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3D Fluid Theory of the Plasma Cascade Instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

The plasma cascade instabilty (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. The actual system is quite complicated and this note discusses a highly simplified model.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates x, y, z, t. The unperturbed plasma has a velocity distribution

$$\mathbf{v}_0(x,y,z,t) = \hat{x}x\omega_x(t) + \hat{y}y\omega_y(t) + \hat{z}z\omega_z(t). \tag{1}$$

The unperturbed density obeys

$$\frac{\partial n_0}{\partial t} + \nabla\cdot(\mathbf{v}_0 n_0) = 0. \tag{2}$$

Taking $n_0 = n_0(t)$ yields

$$\frac{dn_0}{dt} + (\omega_x(t) + \omega_y(t) + \omega_z(t))n_0 = 0. \tag{3}$$

Defining $\omega_\alpha(t) = \dot{\Phi}_\alpha(t)$, where the dot denotes a time derivative, gives $n_0(t) = \hat{n}_0 \exp(-\Phi_x(t) - \Phi_y(t) - \Phi_z(t))$. This defines our time dependent unperturbed distribution. We will not dwell on how the backround velocity distribution is generated and just mention it is a mixture of focusing from magnets, cavities and space charge forces. For cooling systems with the beam propagating along z it is likely that $\omega_z \approx 0$ but we keep it to allow for general calculations.

Now consider a pertubation $\mathbf{v}_1(x,y,z,t)$, $n_1(x,y,z,t)$. We work to first order in perturbation theory so the force and particle conservation equations are

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 = q\mathbf{E}_1/m, \tag{4}$$

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(\mathbf{v}_1 n_0 + \mathbf{v}_0 n_1) = 0. \tag{5}$$

We are going to solve equations (4) and (5) using Fourier transforms with time dependent spatial wave numbers.

$$\mathbf{v}_1 = \tilde{\mathbf{v}}(t)\exp[ik_x\lambda_x(t)x + ik_y\lambda_y(t)y + ik_z\lambda_z(t)z] \equiv \tilde{\mathbf{v}}(t)\exp[i\Psi], \tag{6}$$

where the time dependent functions $\lambda_\alpha(t)$ remain to be determined. We also take $n_1 = \tilde{n}(t)\exp[i\Psi]$. Inserting these in (4) and (5) and defining $\mathbf{E}_1 = \tilde{\mathbf{E}}(t)\exp(i\Psi)$ gives

$$\frac{\partial}{\partial t}\left[\tilde{v}_x(t)e^{i\Psi}\right] + \tilde{v}_x\omega_x(t)e^{i\Psi} + \left\{x\omega_x\frac{\partial}{\partial x} + y\omega_y\frac{\partial}{\partial y} + z\omega_z\frac{\partial}{\partial z}\right\}\left[\tilde{v}_x e^{i\Psi}\right] = e^{i\Psi}q\tilde{E}_x/m \tag{7}$$

$$\frac{\partial}{\partial t}\left[\tilde{v}_y(t)e^{i\Psi}\right] + \tilde{v}_y\omega_y(t)e^{i\Psi} + \left\{x\omega_x\frac{\partial}{\partial x} + y\omega_y\frac{\partial}{\partial y} + z\omega_z\frac{\partial}{\partial z}\right\}\left[\tilde{v}_y e^{i\Psi}\right] = e^{i\Psi}q\tilde{E}_y/m \tag{8}$$

$$\frac{\partial}{\partial t}\left[\tilde{v}_z(t)e^{i\Psi}\right] + \tilde{v}_z\omega_z(t)e^{i\Psi} + \left\{x\omega_x\frac{\partial}{\partial x} + y\omega_y\frac{\partial}{\partial y} + z\omega_z\frac{\partial}{\partial z}\right\}\left[\tilde{v}_z e^{i\Psi}\right] = e^{i\Psi}q\tilde{E}_z/m \tag{9}$$

$$\frac{\partial}{\partial t}\left[\tilde{n}_1(t)e^{i\Psi}\right] + \tilde{n}_1(\omega_x+\omega_y+\omega_z)e^{i\Psi} + \left\{x\omega_x\frac{\partial}{\partial x} + y\omega_y\frac{\partial}{\partial y} + z\omega_z\frac{\partial}{\partial z}\right\}\left[\tilde{n}_1 e^{i\Psi}\right]$$
$$+n_0(t)\left[\tilde{v}_x\frac{\partial}{\partial x} + \tilde{v}_y\frac{\partial}{\partial y} + \tilde{v}_z\frac{\partial}{\partial z}\right]e^{i\Psi} = 0 \tag{10}$$

It is now clear how to choose the λ_αs. We demand

$$\frac{\partial}{\partial t}e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} e^{i\Psi} = 0, \tag{11}$$

which leads to

$$\dot{\lambda}_\alpha + \omega_\alpha \lambda_\alpha = 0, \tag{12}$$

where $\alpha = x, y, z$. The solution is $\lambda_\alpha(t) = \exp(-\Phi_\alpha(t))$. Equations (7) through (10) become

$$\dot{\tilde{v}}_x + \omega_x \tilde{v}_x = q\tilde{E}_x/m \tag{13}$$

$$\dot{\tilde{v}}_y + \omega_y \tilde{v}_y = q\tilde{E}_y/m \tag{14}$$

$$\dot{\tilde{v}}_z + \omega_z \tilde{v}_x = q\tilde{E}_z/m \tag{15}$$

$$\dot{\tilde{n}} + (\omega_x + \omega_y + \omega_z)\tilde{n} + n_0(t)(ik_x\lambda_x\tilde{v}_x + ik_y\lambda_y\tilde{v}_y + ik_z\lambda_z\tilde{v}_z) = 0 \tag{16}$$

To close the equations we use Gauss' law, $\nabla \cdot \mathbf{E} = 4\pi qn$. Since everything varies as $\exp(i\Psi) \equiv \exp(i\mathbf{K}\cdot\mathbf{r})$ we have

$$\mathbf{E} = -\frac{i\mathbf{K}}{K^2} 4\pi qn + \mathbf{E}_{drive,\mathbf{K}}, \tag{17}$$

where $\mathbf{E}_{drive,\mathbf{K}}$ is the spatial Fourier component of $\mathbf{r}Q/r^3$, the electric field due to a driving ion.

Now

$$\frac{Q\mathbf{r}}{r^3} = -i\lambda_x\lambda_y\lambda_z \frac{4\pi Q}{(2\pi)^3} \int d^3k \frac{(k_x\lambda_x, k_y\lambda_y, k_z\lambda_z)}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} e^{ik_x\lambda_x x + ik_y\lambda_y y + ikz\lambda_z z},$$

which is easily checked by taking the divergence, using Gauss law on the left side and the definition of the 3 dimensional delta function on the right side.

To keep our dynamics correct we need to sum quantities according to

$$F_1(x, y, z, t) = \int d^3k \tilde{F}(\mathbf{k}, t) \exp(ik_x\lambda_x x + \ldots)$$

where F can be any of our small quantities. With this convention the terms in equations (13) through (15) are related to the perturbed density and the external drive via

$$\tilde{\mathbf{E}} = (k_x\lambda_x, k_y\lambda_y, k_z\lambda_z) \left\{ \frac{-4\pi i[q\tilde{n} + H(t)Q\lambda_x\lambda_y\lambda_z]}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} \right\}. \tag{18}$$

Where $H(t)$ is the Heavyside function so that the ion is introduced at rest at $t = 0$. For cooling both $\tilde{v}$ and $\tilde{n}$ vanish at $t = 0$ so equations (13) through (16) are well posed ordinary differential equations. Numerically integrating these equations as well as a Vlasov analysis are left for future work.

[1] V. N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, arxiv:1802.08677, February 2018

[2] V. N. Litvinenko, G. Wang, Y. Jing, D. Kayran, J. Ma, I. Petrushina,I. Pinayev, K. Shih, arxiv:1902.10846, February 2019