

## 3D Fluid theory of the plasma cascade instability

M. Blaskiewicz

July 2019

Collider Accelerator Department  
**Brookhaven National Laboratory**

**U.S. Department of Energy**

USDOE Office of Science (SC), Nuclear Physics (NP) (SC-26)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No. DE-SC0012704 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

## **DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

# 3D Fluid Theory of the Plasma Cascade Instability

M. Blaskiewicz

BNL 911B, Upton, NY 11973, USA

The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. Existing theory is limited to wave propagation that is orthogonal to the advective compression direction. This note provides a theory allowing for wave propagation in arbitrary directions.

The plasma cascade instability (PCI) [1, 2] is a proposed mechanism for microbunching in electron beams without dipole magnets. If the theory bears out this process may well be very widespread, contributing to enhanced noise in a variety of systems employing electron beams. The actual system is quite complicated and this note discusses a highly simplified model.

Consider a homogeneous, infinite, electron plasma. We use Cartesian coordinates  $x, y, z, t$ . The unperturbed plasma has a velocity distribution

$$\mathbf{v}_0(x, y, z, t) = \hat{x}x\omega_x(t) + \hat{y}y\omega_y(t) + \hat{z}z\omega_z(t). \quad (1)$$

The unperturbed density obeys

$$\frac{\partial n_0}{\partial t} + \nabla \cdot (\mathbf{v}_0 n_0) = 0. \quad (2)$$

Taking  $n_0 = n_0(t)$  yields

$$\frac{dn_0}{dt} + (\omega_x(t) + \omega_y(t) + \omega_z(t))n_0 = 0. \quad (3)$$

Defining  $\omega_\alpha(t) = \dot{\Phi}_\alpha(t)$ , where the dot denotes a time derivative, gives  $n_0(t) = \hat{n}_0 \exp(-\Phi_x(t) - \Phi_y(t) - \Phi_z(t))$ . This defines our time dependent unperturbed distribution. We will not dwell on how the background velocity distribution is generated and just mention it is a mixture of focusing from magnets, cavities and space charge forces. For cooling systems with the beam propagating along  $z$  it is likely that  $\omega_z \approx 0$  but we keep it to allow for general calculations.

Now consider a perturbation  $\mathbf{v}_1(x, y, z, t)$ ,  $n_1(x, y, z, t)$ . We work to first order in perturbation theory so the force and particle conservation equations are

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_1 = q \mathbf{E}_1 / m, \quad (4)$$

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (\mathbf{v}_1 n_0 + \mathbf{v}_0 n_1) = 0. \quad (5)$$

We are going to solve equations (4) and (5) using Fourier transforms with time dependent spatial wave numbers.

$$\mathbf{v}_1 = \tilde{\mathbf{v}}(t) \exp[ik_x \lambda_x(t)x + ik_y \lambda_y(t)y + ik_z \lambda_z(t)z] \equiv \tilde{\mathbf{v}}(t) \exp[i\Psi], \quad (6)$$

where the time dependent functions  $\lambda_\alpha(t)$  remain to be determined. We also take  $n_1 = \tilde{n}(t) \exp[i\Psi]$ . Inserting these in (4) and (5) and defining  $\mathbf{E}_1 = \tilde{\mathbf{E}}(t) \exp(i\Psi)$  gives

$$\frac{\partial}{\partial t} [\tilde{v}_x(t)e^{i\Psi}] + \tilde{v}_x \omega_x(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_x e^{i\Psi}] = e^{i\Psi} q \tilde{E}_x / m \quad (7)$$

$$\frac{\partial}{\partial t} [\tilde{v}_y(t)e^{i\Psi}] + \tilde{v}_y \omega_y(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_y e^{i\Psi}] = e^{i\Psi} q \tilde{E}_y / m \quad (8)$$

$$\frac{\partial}{\partial t} [\tilde{v}_z(t)e^{i\Psi}] + \tilde{v}_z \omega_z(t)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{v}_z e^{i\Psi}] = e^{i\Psi} q \tilde{E}_z / m \quad (9)$$

$$\begin{aligned} \frac{\partial}{\partial t} [\tilde{n}_1(t)e^{i\Psi}] + \tilde{n}_1(\omega_x + \omega_y + \omega_z)e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} [\tilde{n}_1 e^{i\Psi}] \\ + n_0(t) \left[ \tilde{v}_x \frac{\partial}{\partial x} + \tilde{v}_y \frac{\partial}{\partial y} + \tilde{v}_z \frac{\partial}{\partial z} \right] e^{i\Psi} = 0 \end{aligned} \quad (10)$$

It is now clear how to choose the  $\lambda_\alpha$ s. We demand

$$\frac{\partial}{\partial t} e^{i\Psi} + \left\{ x\omega_x \frac{\partial}{\partial x} + y\omega_y \frac{\partial}{\partial y} + z\omega_z \frac{\partial}{\partial z} \right\} e^{i\Psi} = 0, \quad (11)$$

which leads to

$$\dot{\lambda}_\alpha + \omega_\alpha \lambda_\alpha = 0, \quad (12)$$

where  $\alpha = x, y, z$ . The solution is  $\lambda_\alpha(t) = \exp(-\Phi_\alpha(t))$ . Equations (7) through (10) become

$$\dot{\tilde{v}}_x + \omega_x \tilde{v}_x = q\tilde{E}_x/m \quad (13)$$

$$\dot{\tilde{v}}_y + \omega_y \tilde{v}_y = q\tilde{E}_y/m \quad (14)$$

$$\dot{\tilde{v}}_z + \omega_z \tilde{v}_z = q\tilde{E}_z/m \quad (15)$$

$$\dot{\tilde{n}} + (\omega_x + \omega_y + \omega_z)\tilde{n} + n_0(t)(ik_x\lambda_x\tilde{v}_x + ik_y\lambda_y\tilde{v}_y + ik_z\lambda_z\tilde{v}_z) = 0 \quad (16)$$

To close the equations we use Gauss' law,  $\nabla \cdot \mathbf{E} = 4\pi qn$ . Since everything varies as  $\exp(i\Psi) \equiv \exp(i\mathbf{K} \cdot \mathbf{r})$  we have

$$\mathbf{E} = -\frac{i\mathbf{K}}{K^2} 4\pi qn + \mathbf{E}_{drive, \mathbf{K}}, \quad (17)$$

where  $\mathbf{E}_{drive, \mathbf{K}}$  is the spatial Fourier component of  $\mathbf{r}Q/r^3$ , the electric field due to a driving ion.

Now

$$\frac{Q\mathbf{r}}{r^3} = -i\lambda_x\lambda_y\lambda_z \frac{4\pi Q}{(2\pi)^3} \int d^3k \frac{(k_x\lambda_x, k_y\lambda_y, k_z\lambda_z)}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} e^{ik_x\lambda_x x + ik_y\lambda_y y + ik_z\lambda_z z},$$

which is easily checked by taking the divergence, using Gauss law on the left side and the definition of the 3 dimensional delta function on the right side.

To keep our dynamics correct we need to sum quantities according to

$$F_1(x, y, z, t) = \int d^3k \tilde{F}(\mathbf{k}, t) \exp(ik_x\lambda_x x + \dots)$$

where  $F$  can be any of our small quantities. With this convention the terms in equations (13) through (15) are related to the perturbed density and the external drive via

$$\tilde{\mathbf{E}} = (k_x\lambda_x, k_y\lambda_y, k_z\lambda_z) \left\{ \frac{-4\pi i[q\tilde{n} + H(t)Q\lambda_x\lambda_y\lambda_z]}{k_x^2\lambda_x^2 + k_y^2\lambda_y^2 + k_z^2\lambda_z^2} \right\}. \quad (18)$$

Where  $H(t)$  is the Heavyside function so that the ion is introduced at rest at  $t = 0$ . For cooling both  $\tilde{v}$  and  $\tilde{n}$  vanish at  $t = 0$  so equations (13) through (16) are well posed ordinary differential equations. Numerically integrating these equations as well as a Vlasov analysis are left for future work.

---

[1] V. N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, arxiv:1802.08677, February 2018

[2] V. N. Litvinenko, G. Wang, Y. Jing, D. Kayran, J. Ma, I. Petrushina, I. Pinayev, K. Shih, arxiv:1902.10846, February 2019