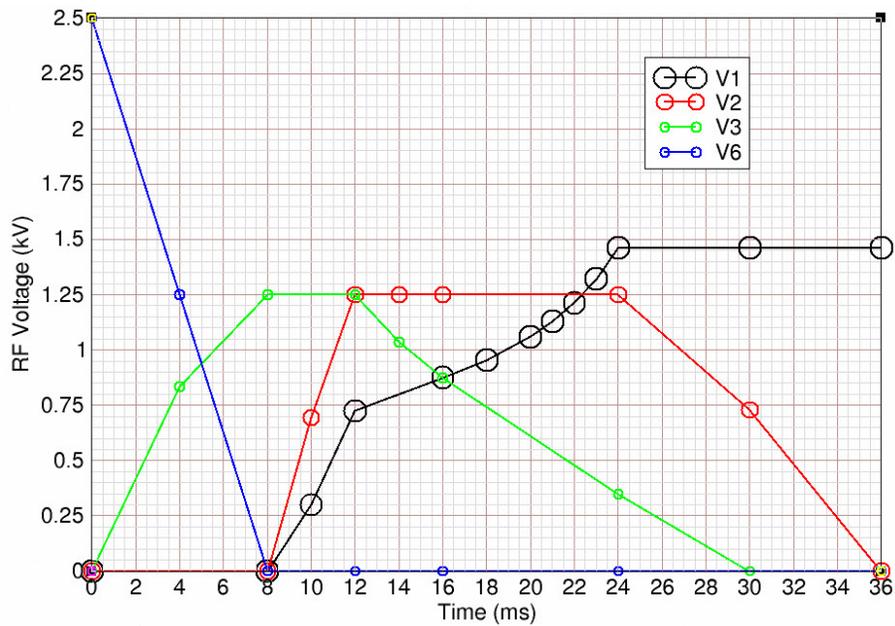


Figure 33: $(V3,V2,V1) = (0.480988, 1.25, 1.21420)$ kV. Time = 22 ms.



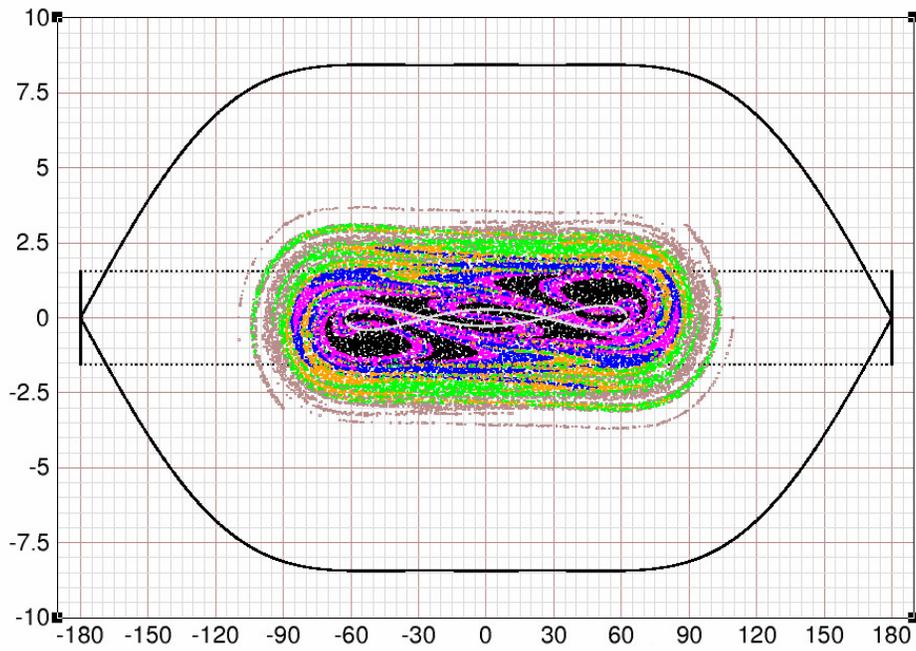
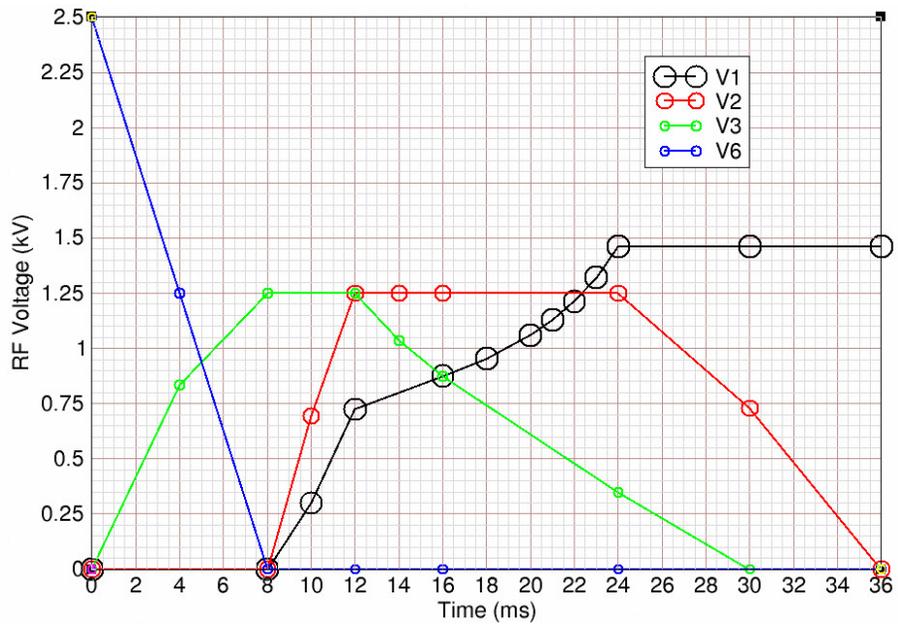


Figure 34: $(V3, V2, V1) = (0.415494, 1.25, 1.32080)$ kV. Time = 23 ms.



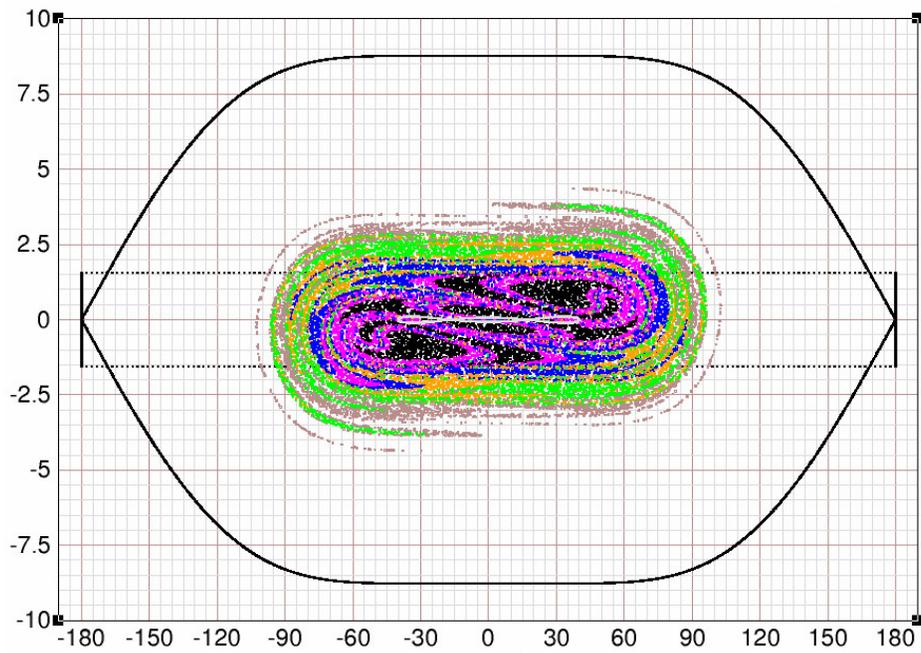
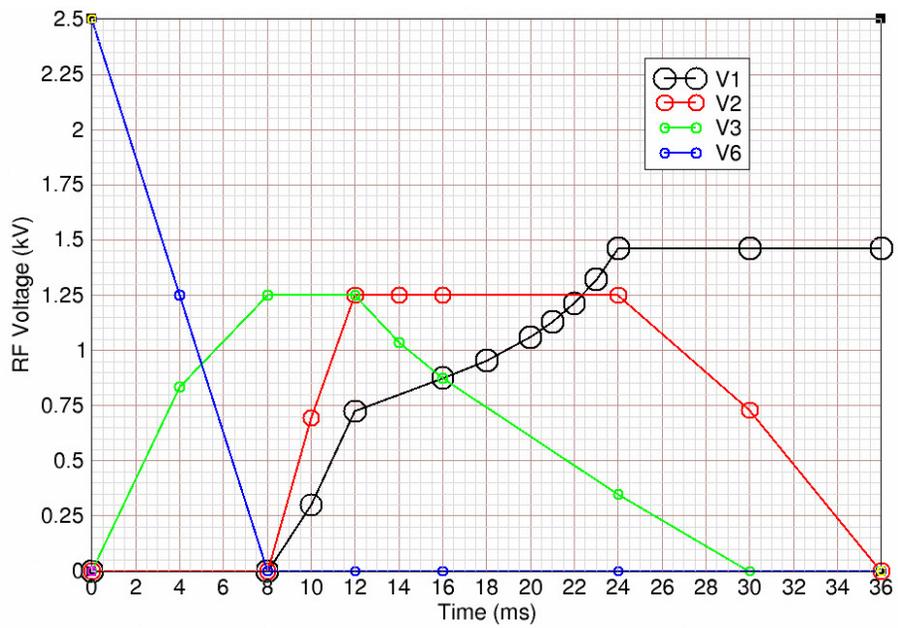


Figure 35: $(V3, V2, V1) = (0.35, 1.25, 1.46045)$ kV. Time = 24 ms.



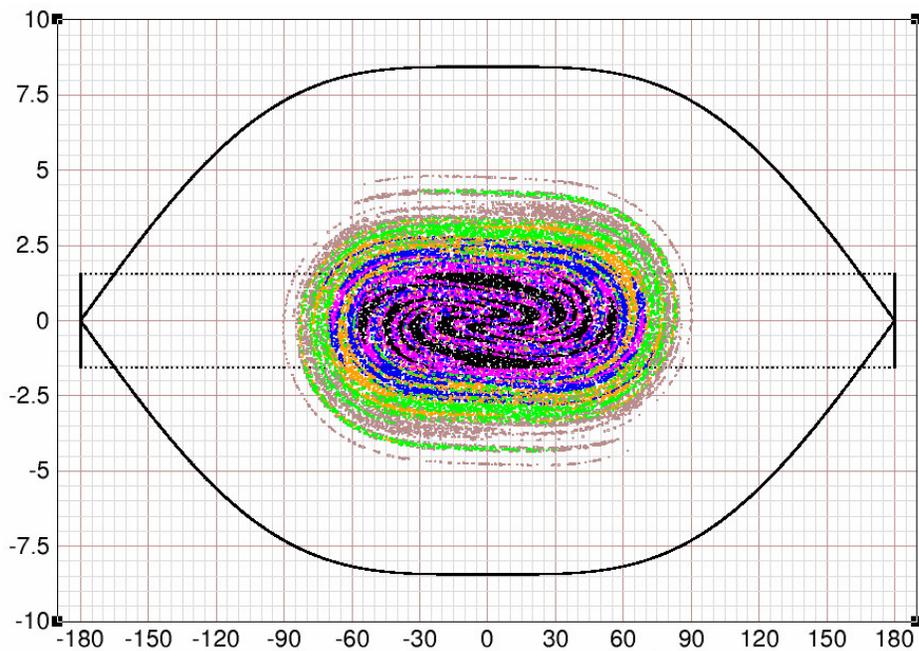
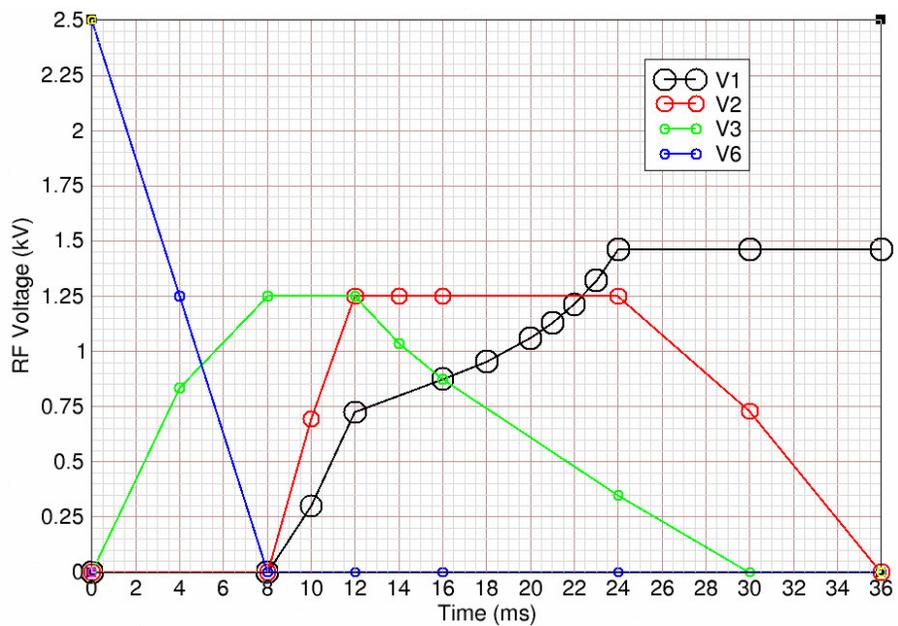


Figure 36: $(V3, V2, V1) = (0, 0.730225, 1.46045)$ kV. Time = 30 ms.



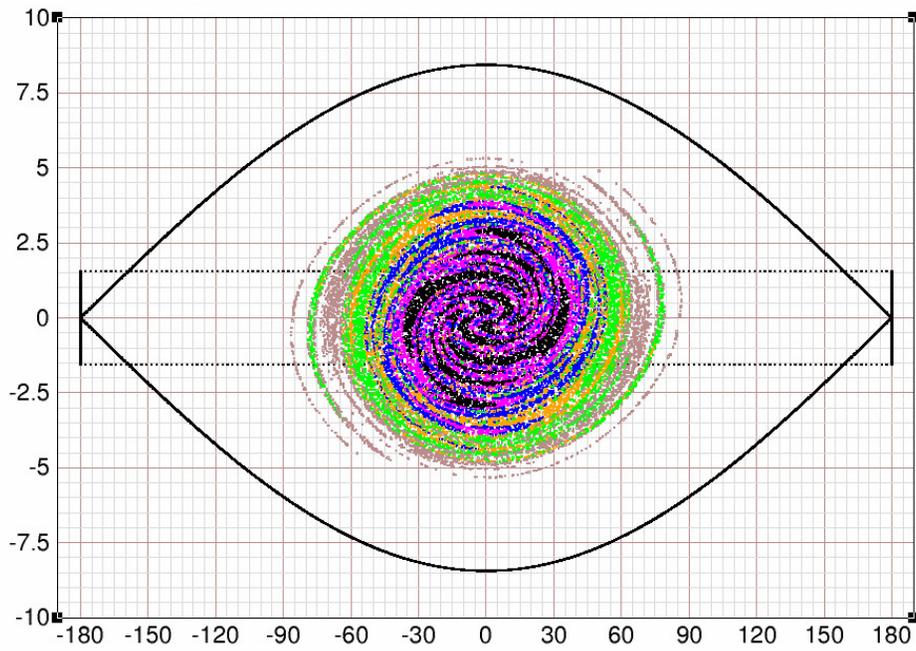
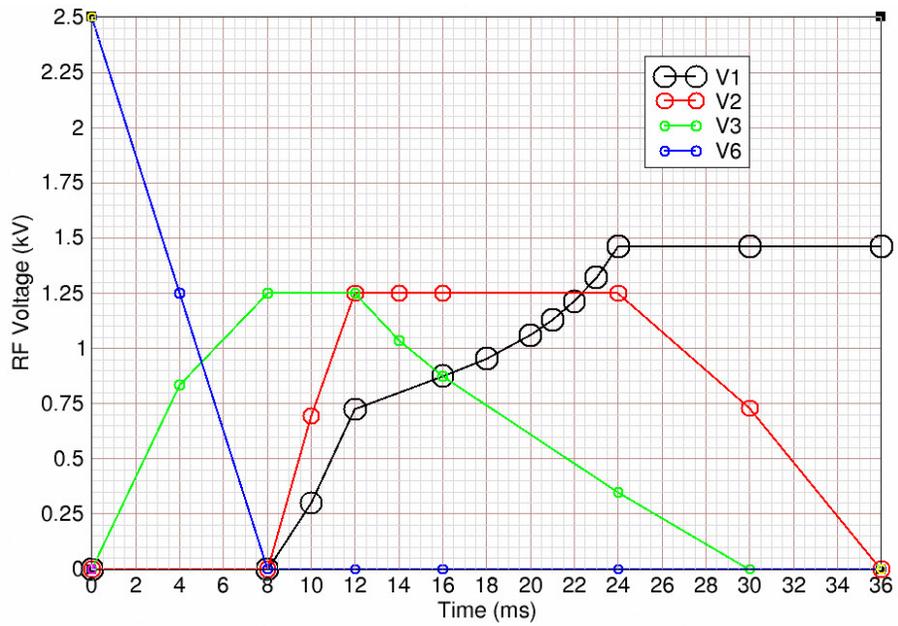


Figure 37: $(V_3, V_2, V_1) = (0, 0, 1.46045)$ kV. Time = 36 ms.



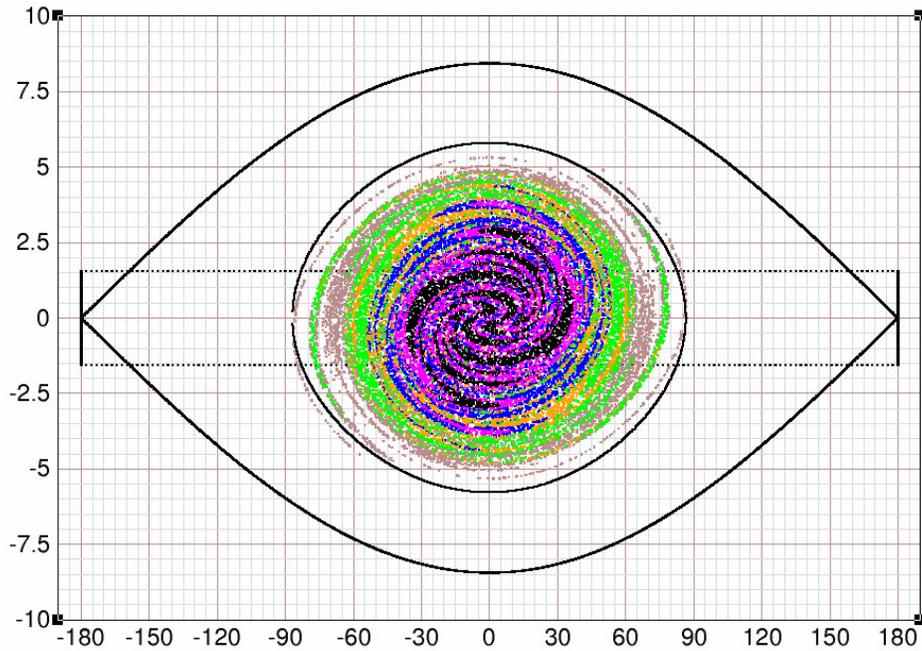
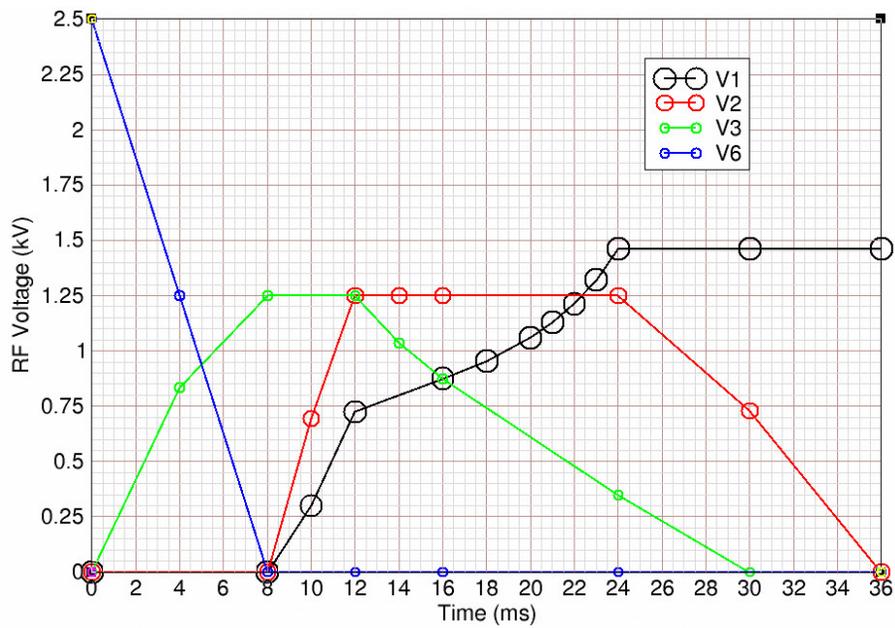


Figure 38: Final merged bunch at 36 ms. The inner black curve is matched to the RF bucket and encloses an area 1.361 times the area of the initial six bunches. There is significant filamentation and mixing of amplitude layers.



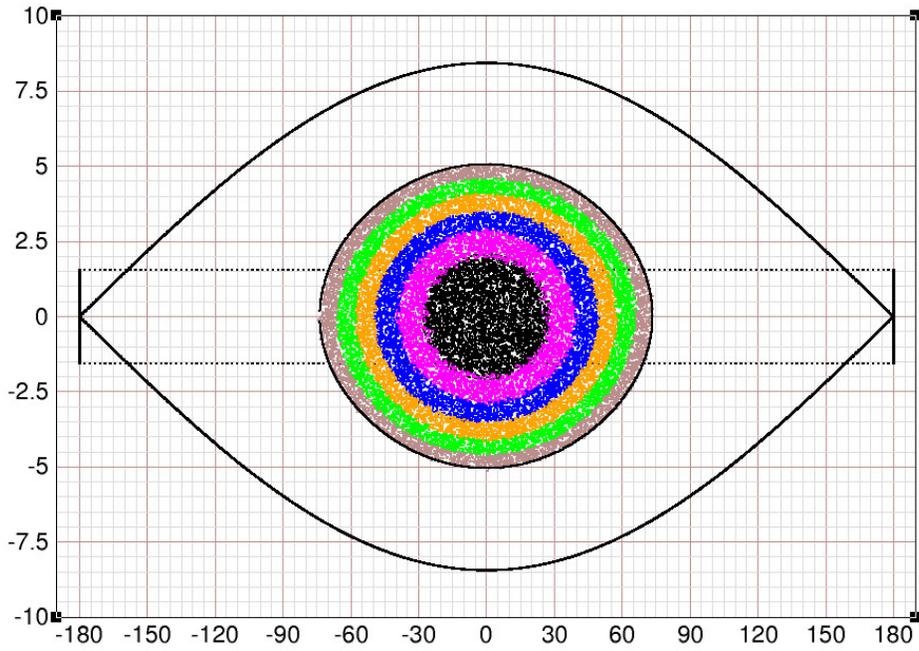
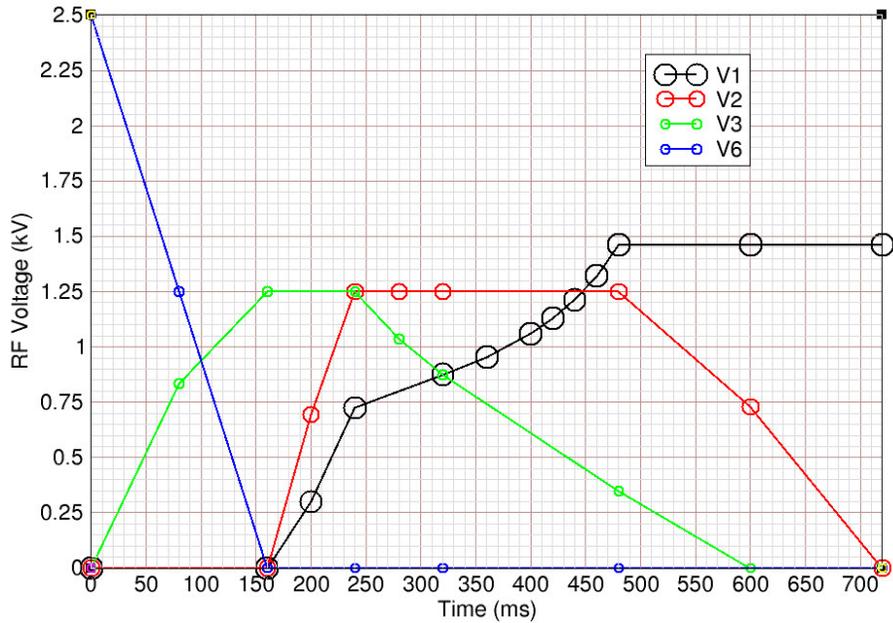


Figure 39: Final merged bunch at 720 ms . The inner black curve is matched to the RF bucket and encloses an area 1.017 times the area of the initial six bunches. There is no filamentation or mixing of amplitude layers.



2 Triple-harmonic RF bucket

A triple-harmonic RF bucket is one in which the three harmonic numbers h , $2h$, and $3h$ are active. For the case in which three bunches are to be merged into one we have, as shown in [6], the “force” function

$$F(\phi) = A_1 \sin \phi - A_2 \sin 2\phi + A_3 \sin 3\phi \quad (7)$$

and associated “potential”

$$U(\phi) = A_1 \cos \phi - \frac{1}{2}A_2 \cos 2\phi + \frac{1}{3}A_3 \cos 3\phi \quad (8)$$

where the amplitudes A_1 , A_2 , A_3 are either zero or positive.

One starts with amplitudes

$$A_1 = A_2 = 0, \quad A_3 > 0 \quad (9)$$

and then A_1 and A_2 are raised from zero. Eventually A_3 and A_2 are brought down to zero, leaving A_1 as the only nonzero amplitude. Ideally this is done adiabatically. One ends up with merged bunches sitting in harmonic h buckets.

According to the notation and conventions used in [6] we have

$$A_1 = \frac{eQV_1}{2\pi h}, \quad A_2 = \frac{eQV_2}{2\pi h}, \quad A_3 = \frac{eQV_3}{2\pi h} \quad (10)$$

where V_1 , V_2 , V_3 are the harmonic h , $2h$, $3h$ voltages respectively.

It is convenient to define

$$Q = \frac{A_2}{A_1}, \quad R = \frac{A_3}{A_1} \quad (11)$$

and work with

$$\mathcal{F} = \frac{F(\phi)}{A_1}, \quad \mathcal{U} = \frac{U(\phi)}{A_1}. \quad (12)$$

We then have

$$\mathcal{F} = \sin \phi - Q \sin 2\phi + R \sin 3\phi \quad (13)$$

$$\mathcal{U} = \cos \phi - \frac{Q}{2} \cos 2\phi + \frac{R}{3} \cos 3\phi \quad (14)$$

$$\mathcal{U}' = -\sin \phi + Q \sin 2\phi - R \sin 3\phi \quad (15)$$

$$\mathcal{U}'' = -\cos \phi + 2Q \cos 2\phi - 3R \cos 3\phi \quad (16)$$

$$\mathcal{U}''' = \sin \phi - 4Q \sin 2\phi + 9R \sin 3\phi \quad (17)$$

$$\mathcal{U}'''' = \cos \phi - 8Q \cos 2\phi + 27R \cos 3\phi \quad (18)$$

$$\mathcal{U}''''' = -\sin \phi + 16Q \sin 2\phi - 81R \sin 3\phi \quad (19)$$

$$\mathcal{U}'''''' = -\cos \phi + 32Q \cos 2\phi - 243R \cos 3\phi \quad (20)$$

and

$$\mathcal{F} = -\mathcal{U}' \quad (21)$$

where the primes denote differentiation with respect to ϕ .

Introducing the notation

$$C = \cos \phi, \quad S = \sin \phi \quad (22)$$

we have the identities

$$\sin 2\phi = 2CS \quad (23)$$

$$\sin 3\phi = 3S - 4S^3 \quad (24)$$

and

$$\cos 2\phi = 2C^2 - 1 \quad (25)$$

$$\cos 3\phi = 4C^3 - 3C \quad (26)$$

which give

$$\mathcal{U} = C - \frac{Q}{2} (2C^2 - 1) + \frac{R}{3} (4C^3 - 3C) \quad (27)$$

$$\mathcal{U}' = -\left\{1 - 2QC + R(4C^2 - 1)\right\} S \quad (28)$$

$$\mathcal{U}'' = -C + 2Q(2C^2 - 1) - 3R(4C^3 - 3C). \quad (29)$$

For phases with subscript “a” we write

$$C_a = \cos \phi_a, \quad S_a = \sin \phi_a \quad (30)$$

and

$$\mathcal{U}_a = C_a - \frac{Q}{2} (2C_a^2 - 1) + \frac{R}{3} (4C_a^3 - 3C_a) \quad (31)$$

$$\mathcal{U}'_a = -\left\{1 - 2QC_a + R(4C_a^2 - 1)\right\} S_a \quad (32)$$

$$\mathcal{U}''_a = -C_a + 2Q(2C_a^2 - 1) - 3R(4C_a^3 - 3C_a). \quad (33)$$

3 Fixed point phases

Fixed point phases ϕ_f are those for which

$$\mathcal{U}'_f = 0. \quad (34)$$

We work exclusively below transition as that is where the 3 to 1 merge is done in both Booster and AGS. Below transition the fixed point phase is stable if

$$\mathcal{U}''_f < 0 \quad (35)$$

and unstable if

$$\mathcal{U}''_f > 0. \quad (36)$$

We use subscripts “s” and “u” for fixed points that are stable and unstable respectively.

Note that the function $-\mathcal{U}(\phi)$ has a local minimum at a stable fixed point and a local maximum at an unstable fixed point. It is for this reason that it is helpful to think in terms of the “potential well” $-\mathcal{U}(\phi)$ rather than $\mathcal{U}(\phi)$.

4 Unstable fixed point phases $\phi_u = \pm\pi$

For phases $\phi = \pm\pi$, equations (15) through (20) give

$$\mathcal{U}' = 0, \quad \mathcal{U}''' = 0, \quad \mathcal{U}'''' = 0 \quad (37)$$

$$\mathcal{U}'' = 1 + 2Q + 3R \quad (38)$$

$$\mathcal{U}'''' = -1 - 8Q - 27R \quad (39)$$

$$\mathcal{U}'''''' = 1 + 32Q + 243R. \quad (40)$$

Here

$$Q \geq 0, \quad R \geq 0 \quad (41)$$

and therefore

$$\mathcal{U}'' > 0. \quad (42)$$

We therefore have unstable fixed point phases

$$\phi_u = \pm\pi \quad (43)$$

The function $-\mathcal{U}(\phi)$ reaches a local maximum at these phases.

5 Fixed point phase $\phi_f = 0$

For phase $\phi = 0$, equations (15) through (20) give

$$\mathcal{U}' = 0, \quad \mathcal{U}''' = 0, \quad \mathcal{U}'''' = 0 \quad (44)$$

$$\mathcal{U}'' = -1 + 2Q - 3R \quad (45)$$

$$\mathcal{U}'''' = 1 - 8Q + 27R \quad (46)$$

$$\mathcal{U}'''''' = -1 + 32Q - 243R. \quad (47)$$

We therefore have fixed point phase

$$\phi_f = 0 \quad (48)$$

which is stable if

$$2Q - 1 - 3R < 0 \quad (49)$$

and unstable if

$$2Q - 1 - 3R > 0. \quad (50)$$

The case

$$2Q - 1 - 3R = 0 \quad (51)$$

will be considered in Section 13.

6 Separatrix and area enclosed

Let us assume that we have stable fixed point phase $\phi = 0$, unstable fixed point phases $\phi = \pm\pi$, and one stable and one unstable fixed point phase between 0 and π such that

$$0 < \phi_u < \phi_s < \pi. \quad (52)$$

We assume that there is an additional phase ϕ_e that satisfies the equation

$$\mathcal{U}(\phi_e) = \mathcal{U}(\phi_u) \quad (53)$$

with

$$0 < \phi_u < \phi_s < \phi_e < \pi. \quad (54)$$

Because of the identities

$$\mathcal{U}(-\phi) = \mathcal{U}(\phi), \quad \mathcal{U}'(-\phi) = -\mathcal{U}'(\phi), \quad \mathcal{U}''(-\phi) = \mathcal{U}''(\phi) \quad (55)$$

we also have unstable and stable fixed point phases $-\phi_u$ and $-\phi_s$, and the phase $-\phi_e$ satisfies

$$\mathcal{U}(-\phi_e) = \mathcal{U}(-\phi_u). \quad (56)$$

The separatrix of interest is the curve $W(\phi)$ given by [6]

$$W^2(\phi) = \frac{2A_1}{a} (\mathcal{U}_u - \mathcal{U}) \quad (57)$$

where

$$a = \left(\frac{\hbar^2 \omega^2 \eta}{\beta^2 E} \right) = \left(\frac{\hbar^2 c^2 \eta}{R^2 E} \right), \quad A_1 = \frac{eQV_1}{2\pi\hbar}. \quad (58)$$

This is a closed curve that consists of a central and two outer lobes. The central lobe extends from $-\phi_u$ to ϕ_u and the outer lobes extend from $-\phi_e$ to $-\phi_u$ and from ϕ_u to ϕ_e . Because of the symmetry $\mathcal{U}(-\phi) = \mathcal{U}(\phi)$, the areas enclosed by the two outer lobes are equal.

The area of the central lobe is

$$\mathcal{A}_c = 2 \left(\frac{2A_1}{|a|} \right)^{1/2} \int_{-\phi_u}^{\phi_u} (\mathcal{U} - \mathcal{U}_u)^{1/2} d\phi \quad (59)$$

and that of each outer lobe is

$$\mathcal{A}_e = 2 \left(\frac{2A_1}{|a|} \right)^{1/2} \int_{\phi_u}^{\phi_e} (\mathcal{U} - \mathcal{U}_u)^{1/2} d\phi. \quad (60)$$

Here it is convenient to express $2A_1/|a|$ in terms of

$$\mathcal{B}_1 = 8 \frac{R}{\hbar c} \left\{ \frac{2eQV_1 E}{\pi \hbar |\eta|} \right\}^{1/2} \quad (61)$$

which is the area of the harmonic h bucket produced by V_1 . This gives

$$2 \left(\frac{2A_1}{|a|} \right)^{1/2} = \frac{\sqrt{2}}{8} \mathcal{B}_1. \quad (62)$$

We can then write

$$\mathcal{A}_c = \frac{\sqrt{2}}{8} \mathcal{B}_1 \mathcal{B}_c, \quad \mathcal{A}_e = \frac{\sqrt{2}}{8} \mathcal{B}_1 \mathcal{B}_e \quad (63)$$

where

$$\mathcal{B}_c = \int_{-\phi_u}^{\phi_u} (\mathcal{U} - \mathcal{U}_u)^{1/2} d\phi \quad (64)$$

and

$$\mathcal{B}_e = \int_{\phi_u}^{\phi_e} (\mathcal{U} - \mathcal{U}_u)^{1/2} d\phi. \quad (65)$$

If parameters Q and R are adjusted so that

$$\mathcal{B}_e = \mathcal{B}_c \quad (66)$$

then the area of each outer lobe of the separatrix will equal the area of the central lobe. In order to have a 3 to 1 merge that preserves the distribution of particles with respect to longitudinal oscillation amplitude, we need to maintain the equality (66) while reducing the total area enclosed by the separatrix as demonstrated in [1]. This must be done adiabatically.

If we have particular values of Q and R for which (66) is satisfied, then it will be satisfied for any voltages V_1, V_2, V_3 that satisfy

$$\frac{V_2}{V_1} = Q, \quad \frac{V_3}{V_1} = R. \quad (67)$$

7 Conditions that give lobes of equal area

The separatrix described in the previous section requires fixed point phases ϕ_u and ϕ_s such that

$$0 < \phi_u < \phi_s < \pi. \quad (68)$$

According to (32) these must satisfy

$$1 - 2QC_f + R(4C_f^2 - 1) = 0 \quad (69)$$

which has solutions

$$C_f = \frac{Q}{4R} \left\{ 1 \pm \sqrt{1 + D} \right\} \quad (70)$$

where

$$D = \frac{4R}{Q^2}(R - 1). \quad (71)$$

This gives two distinct and real fixed point phases provided

$$1 + D > 0 \quad (72)$$

and

$$|C_f| < 1. \quad (73)$$

With the help of the identity

$$(2R - 1)^2 = 4R^2 - 4R + 1 \quad (74)$$

we have

$$1 + D = \frac{1}{Q^2} \left\{ Q^2 + (2R - 1)^2 - 1 \right\} \quad (75)$$

and the condition (72) becomes

$$Q^2 + (2R - 1)^2 > 1. \quad (76)$$

In **Sections 8, 9, 10, 11** we consider the special cases $R = Q$, $R = 1$, $R = 1/2$, and $Q = 1$ respectively. It is shown that the values of Q and R for which (66) is satisfied are

$$R = Q = 1.724\,5964\,8340 \quad (77)$$

$$R = 1, \quad Q = 1.430\,2845\,6621 \quad (78)$$

$$R = 1/2, \quad Q = 1.119\,7590\,0273 \quad (79)$$

$$Q = 1, \quad R = 0.3657\,2039\,31952 \quad (80)$$

for the four cases. The corresponding phases ϕ_u, ϕ_s, ϕ_e are

$$\phi_u = 48.75^\circ, \quad \phi_s = 99.17^\circ, \quad \phi_e = 124.65^\circ \quad (81)$$

$$\phi_u = 44.345^\circ, \quad \phi_s = 90^\circ, \quad \phi_e = 110.95^\circ \quad (82)$$

$$\phi_u = 35.73^\circ, \quad \phi_s = 72.06^\circ, \quad \phi_e = 86.79^\circ \quad (83)$$

$$\phi_u = 29.87^\circ, \quad \phi_s = 60^\circ, \quad \phi_e = 71.55^\circ \quad (84)$$

respectively.

8 Lobes of equal area obtained with $R = Q$

For the case in which $R = Q$ we have

$$C_f = \frac{1}{4} \left\{ 1 \pm \sqrt{1 + D} \right\} \quad (85)$$

$$1 + D = 5 - \frac{4}{Q} \quad (86)$$

$$\mathcal{U}_f'' = -C_f + 2Q(2C_f^2 - 1) - 3Q(4C_f^3 - 3C_f) \quad (87)$$

and

$$\mathcal{U}_f'' = -C_f - Q\{12C_f^3 - 4C_f^2 - 9C_f + 2\}. \quad (88)$$

Here $Q > 0$ and we require

$$1 + D > 0. \quad (89)$$

This gives

$$0 < 5 - \frac{4}{Q} < 5 \quad (90)$$

$$0 < \sqrt{5 - \frac{4}{Q}} < \sqrt{5} \quad (91)$$

and

$$1 \leq 1 + \sqrt{5 - \frac{4}{Q}} \leq 1 + \sqrt{5}. \quad (92)$$

For the upper sign in (85) we therefore have

$$\frac{1}{4} \leq C_f \leq \frac{1 + \sqrt{5}}{4} \quad (93)$$

which gives unstable fixed point phases

$$75.52^\circ \geq \phi_u \geq 36^\circ. \quad (94)$$

From (91) we also have

$$-\sqrt{5} \leq -\sqrt{5 - \frac{4}{Q}} \leq 0 \quad (95)$$

which gives

$$1 - \sqrt{5} \leq 1 - \sqrt{5 - \frac{4}{Q}} \leq 1. \quad (96)$$

For the lower sign in (85) we therefore have

$$\frac{1 - \sqrt{5}}{4} \leq C_f \leq \frac{1}{4} \quad (97)$$

which gives stable fixed point phases

$$108^\circ \geq \phi_s \geq 75.52^\circ. \quad (98)$$

By numerical integration of (64) and (65) one finds that (66) is satisfied if

$$R = Q = 1.724\,5964\,8340. \quad (99)$$

The corresponding phases ϕ_u, ϕ_s, ϕ_e are

$$\phi_u = 48.75^\circ, \quad \phi_s = 99.17^\circ, \quad \phi_e = 124.65^\circ. \quad (100)$$

9 Lobes of equal area obtained with $R = 1$

For the case in which

$$R = 1, \quad 0 < Q < 2 \quad (101)$$

equations (71), (70), and (33) become

$$D = 0 \quad (102)$$

$$C_f = \frac{Q}{4} (1 \pm 1) \quad (103)$$

and

$$\mathcal{U}_f'' = -C_f + 2Q(2C_f^2 - 1) - 3(4C_f^3 - 3C_f). \quad (104)$$

Taking the minus sign in (103) gives

$$C_f = 0, \quad \mathcal{U}_f'' = -2Q < 0 \quad (105)$$

and stable fixed point phases

$$\phi_s = \pm 90^\circ. \quad (106)$$

Taking the plus sign in (103) gives

$$C_f = \frac{Q}{2}, \quad C_f^2 = \frac{Q^2}{4}, \quad C_f^3 = \frac{Q^3}{8} \quad (107)$$

$$\mathcal{U}_f'' = -\frac{Q}{2} + Q^3 - 2Q - \frac{3Q^3}{2} + \frac{9Q}{2} \quad (108)$$

$$\mathcal{U}_f'' = 2Q - \frac{Q^3}{2} = \frac{Q}{2} (4 - Q^2) \quad (109)$$

$$\mathcal{U}_f'' = -\frac{Q}{2} (Q - 2)(Q + 2) > 0 \quad (110)$$

and unstable fixed point phases

$$\phi_u = \pm \arccos(Q/2). \quad (111)$$

By numerical integration of (64) and (65) one finds that (66) is satisfied if

$$Q = 1.430\,2845\,6621. \quad (112)$$

The corresponding phases ϕ_u, ϕ_s, ϕ_e are

$$\phi_u = 44.345^\circ, \quad \phi_s = 90^\circ, \quad \phi_e = 110.95^\circ. \quad (113)$$

10 Lobes of equal area obtained with $R = 1/2$

For the case in which

$$R = \frac{1}{2}, \quad 1 < Q < \frac{5}{4} \quad (114)$$

we have

$$C_f = \frac{Q}{2} \left\{ 1 \pm \sqrt{1+D} \right\} \quad (115)$$

$$1+D = 1 - \frac{1}{Q^2} = \frac{1}{Q^2} (Q^2 - 1) \quad (116)$$

and

$$C_f = \frac{1}{2} \left\{ Q \pm \sqrt{Q^2 - 1} \right\} \quad (117)$$

where

$$1 < Q + \sqrt{Q^2 - 1} < 2 \quad (118)$$

and

$$Q - \frac{3}{4} < Q - \sqrt{Q^2 - 1} < Q. \quad (119)$$

By numerical integration of (64) and (65) one finds that (66) is satisfied if

$$Q = 1.119\,7590\,0273. \quad (120)$$

The corresponding phases ϕ_u, ϕ_s, ϕ_e are

$$\phi_u = 35.73^\circ, \quad \phi_s = 72.06^\circ, \quad \phi_e = 86.79^\circ. \quad (121)$$

11 Lobes of equal area obtained with $Q = 1$

For the case in which

$$Q = 1, \quad \frac{1}{3} < R < \frac{1}{2} \quad (122)$$

equations (71), (70), and (33) give

$$D = 4R(R - 1) < 0 \quad (123)$$

$$1 + D = (2R - 1)^2 < 1 \quad (124)$$

$$C_f = \frac{1}{4R} \{1 \pm (2R - 1)\} \quad (125)$$

and

$$\mathcal{U}_f'' = -C_f + 2(2C_f^2 - 1) - 3R(4C_f^3 - 3C_f). \quad (126)$$

Taking the plus sign in (125) gives

$$C_f = \frac{1}{2}, \quad \mathcal{U}_f'' = 3\left(R - \frac{1}{2}\right) < 0 \quad (127)$$

and stable fixed point phase

$$\phi_s = 60^\circ. \quad (128)$$

Taking the minus sign in (125) gives

$$C_f = \frac{1 - R}{2R} < 1 \quad (129)$$

and unstable fixed point phase

$$\phi_u = \arccos\left(\frac{1 - R}{2R}\right). \quad (130)$$

By numerical integration of (64) and (65) one finds that (66) is satisfied if

$$R = 0.3657\,2039\,31952. \quad (131)$$

The corresponding phases ϕ_u , ϕ_s , ϕ_e are

$$\phi_u = 29.87^\circ, \quad \phi_s = 60^\circ, \quad \phi_e = 71.55^\circ. \quad (132)$$

12 Assorted values of parameters (R, Q) that give lobes of equal area

Here we give various voltages (V_3, V_2, V_1) and associated parameters

$$R = \frac{V_3}{V_1}, \quad Q = \frac{V_2}{V_1} \quad (133)$$

for which (66) is satisfied. These have been obtained by numerical integration of (64) and (65). The associated phases (ϕ_u, ϕ_s, ϕ_e) are given in degrees. The single-lobe areas are given in units of the area of the single harmonic $3h$ bucket obtained with

$$(V_3, V_2, V_1) = (1.25, 0, 0). \quad (134)$$

In order of decreasing lobe area we have

$$\underline{(V_3, V_2, V_1) = (1.25, 0, 0)} \quad (135)$$

$$(\phi_u, \phi_s, \phi_e) = (60, 120, 180) \quad (136)$$

$$\text{Area} = 1.00000 \quad (137)$$

$$\underline{(V_3, V_2, V_1) = (1.25, 0.695, 0.3001)} \quad (138)$$

$$(R, Q) = (4.16527824059, 2.31589470177) \quad (139)$$

$$(\phi_u, \phi_s, \phi_e) = (53.3808, 108.5719, 140.8947) \quad (140)$$

$$\text{Area} = 0.68125 \quad (141)$$

$$\underline{(V_3, V_2, V_1) = (1.25, 1.25, 0.72480723)} \quad (142)$$

$$(R, Q) = (1.72459648340, 1.72459648340) \quad (143)$$

$$(\phi_u, \phi_s, \phi_e) = (48.7524, 99.1671, 124.6547) \quad (144)$$

$$\text{Area} = 0.49557 \quad (145)$$

$$\underline{(V_3, V_2, V_1)} = (1.03455, 1.25, 0.799380) \quad (146)$$

$$(R, Q) = (1.29419049764, 1.56371187670) \quad (147)$$

$$(\phi_u, \phi_s, \phi_e) = (46.6176, 94.7459, 117.8746) \quad (148)$$

$$\text{Area} = 0.384 \quad (149)$$

$$\underline{(V_3, V_2, V_1)} = (0.87395196, 1.25, 0.87395196) \quad (150)$$

$$(R, Q) = (1, 1.43028456621) \quad (151)$$

$$(\phi_u, \phi_s, \phi_e) = (44.3451, 90, 110.9511) \quad (152)$$

$$\text{Area} = 0.29409 \quad (153)$$

$$\underline{(V_3, V_2, V_1)} = (0.742964, 1.25, 0.9536) \quad (154)$$

$$(R, Q) = (0.779114932886, 1.31082214765) \quad (155)$$

$$(\phi_u, \phi_s, \phi_e) = (41.7336, 84.5500, 103.3358) \quad (156)$$

$$\text{Area} = 0.21679 \quad (157)$$

$$\underline{(V_3, V_2, V_1)} = (0.611976, 1.25, 1.06050) \quad (158)$$

$$(R, Q) = (0.577063649222, 1.17868929750) \quad (159)$$

$$(\phi_u, \phi_s, \phi_e) = (37.9007, 76.5731, 92.6497) \quad (160)$$

$$\text{Area} = 0.13726 \quad (161)$$

$$\underline{(V_3, V_2, V_1)} = (0.558155816095, 1.25, 1.11631163219) \quad (162)$$

$$(R, Q) = (0.5, 1.11975900273) \quad (163)$$

$$(\phi_u, \phi_s, \phi_e) = (35.7276, 72.0639, 86.7878) \quad (164)$$

$$\text{Area} = 0.10483 \quad (165)$$

$$\underline{(V_3, V_2, V_1)} = (0.546482, 1.25, 1.12962) \quad (166)$$

$$(R, Q) = (0.483775074804, 1.10656681008) \quad (167)$$

$$(\phi_u, \phi_s, \phi_e) = (35.1885, 70.9484, 85.3541) \quad (168)$$

$$\text{Area} = 0.09791 \quad (169)$$

$$\underline{(V_3, V_2, V_1)} = (0.480988, 1.25, 1.21420) \quad (170)$$

$$(R, Q) = (0.396135727228, 1.02948443420) \quad (171)$$

$$(\phi_u, \phi_s, \phi_e) = (31.5443, 63.4375, 75.8393) \quad (172)$$

$$\text{Area} = 0.06047 \quad (173)$$

$$\underline{(V_3, V_2, V_1)} = (0.457150491494, 1.25, 1.25) \quad (174)$$

$$(R, Q) = (0.3657203931952, 1) \quad (175)$$

$$(\phi_u, \phi_s, \phi_e) = (29.8692, 60, 71.5536) \quad (176)$$

$$\text{Area} = 0.04777 \quad (177)$$

$$\underline{(V_3, V_2, V_1)} = (0.415494, 1.25, 1.32080) \quad (178)$$

$$(R, Q) = (0.314577528770, 0.946396123561) \quad (179)$$

$$(\phi_u, \phi_s, \phi_e) = (26.2461, 52.6032, 62.4486) \quad (180)$$

$$\text{Area} = 0.02758 \quad (181)$$

$$\underline{(V_3, V_2, V_1)} = (0.35, 1.25, 1.46045) \quad (182)$$

$$(R, Q) = (0.239652162005, 0.855900578589) \quad (183)$$

$$(\phi_u, \phi_s, \phi_e) = (17.0159, 33.9535, 39.9929) \quad (184)$$

$$\text{Area} = 0.00461 \quad (185)$$

$$\underline{(V_3, V_2, V_1)} = (0.33, 1.25, 1.5124031) \quad (186)$$

$$(R, Q) = (0.218195797139, 0.826499231587) \quad (187)$$

$$(\phi_u, \phi_s, \phi_e) = (11.9134, 23.7262, 27.8728) \quad (188)$$

$$\text{Area} = 0.001087 \quad (189)$$

$$\underline{(V_3, V_2, V_1)} = (0.3130, 1.25, 1.56100205945) \quad (190)$$

$$(R, Q) = (0.200512227454, 0.800767681524) \quad (191)$$

$$(\phi_u, \phi_s, \phi_e) = (2.0582, 4.0924, 4.7965) \quad (192)$$

$$\text{Area} = 0.9523 \times 10^{-6} \quad (193)$$

and finally

$$\underline{(V_3, V_2, V_1)} = (0.3125, 1.25, 1.5625) \quad (194)$$

$$(R, Q) = (1/5, 4/5) \quad (195)$$

$$(\phi_u, \phi_s, \phi_e) = (0, 0, 0) \quad (196)$$

$$\text{Area} = 0. \quad (197)$$

The sequence of R and Q values and corresponding voltages given above can be used to produce a 3 to 1 merge that preserves the distribution of particles with respect to longitudinal oscillation amplitude. This is accomplished by adiabatically reducing the area enclosed by the three-lobed separatrix while keeping the areas of the individual lobes equal to one another. The voltages can be multiplied by a common scaling factor to make the scheme work for any given longitudinal emittance in any given machine.

13 The limit of zero lobe area

The data in the previous section show that the limit of zero lobe area is reached when

$$(R, Q) = (1/5, 4/5). \quad (198)$$

Putting these values into the equations

$$D = \frac{4R}{Q^2}(R - 1) \quad (199)$$

$$C_f = \frac{Q}{4R} \{1 \pm \sqrt{1 + D}\} \quad (200)$$

gives

$$D = -1, \quad C_f = 1. \quad (201)$$

This shows that for parameter values $(R, Q) = (1/5, 4/5)$ there are no fixed point phases ϕ_u and ϕ_s that satisfy

$$0 < \phi_u < \phi_s < \pi. \quad (202)$$

If, however, R and Q are moved slightly away from these values, one can obtain ϕ_u and ϕ_s that satisfy (202). This is shown by (191) and (192).

Further insight can be gained by returning to **Section 5** and examining the case for which

$$\mathcal{U}'' = 2Q - 1 - 3R = 0. \quad (203)$$

Multiplying (203) by nine gives

$$27R = 18Q - 9, \quad (204)$$

which inserted into (46) gives

$$\mathcal{U}'''' = 10Q - 8. \quad (205)$$

If we now take

$$Q = 4/5 \quad (206)$$

we have

$$\mathcal{U}'''' = 0 \quad (207)$$

and (203) gives

$$R = 1/5. \quad (208)$$

According to (44) through (47) we then have, at phase $\phi = 0$,

$$\mathcal{U}' = \mathcal{U}'' = \mathcal{U}''' = \mathcal{U}'''' = \mathcal{U}''''' = 0 \quad (209)$$

and

$$\mathcal{U}'''''' = -24. \quad (210)$$

We therefore have stable fixed point phase $\phi_s = 0$. This also can be seen by substituting

$$Q = 4/5, \quad R = 1/5 \quad (211)$$

into (28) which gives, for any phase ϕ ,

$$-\mathcal{U}' = \left\{ 1 - \frac{8}{5}C + \frac{1}{5}(4C^2 - 1) \right\} S \quad (212)$$

where

$$C = \cos \phi, \quad S = \sin \phi. \quad (213)$$

This reduces to

$$-\mathcal{U}' = \frac{4}{5}(C - 1)^2 S \quad (214)$$

which shows that for

$$-\pi < \phi < 0 \quad (215)$$

the function $-\mathcal{U}$ is monotonically decreasing, while for

$$0 < \phi < \pi \quad (216)$$

it is monotonically increasing. The function $-\mathcal{U}(\phi)$ therefore has a broad minimum at $\phi = 0$ with no fixed point phases between 0 and π . (We note in passing that this is just what one would want to have in order to obtain a flat bunch sitting in a triple harmonic bucket.) If the special conditions that produce this broad minimum are slightly perturbed then one can obtain two small phases ϕ_u and ϕ_s that satisfy (202).

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