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July 2019

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**U.S. Department of Energy**

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

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# 3D Fluid Theory of the Plasma Cascade Instability

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The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. The actual system is quite complicated and this note discusses a highly simplified model.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates  $x, y, z, t$ . The unperturbed plasma has a velocity distribution

$$\mathbf{v}_0(x, y, z, t) = \hat{x}x\omega_x(t) + \hat{y}y\omega_y(t) + \hat{z}z\omega_z(t). \quad (1)$$

The unperturbed density obeys

$$\frac{\partial n_0}{\partial t} + \nabla \cdot (\mathbf{v}_0 n_0) = 0. \quad (2)$$

Taking  $n_0 = n_0(t)$  yields

$$\frac{dn_0}{dt} + (\omega_x(t) + \omega_y(t) + \omega_z(t))n_0 = 0. \quad (3)$$

Defining  $\omega_\alpha(t) = \dot{\Phi}_\alpha(t)$ , where the dot denotes a time derivative, gives  $n_0(t) = \hat{n}_0 \exp(-\Phi_x(t) - \Phi_y(t) - \Phi_z(t))$ . This defines our time dependent unperturbed distribution. We will not dwell on how the background velocity distribution is generated and just mention it is a mixture of focusing from magnets, cavities and space charge forces. For cooling systems with the beam propagating along  $z$  it is likely that  $\omega_z \approx 0$  but we keep it to allow for general calculations.

Now consider a perturbation  $\mathbf{v}_1(x, y, z, t)$ ,  $n_1(x, y, z, t)$ . We work to first order in perturbation theory so the force and particle conservation equations are

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1 = q\mathbf{E}_1/m, \quad (4)$$

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (\mathbf{v}_1 n_0 + \mathbf{v}_0 n_1) = 0. \quad (5)$$

We are going to solve equations (4) and (5) using Fourier transforms with time dependent spatial wave numbers.

$$\mathbf{v}_1 = \tilde{\mathbf{v}}(t) \exp[ik_x \lambda_x(t)x + ik_y \lambda_y(t)y + ik_z \lambda_z(t)z] \equiv \tilde{\mathbf{v}}(t) \exp[i\Psi], \quad (6)$$

where the time dependent functions  $\lambda_\alpha(t)$  remain to be determined. We also take  $n_1 = \tilde{n}(t) \exp[i\Psi]$ . Inserting these in (4) and (5) and defining  $\mathbf{E}_1 = \tilde{\mathbf{E}}(t) \exp(i\Psi)$  gives

$$\frac{\partial}{\partial t} [\tilde{v}_x(t)e^{i\Psi}] + \tilde{v}_x\omega_x(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_x e^{i\Psi}] = e^{i\Psi} q\tilde{E}_x/m \quad (7)$$

$$\frac{\partial}{\partial t} [\tilde{v}_y(t)e^{i\Psi}] + \tilde{v}_y\omega_y(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_y e^{i\Psi}] = e^{i\Psi} q\tilde{E}_y/m \quad (8)$$

$$\frac{\partial}{\partial t} [\tilde{v}_z(t)e^{i\Psi}] + \tilde{v}_z\omega_z(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_z e^{i\Psi}] = e^{i\Psi} q\tilde{E}_z/m \quad (9)$$

$$\begin{aligned} \frac{\partial}{\partial t} [\tilde{n}_1(t)e^{i\Psi}] + \tilde{n}_1(\omega_x + \omega_y + \omega_z)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{n}_1 e^{i\Psi}] \\ + n_0(t) \left[ \tilde{v}_x \frac{\partial}{\partial x} + \tilde{v}_y \frac{\partial}{\partial y} + \tilde{v}_z \frac{\partial}{\partial z} \right] e^{i\Psi} = 0 \end{aligned} \quad (10)$$

It is now clear how to choose the  $\lambda_\alpha$ s. We demand

$$\frac{\partial}{\partial t} e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} e^{i\Psi} = 0, \quad (11)$$

which leads to

$$\dot{\lambda}_\alpha + \omega_\alpha \lambda_\alpha = 0, \quad (12)$$

where  $\alpha = x, y, z$ . The solution is  $\lambda_\alpha(t) = \exp(-\Phi_\alpha(t))$ . Equations (7) through (10) become

$$\dot{\tilde{v}}_x + \omega_x \tilde{v}_x = q\tilde{E}_x/m \quad (13)$$

$$\dot{\tilde{v}}_y + \omega_y \tilde{v}_y = q\tilde{E}_y/m \quad (14)$$

$$\dot{\tilde{v}}_z + \omega_z \tilde{v}_z = q\tilde{E}_z/m \quad (15)$$

$$\dot{\tilde{n}} + (\omega_x + \omega_y + \omega_z)\tilde{n} + n_0(t)(ik_x\lambda_x\tilde{v}_x + ik_y\lambda_y\tilde{v}_y + ik_z\lambda_z\tilde{v}_z) = 0 \quad (16)$$

To close the equations we use Gauss' law,  $\nabla \cdot \mathbf{E} = 4\pi qn$ . Since everything varies as  $\exp(i\Psi) \equiv \exp(i\mathbf{K} \cdot \mathbf{r})$  we have

$$\mathbf{E} = -\frac{i\mathbf{K}}{K^2} 4\pi qn + \mathbf{E}_{drive, \mathbf{K}}, \quad (17)$$

where  $\mathbf{E}_{drive, \mathbf{K}}$  is the spatial Fourier component of  $\mathbf{r}Q/r^3$ , the electric field due to a driving ion.

Now

$$\frac{Q\mathbf{r}}{r^3} = -i\lambda_x\lambda_y\lambda_z \frac{4\pi Q}{(2\pi)^3} \int d^3k \frac{(k_x\lambda_x, k_y\lambda_y, k_z\lambda_z)}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} e^{ik_x\lambda_x x + ik_y\lambda_y y + ik_z\lambda_z z},$$

which is easily checked by taking the divergence, using Gauss law on the left side and the definition of the 3 dimensional delta function on the right side.

To keep our dynamics correct we need to sum quantities according to

$$F_1(x, y, z, t) = \int d^3k \tilde{F}(\mathbf{k}, t) \exp(ik_x\lambda_x x + \dots)$$

where  $F$  can be any of our small quantities. With this convention the terms in equations (13) through (15) are related to the perturbed density and the external drive via

$$\tilde{\mathbf{E}} = (k_x\lambda_x, k_y\lambda_y, k_z\lambda_z) \left\{ \frac{-4\pi i[q\tilde{n} + H(t)Q\lambda_x\lambda_y\lambda_z]}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} \right\}. \quad (18)$$

Where  $H(t)$  is the Heavyside function so that the ion is introduced at rest at  $t = 0$ . For cooling both  $\tilde{v}$  and  $\tilde{n}$  vanish at  $t = 0$  so equations (13) through (16) are well posed ordinary differential equations. Numerically integrating these equations as well as a Vlasov analysis are left for future work.

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[1] V. N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, arxiv:1802.08677, February 2018

[2] V. N. Litvinenko, G. Wang, Y. Jing, D. Kayran, J. Ma, I. Petrushina, I. Pinayev, K. Shih, arxiv:1902.10846, February 2019