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HOM Induced Energy Spread and Emittance Growth in LEReC system

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Introduction

To map the QCD phase diagram, especially to search the QCD critical point using the Relativistic Heavy Ion Collider (RHIC), significant luminosity improvement at energies below $\gamma = 10.7$ is required, which can be achieved with the help of an electron cooling upgrade called Low Energy RHIC electron Cooler (LEReC) (1).

An electron accelerator for LEReC linear accelerator (Linac) consists of a DC photoemission gun and a 704 MHz SRF booster cavity. The booster cavity is converted from the SRF gun of the ERL project (2). In LEReC Phase I (electron kinetic energies up to 2 MeV) a one cell 704 MHz normal conducting cavity and a 3-cell third harmonic (2.1 GHz) normal conducting cavity will be added to de-chirp the energy spread and to compensate its non-linearity.

The electron beam in the LEReC Linac is relatively soft, with its energy at around 2 MeV. The electron beam can be disturbed by the Higher Order Modes (HOMs) in the cavities, especially when considering the energy spread should be limited to $5e-4$ rms in dp/p , and the rms emittance should be limited to 2.5 mm-mrad. In this case the HOMs in the 704 MHz SRF booster cavity, and in the 2.1 GHz and 704 MHz normal conducting cavities should be carefully evaluated to ensure these limitations. In this paper, we calculate the energy spread from the longitudinal modes of the 704 MHz SRF cavity, as well as the 2.1 GHz and 704 MHz normal conducting cavities, we also calculate the emittance growth from the transverse modes of the 704 MHz SRF cavity. The HOM power estimation will be discussed elsewhere.

Energy spread from the longitudinal modes

To calculate the energy spread from the longitudinal modes, Eigen mode simulation is first done using CST Microwave Studio™, with the simulation frequency ranges from the fundamental mode to the first longitudinal cut-off of the beam pipe. Then the single bunch wake potential is constructed using the Eigen mode simulation results. This single bunch wake potential from the Eigen mode is further compared with the CST Particle Studio™ result. The multi bunch wake potential is calculated by shifting – adding the single bunch wake potential.

1, Eigen mode simulation

For the 2.1 GHz warm cavity, the beam pipe is 1.925" in diameter, with the cut-off frequency 3.60 GHz for TE_{11} mode and 4.70 GHz for TM_{01} mode. We calculate the HOMs using CST Microwave Studio™ Eigen mode simulation with frequency up to 4.70 GHz.

For the 704 MHz warm cavity, the beam pipe is 82 mm in diameter, with the cut-off frequency 2.14 GHz for TE₁₁ mode and 2.80 GHz for TM₀₁ mode. We calculate the HOMs using CST Microwave Studio™ Eigen mode simulation with frequency up to 2.80 GHz.

For the 704 MHz SRF booster cavity, the downstream (closer to 2.1 GHz warm cavity) side beam pipe is 50 mm in radius, and this pipe is further tapered to 30.2 mm radius before 2.1 GHz cavity. The beam pipe cut-off frequency for 30.2 mm radius is 2.91 GHz for TE₁₁ mode and 3.81 GHz for TM₀₁ mode. We calculate the HOMs using CST Microwave Studio™ Eigen mode simulation with frequency up to 3.81 GHz.

For all the longitudinal modes, we use the **accelerator definition** R/Q at $\beta=1$:

$$\left(\frac{R}{Q}\right) = \frac{|V_z|^2}{\omega U}$$

2, Single bunch wake potential

For each longitudinal HOM, the impedance is given by:

$$Z_{\parallel}(\omega) = \frac{R_s}{1 + jQ_r(\omega/\omega_r - \omega_r/\omega)}$$

where R_s is the shunt impedance in accelerator definition, Q_r the quality factor and ω_r the resonant frequency. The wake function can be found from its Inverse Fourier transform, and its real part with $Q_r \gg 1$ (narrow band HOMs) is (3):

$$G_{\parallel}(\tau) = \frac{\omega_r R_s}{2Q_r} e^{-\frac{\omega_r \tau}{2Q_r}} \cos(\omega_r \tau)$$

Please note in (3) the impedance is in circuit definition.

The single bunch is noted as $Q\lambda(t)$, the $\lambda(t)$ will be defined later for bunch with different shape, and Q the charge of the bunch.

The wake potential is the convolution of the wake function and the normalized density (3):

$$W_{\parallel}(\tau) = \int_0^{\infty} dt G_{\parallel}(t) \lambda(\tau - t) = \int_{-\infty}^{\tau} dt G_{\parallel}(\tau - t) \lambda(t)$$

a) Delta bunch

For a simplified model with a single bunch $Q\lambda(t)$ in dirac delta form with Q charge at $t=0$, with normalized density:

$$\lambda(t) = \begin{cases} na; & t = 0 \\ 0; & t > 0 \end{cases}$$

$$\text{and } \int_{-\infty}^{+\infty} \lambda(t) = 1.$$

After integration the wake potential can be written:

$$W_{\parallel}(\tau) = \frac{\omega_r R_s}{2Q_r} e^{-\frac{\omega_r \tau}{2Q_r}} \text{Cos}[\omega_r \tau] H[\tau]$$

H is the Heaviside step function with

$$H(t) = \begin{cases} 0; & t < 0 \\ 1; & t > 0 \end{cases}$$

b) Rectangular bunch

For a simplified model with a single bunch $Q\lambda(t)$ in rectangular form with Q charge uniformly distributed in $[0, T]$, with normalized density:

$$\lambda(t) = \begin{cases} 1/T; & 0 \leq t \leq T \\ 0; & t > T \end{cases}$$

$$\text{and } \int_{-\infty}^{+\infty} \lambda(t) = 1.$$

After integration the wake potential can be written (use $Q_r \gg 1$ again):

$$W_{\parallel}(\tau) = \begin{cases} \frac{1}{T} \frac{R_s}{4Q_r^2} e^{-\frac{\omega_r \tau}{2Q_r}} \left[e^{\frac{\omega_r \tau}{2Q_r}} - \cos(\omega_r \tau) + 2Q_r \sin(\omega_r \tau) \right] & ; 0 \leq \tau \leq T \\ \frac{1}{T} \frac{R_s}{4Q_r^2} e^{-\frac{\omega_r \tau}{2Q_r}} \left\{ \begin{array}{l} e^{\frac{\omega_r T}{2Q_r}} \cos[\omega_r(T - \tau)] \\ + 2Q_r e^{\frac{\omega_r T}{2Q_r}} \sin[\omega_r(T - \tau)] - \cos(\omega_r \tau) + 2Q_r \sin(\omega_r \tau) \end{array} \right\} & ; \tau > T \end{cases}$$

c) Gaussian bunch

For a simplified model with a single bunch $Q\lambda(t)$ in (half of the) Gaussian form with Q charge, with normalized density:

$$\lambda(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t)^2}{2\sigma^2}}$$

$$\text{and } \int_{-\infty}^{+\infty} \lambda(t) = 1.$$

After integration the wake potential can be written (use $Q_r \gg 1$ again) (3):

$$W_{\parallel}(\tau) = \frac{\omega_r R_s}{4Q_r} e^{-\frac{\tau^2}{2\sigma^2}} \operatorname{Re} \left\{ \left(\frac{j}{2Q_r} + 1 \right) w \left(\frac{z_1}{\sqrt{2}} \right) \right\}$$

with $z_1 = \omega_r \sigma + j \left[\frac{\omega_r \sigma}{2Q_r} - \frac{\tau}{\sigma} \right]$ and w-function related to the error function of complex argument:

$$\int_0^{+\infty} dt e^{-a^2 t^2 + jzt} = \frac{\sqrt{\pi}}{2a} w \left(\frac{z}{2a} \right)$$

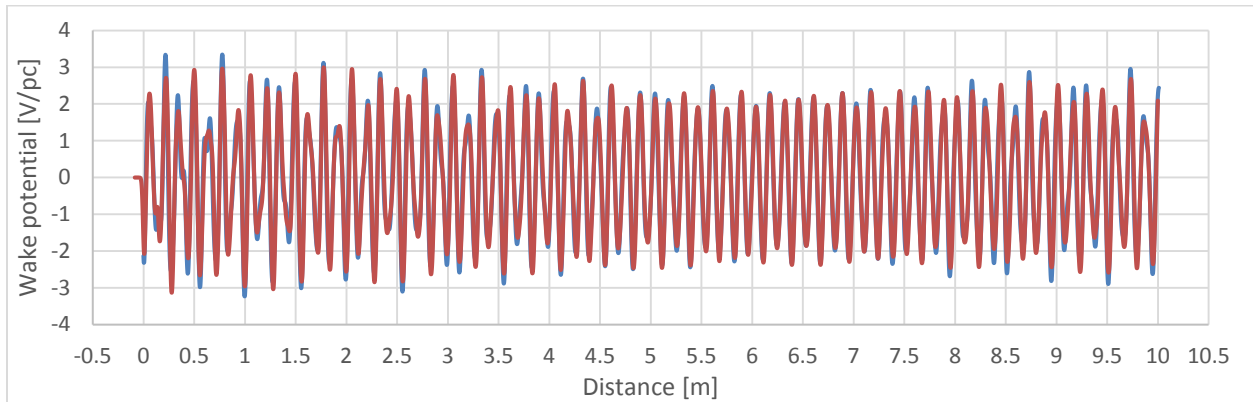
For bunch with any shape, while $\tau \rightarrow \infty$, and for narrow band HOMs with $Q_r \gg 1$, we have,

$$\begin{aligned} W_{\parallel}(\tau) &= \int_{-\infty}^{\tau} dt G_{\parallel}(\tau - t) \lambda(t) = \operatorname{Re} \left[\int_{-\infty}^{\tau} dt \frac{\omega_r R_s}{2Q_r} e^{\left(-\frac{1}{2Q_r} + i\right) \omega_r (\tau - t)} \lambda(t) \right] \\ &= \frac{\omega_r R_s}{2Q_r} \operatorname{Re} \left[e^{\left(-\frac{1}{2Q_r} + i\right) \omega_r \tau} \int_{-\infty}^{\tau} dt e^{-\left(-\frac{1}{2Q_r} + i\right) \omega_r t} \lambda(t) \right] \\ &= \frac{\omega_r R_s}{2Q_r} \operatorname{Re} \left[e^{\left(-\frac{1}{2Q_r} + i\right) \omega_r \tau} \tilde{I}(\omega_r) \right] \end{aligned}$$

3, Single bunch wake potential from CST Particle Studio™

In this simulation, we use 100 pC bunch charge with beam velocity at the speed of light, and 5 mm bunch length to get accurate results on higher frequencies up to 20 GHz (versus actual 10 mm with 10 GHz). We choose indirect interfaces wake integration method. Fig. 1 and Fig. 2 show the single bunch wake potential for 2.1 GHz warm cavity (top); 704 MHz warm cavity (middle); and 704 MHz booster SRF cavity (bottom). Red curves are from the reconstruction of Eigen mode simulation, and blue curves are from CST Particle Studio™. Fig. 1 is with 10 m wake length and Fig. 2 is with 1 m wake length.

(Show the difference between reconstruction and Particle Studio)



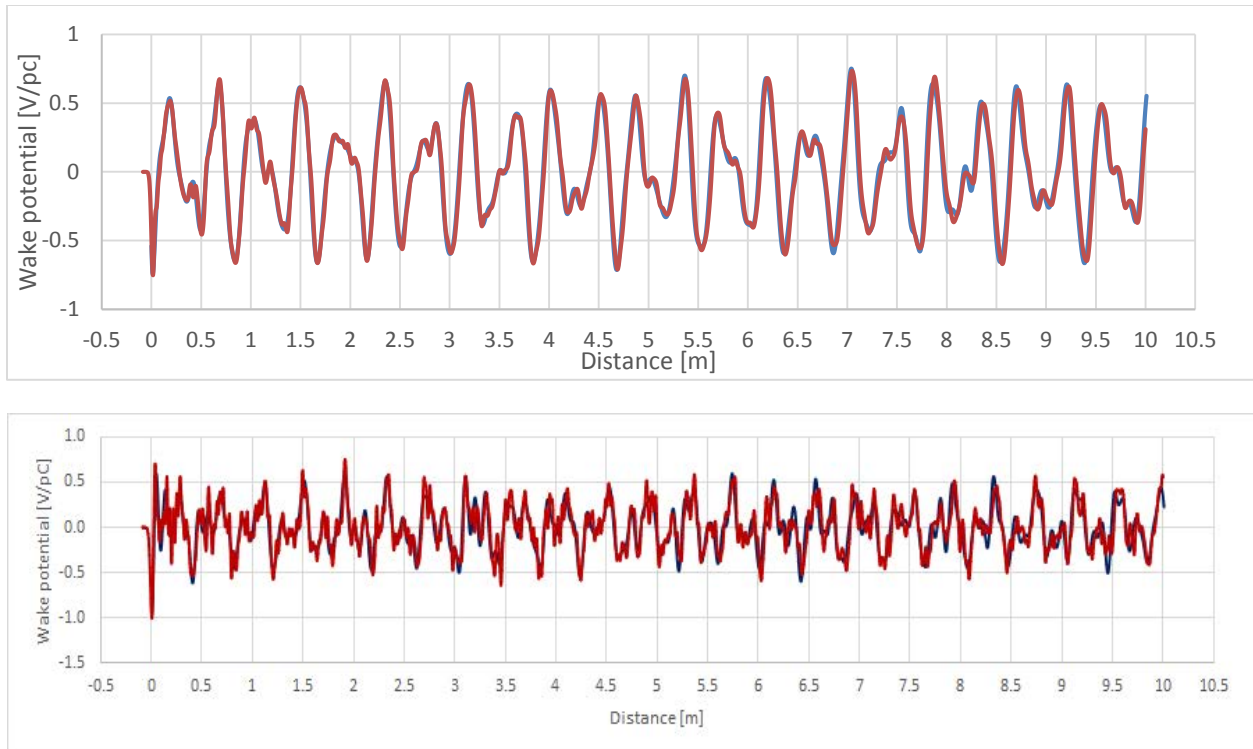


Fig. 1 Single bunch wake potential in 10m for a) 2.1 GHz warm cavity (top); b) 704 MHz warm cavity (middle); c) 704 MHz booster SRF cavity (bottom). Blue curves are from the reconstruction of Eigen mode simulation, and red curves are from CST Particle Studio™.

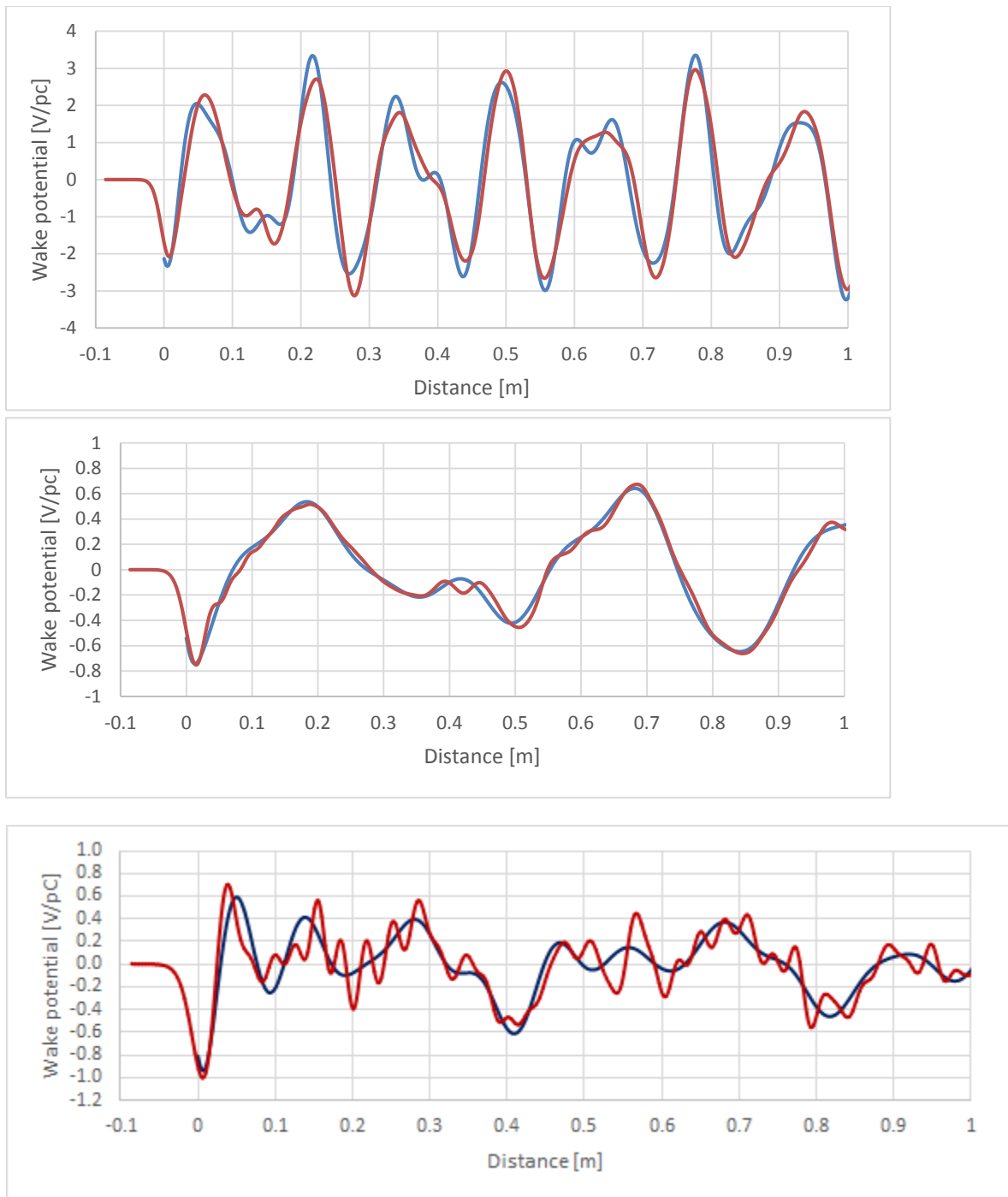


Fig. 2 Single bunch wake potential in 1m for a) 2.1 GHz warm cavity (top); b) 704 MHz warm cavity (middle); c) 704 MHz booster SRF cavity (bottom). Blue curves are from the reconstruction of Eigen mode simulation, and red curves are from CST Particle Studio™.

4, Multi bunch multi train voltage deviation from wake potential envelope (i.e. fundamental mode)

For the above mentioned single bunch with $Q=100pC$, we have 31 bunches in a train with $f_1=703.518$ MHz (corresponds to T_1), and $f_2=9.1$ MHz (corresponds to T_2) between trains.

In this case the envelope of the single rectangular bunch from all the HOMs below longitudinal cut-off frequency is:

$$SingleBunch_EnW_{\parallel}(\tau) = \sum_{HOMs} F \frac{1}{T} \frac{R_s}{2Q_r} e^{-\frac{\omega_r \tau}{2Q_r}}$$

The envelope of the multi (30 or 31, use 31 as an example) rectangular bunch single train is:

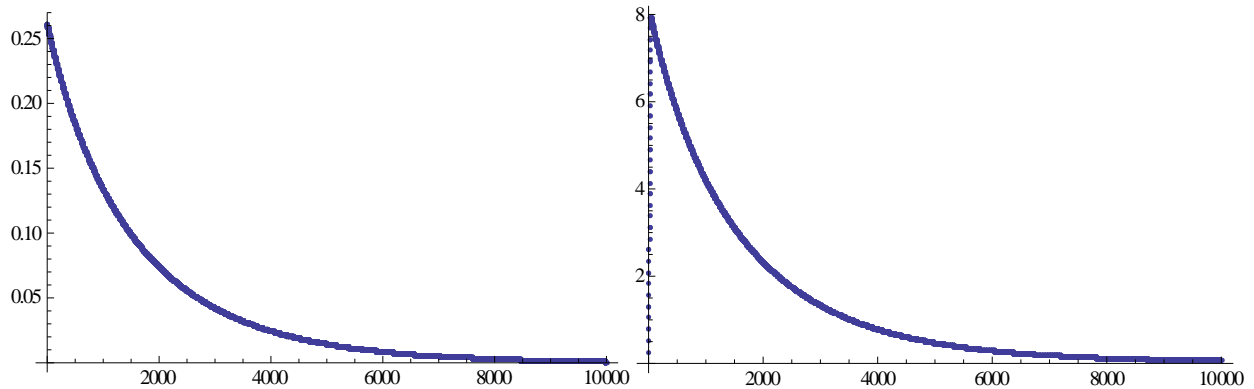
$$MultiBunch_EnW_{\parallel}(\tau) = \sum_{N=1}^{31} \sum_{HOMs} F \frac{1}{T} \frac{R_s}{2Q_r} e^{-\frac{\omega_r[\tau-(N-1)T_1]}{2Q_r}}$$

The envelope of the multi train is:

$$MultiTrain_EnW_{\parallel}(\tau) = \sum_k MultiBunch_EnW_{\parallel}(\tau - (k - 1)T_2)$$

Here k is a number large enough to guarantee the wake potential to saturate.

For the 2.1 GHz cavity the voltage envelope from all the HOMs below longitudinal cut-off frequency is shown in Fig. 3.



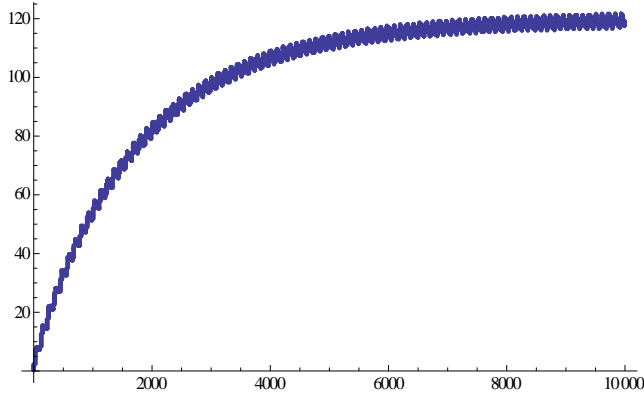


Fig. 3 Envelope of the voltage from wake field of HOMs in 2.1 GHz cavity, with x axis the time in nS, and y axis the voltage in kV: a) Single bunch single train (top left); b) Multi bunch single train (top right); c) multi bunch multi train (bottom).

Using the envelope that assumes all the modes will “beat” the frequency f_1 , we got a ± 122.6 kV voltage deviation (Fig. 3(c)). In this case we need to evaluate the voltage fluctuation with the fine structure of the wake potential. Please note the fundamental mode is not included in this simulation.

For a single bunch, the loss factor from all longitudinal modes below beam pipe longitudinal cut-off frequency is 1.46 V/pC, majorly from the 3.023 GHz HOM (with $R/Q=114/2$ ohm and $Q=18000$, 0.54 V/pC). With 100 pC in a bunch, the envelope of the voltage will be 0.29 kV (Fig. 3(a)). The decay time of these HOMs are around 1600 nS, and the train is in 9.1 MHz frequency (110 nS), so after 14 trains the first train decays to $1/e$ of its original potential. In this case 31 bunches in a train will boost the envelope of the voltage to 8.99 kV (Fig. 3(b)), and 16 trains will boost the voltage envelope to 125.9 kV, close to the number we get above at ± 122.6 kV.

Since for fundamental mode, the mode will “beat” both frequencies f_1 and f_2 , the voltage from the fundamental mode is going to follow the trend of the envelope, not fluctuating within envelope. Fig. 4 shows the envelope of the voltage from multi bunch multi train wake field of the fundamental mode, from where one can see that the voltage changes from 102.7 kV to 107.1 kV. Please note the FPC is set to be critically coupled to the cavity, so the loaded quality factor $Q_r=Q_0/2$.

For a single bunch, the wake potential from the fundamental mode (2.1 GHz with $R/Q=315/2$ ohm and $Q=10490$) is 1.04 V/pC. With 100 pC in a bunch, the envelope of the voltage will be 0.21 kV. The voltage decay of the fundamental mode to $1/e$ is around 1590 nS (with $\tau = 795$ nS), and the train is in 9.1 MHz frequency (110 nS). So after 14 trains the first train decays to $1/e$ of its original potential. In this case 31 bunches in a train will boost the envelope of the voltage to 6.5 , and 14 trains will boost the voltage envelope to 91.2 kV, close to the number we get above at $102.7 \sim 107.1$ kV. With the maximum of the envelope to be 107.1 kV, the voltage can be dropped to $107.1 * \text{Exp}[-(T_2 - 31 * T_1) \omega_r / 2 / Q_r] = 102.7$ kV. The fluctuation comes from the

voltage drop after the 31 bunches pass and before the next 31 bunches come in, it can also be roughly estimated from $q \cdot \omega \cdot R/Q = 100 \text{pC} \cdot 31 \cdot 2 \cdot \text{Pi} \cdot 2.11 \text{GHz} \cdot 315 / 2 \text{ohm} = 6.5 \text{ kV}$.

The voltage fluctuation from the fundamental mode needs to be corrected by the 9 MHz cavity, and/or with the RF power regulation. The following voltage fluctuation simulation only consider the HOMs, fundamental mode is not included.

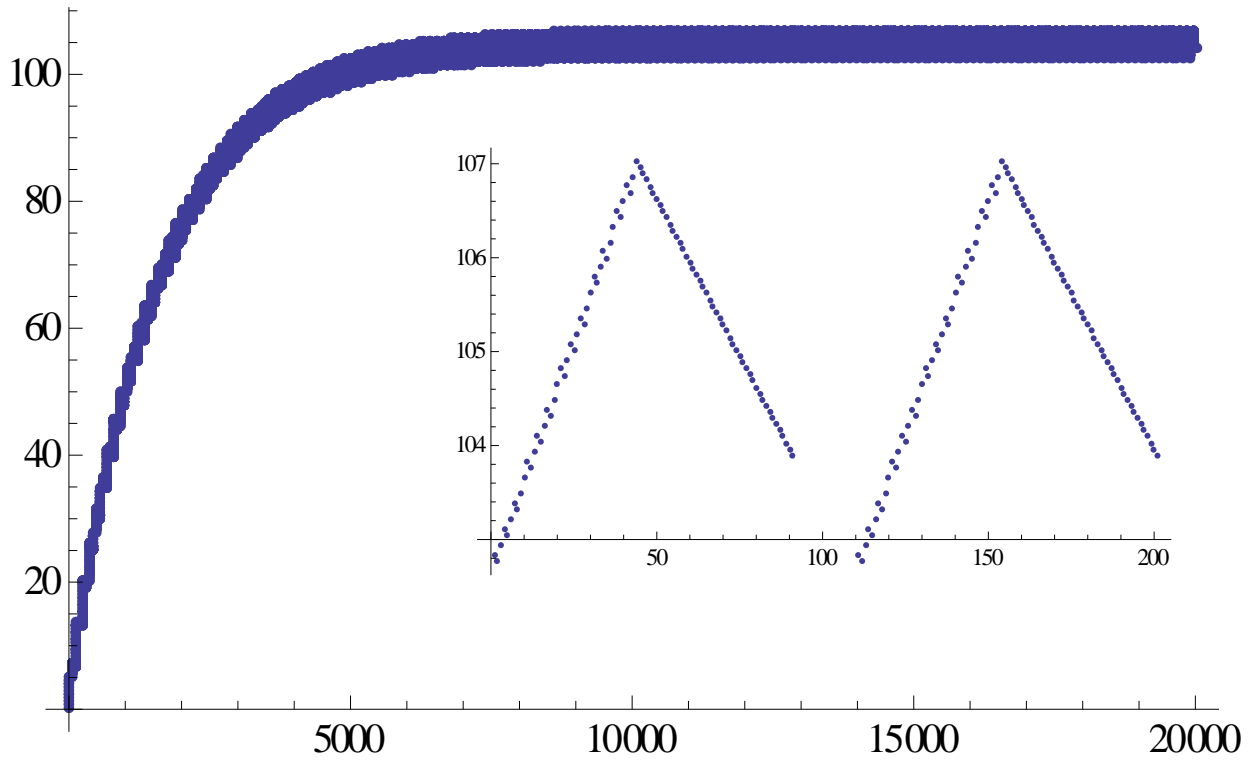


Fig. 4 Envelope of the voltage from multi bunch multi train wake field of the fundamental mode in 2.1 GHz cavity, with x axis the time in nS, and y axis the voltage in kV. The small plot shows the zoom in between 19800 and 20000 nS.

5, Multi bunch multi train voltage deviation from wake potential fine structure

The multi bunch (31 bunches for example) single train wake potential is:

$$MultiBunch_{W_{\parallel}}(\tau) = \sum_{N=1}^{31} \sum_{HOMs} W_{\parallel}(\tau - (N - 1)T_1)$$

and the multi train wake potential is:

$$MultiTrain_{W_{\parallel}}(\tau) = \sum_k MultiBunch_{W_{\parallel}}(\tau - (k - 1)T_2)$$

Here k is a number large enough to guarantee the wake potential to saturate. Considering that it is not easy to accurately simulate the frequency of the HOMs, we assume that **the modes that are close (± 20 MHz) to the multiple of the frequency f_1 will beat the multiple, and all modes will beat the frequency f_2 .**

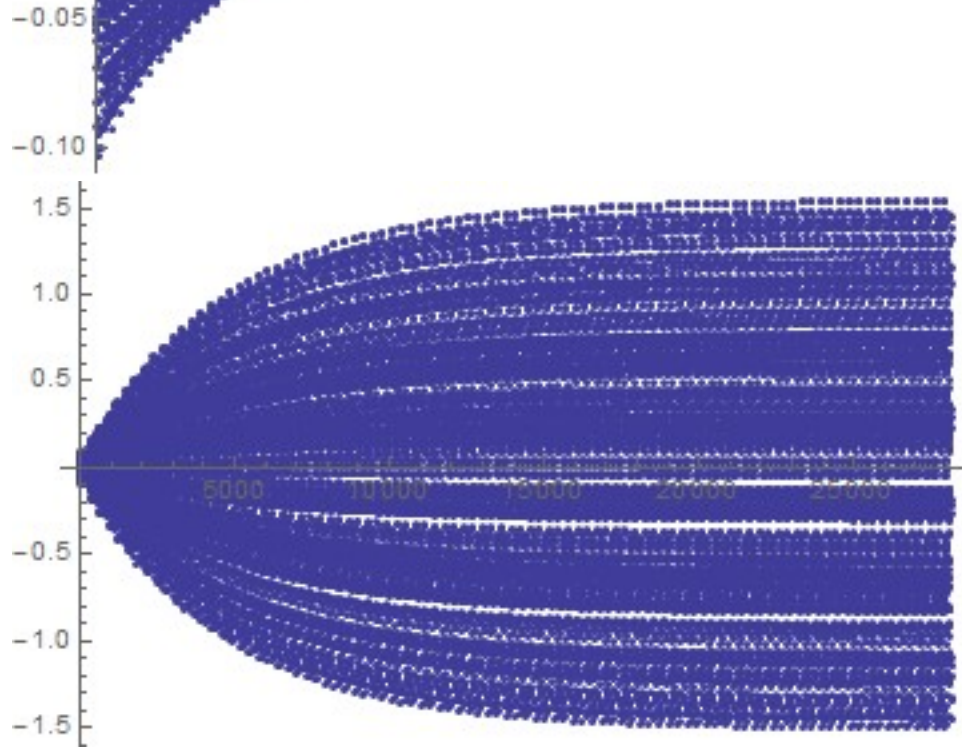
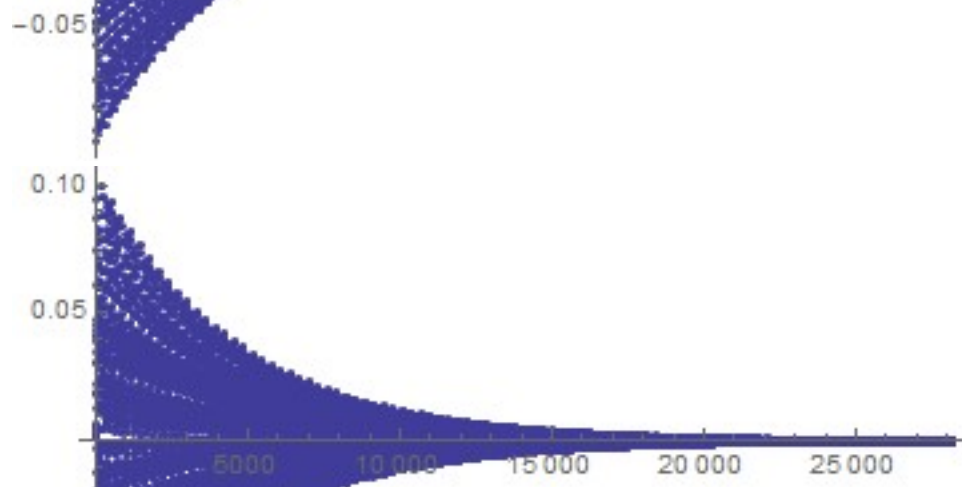
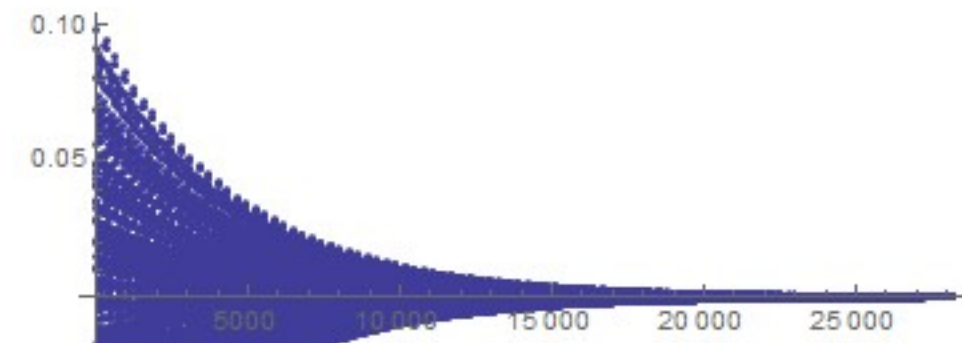
In the following section, we use Gaussian bunch to analyze the voltage fluctuation in the 2.1 GHz warm cavity.

First, voltage fluctuation from each HOM is calculated, and the modes that give more than 0.1 kV are listed below in Table 1:

Table 1 Voltage fluctuation from critical HOMs in the 2.1 GHz warm cavity.

Freq [GHz]	Q_0	R/Q [Ω]	Voltage [kV]
3.2574	15882	10.5	0.12
3.2808	17602	73.8	1.1
3.3034	15265	34.6	0.36
		Total	1.64

The voltage fluctuation from all HOMs is shown in Fig. 5:



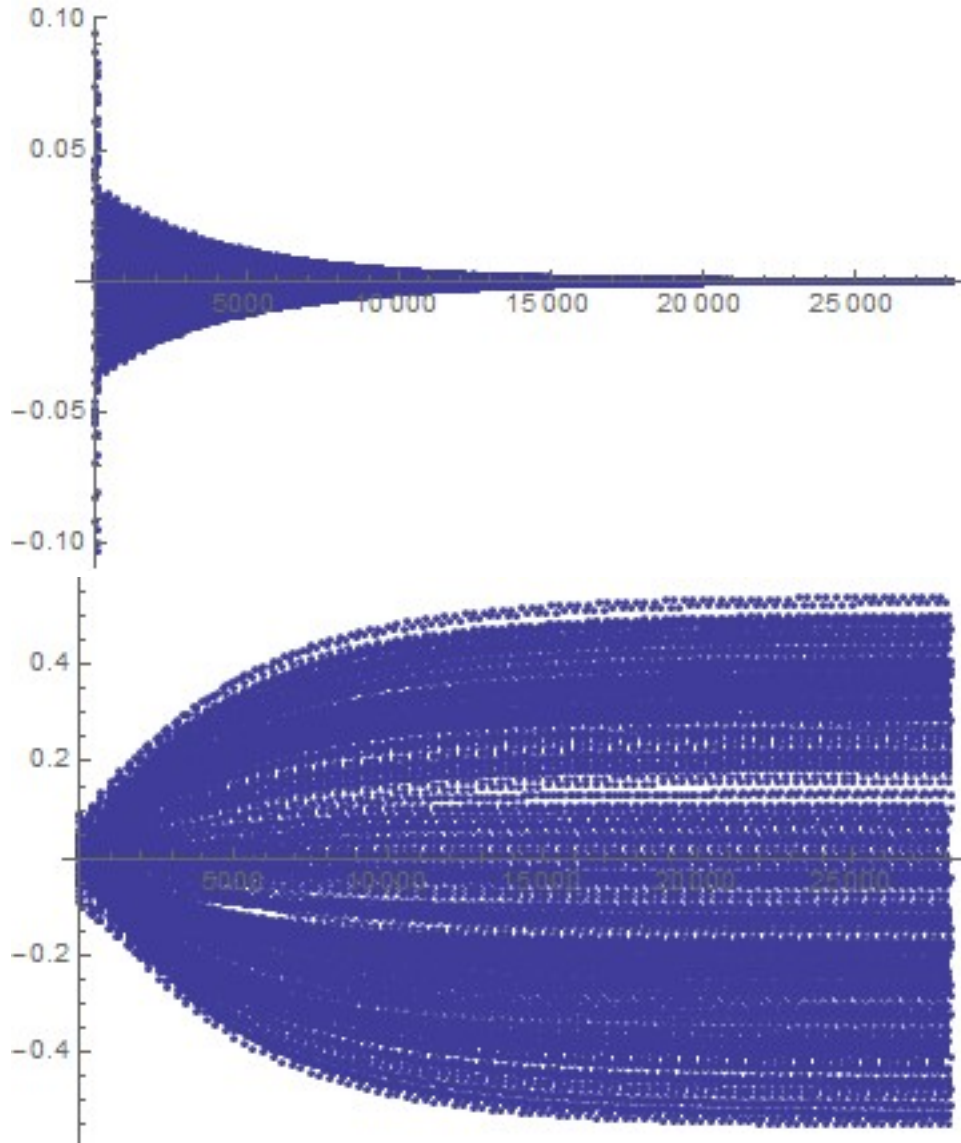


Fig. 5 Voltage from wake field of HOMs in 2.1 GHz cavity, with x axis the time in 0.355332 nS ($T_1/4$) steps, and y axis the voltage in kV: a) single bunch single train; b) multi (31) bunch single train; c) multi (31) bunch multi train; d) multi (30) bunch single train; e) multi (30) bunch multi train. The unacceptable voltage fluctuation in (c) should be suppressed by shifting the high R/Q HOM(s) away from the multiple of 9MHz.

One might notice that the multi bunch multi train patterns for 31 bunches per train and for 30 bunches per train are quite different, the voltage from 30 bunches configuration is significantly lower than that from 31 bunches. The reason is that 3.2808 GHz is 4.663 times of the 704 MHz.

Let us consider a mode 2/3 times of the fundamental. A single bunch gives $\text{Cos}[\tau]$ voltage fluctuation: $\text{Cos}[\tau]$. Two bunches give: $\text{Cos}[\tau] + \text{Cos}[\tau - 2/3 * 2\pi] * \text{UnitStep}[\tau - 2/3 * 2\pi]$. And three

bunches give: $\text{Cos}[\tau] + \text{Cos}[\tau - 2/3 * 2\pi] * \text{UnitStep}[\tau - 2/3 * 2\pi] + \text{Cos}[\tau - 4/3 * 2\pi] * \text{UnitStep}[\tau - 4/3 * 2\pi]$.
 The voltage fluctuation from this mode will get cancelled every three bunches.

In our case, the voltage get heavily reduced, even it did not get fully cancelled, for 3.2808 GHz mode due to a small times slip with 30 bunches configuration. For the configuration with 31 bunches, the last bunch produces a relatively large voltage fluctuation.

The field pattern of the 3.2808 GHz HOM in the 2.1 GHz warm cavity is shown in Fig. 6, it is a TM_{011} mode.

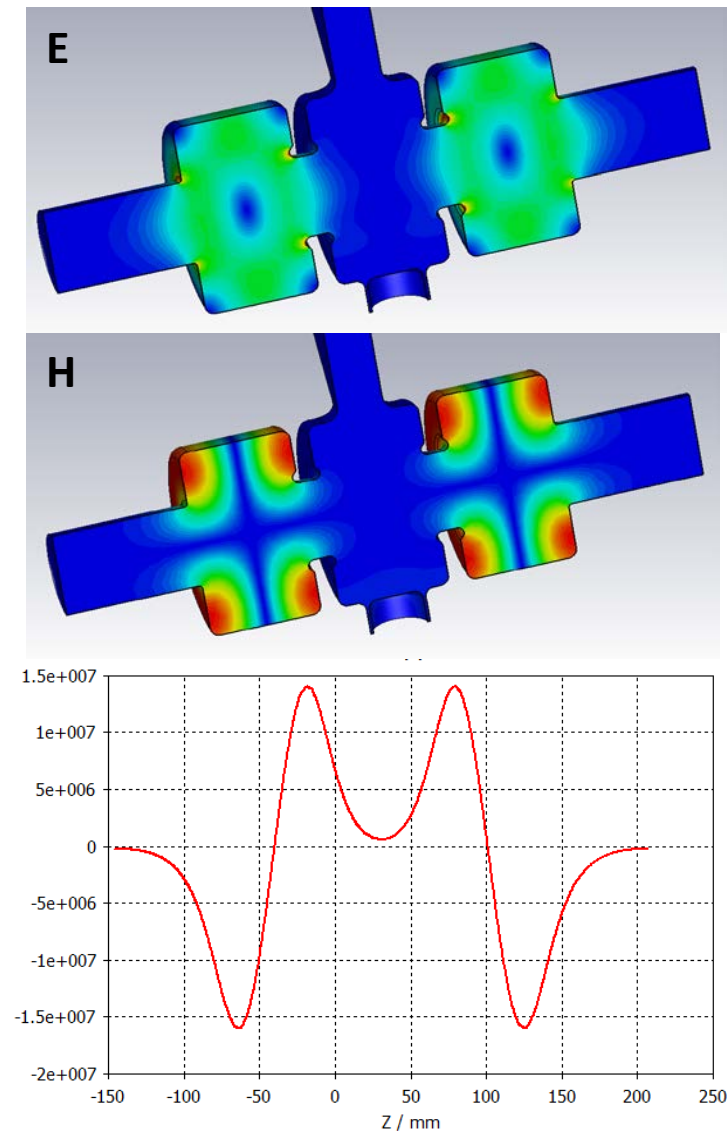


Fig. 6 Field pattern of the 3.2808 GHz HOM in the 2.1 GHz warm cavity: a) E field on the cross section; b) H field on the cross section; c) E field along the center axis, with x axis the length in mm and y axis the E field strength in V/m.

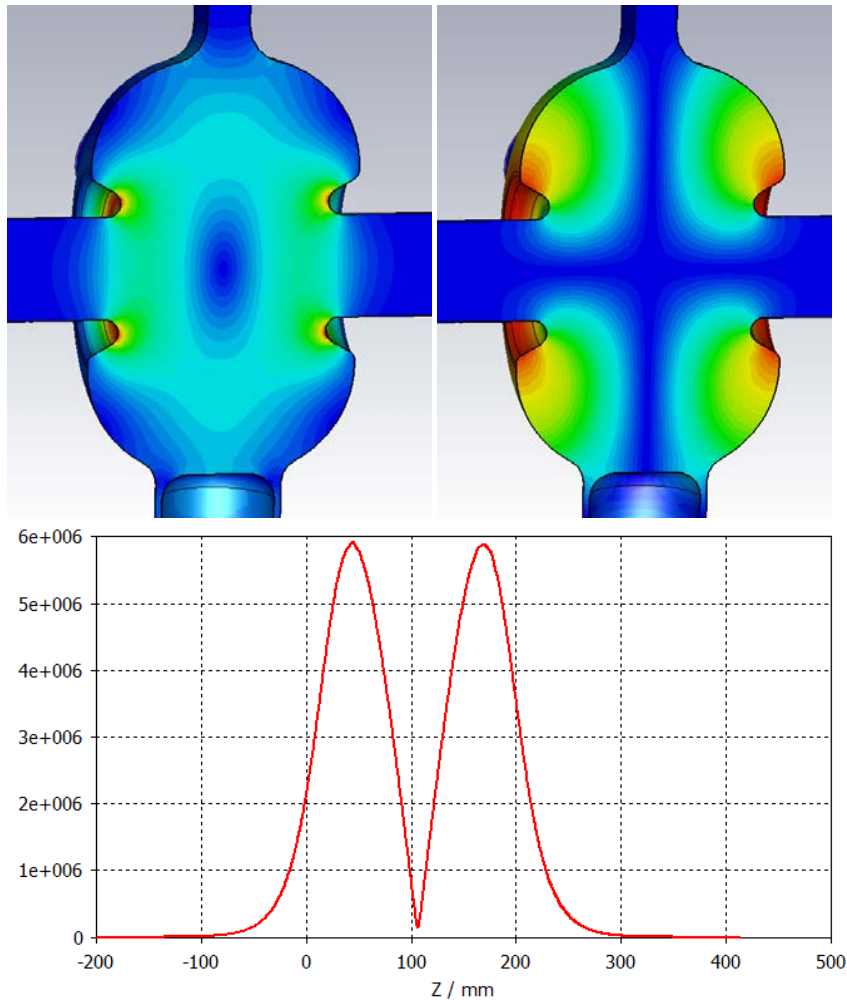


Fig. 7 Field pattern of the 1.087 GHz HOM in the 704 MHz warm cavity: a) E field on the cross section; b) H field on the cross section; c) E field along the center axis, with x axis the length in mm and y axis the E field strength in V/m.

Please note that even it is not obvious, the voltage fluctuation reduces while the resonance frequency moves far away from the harmonic of f_2 .

The 31 bunch configuration gives a ± 1.57 kV peak to peak voltage fluctuation (Fig. 5(c)), corresponding to $\pm 7.9e-4$ dp/p peak to peak. If we shift the 3.2808 GHz mode 0.5 MHz away from the harmonic of the 9.14 MHz gives 0.57 kV total fluctuation, corresponding to $\pm 2.9e-4$ dp/p peak to peak.

If we use 30 bunches instead of 31 bunches, the voltage changes to 0.55 kV (Fig. 5(e)). If we shift the 3.2808 GHz mode 0.5 MHz away from the harmonic of the 9.14 MHz, the 30-bunch configuration gives 0.32 kV fluctuation. It is less than the voltage fluctuation of 31-bunch configuration.

For the 704 MHz warm cavity, the 31 bunch configuration gives a 0.86 kV voltage fluctuation and the 30 bunch configuration gives 2.04 kV. With the 1.087 GHz mode 1 MHz away from the harmonic of the 9.14 MHz, the voltage fluctuation changes to 0.43 kV for 31 bunch configuration and 0.62 kV for 30 bunch configuration, corresponding to maximum $\pm 3.1e-4$ dp/p peak to peak regardless the configuration. The field pattern of the 1.087 GHz HOM in the 704 MHz warm cavity is shown in Fig. 7, it is a TM_{011} mode.

For the 704 MHz booster cavity, the 31 bunch configuration gives a 4.68 kV voltage fluctuation and the 30 bunch configuration gives 4.92 kV. With the 1.488 GHz (measured to be at 1.478 GHz) TM_{020} mode 0.7 MHz away from the harmonic of the 9.14 MHz, the voltage fluctuation changes to 0.50 kV for 31 bunch configuration and 0.60 kV for 30 bunch configuration, corresponding to maximum $\pm 3.0e-4$ dp/p peak to peak regardless the configuration. The field pattern of the 1.478 GHz HOM in the 704 MHz SRF booster cavity is shown in Fig. 8.

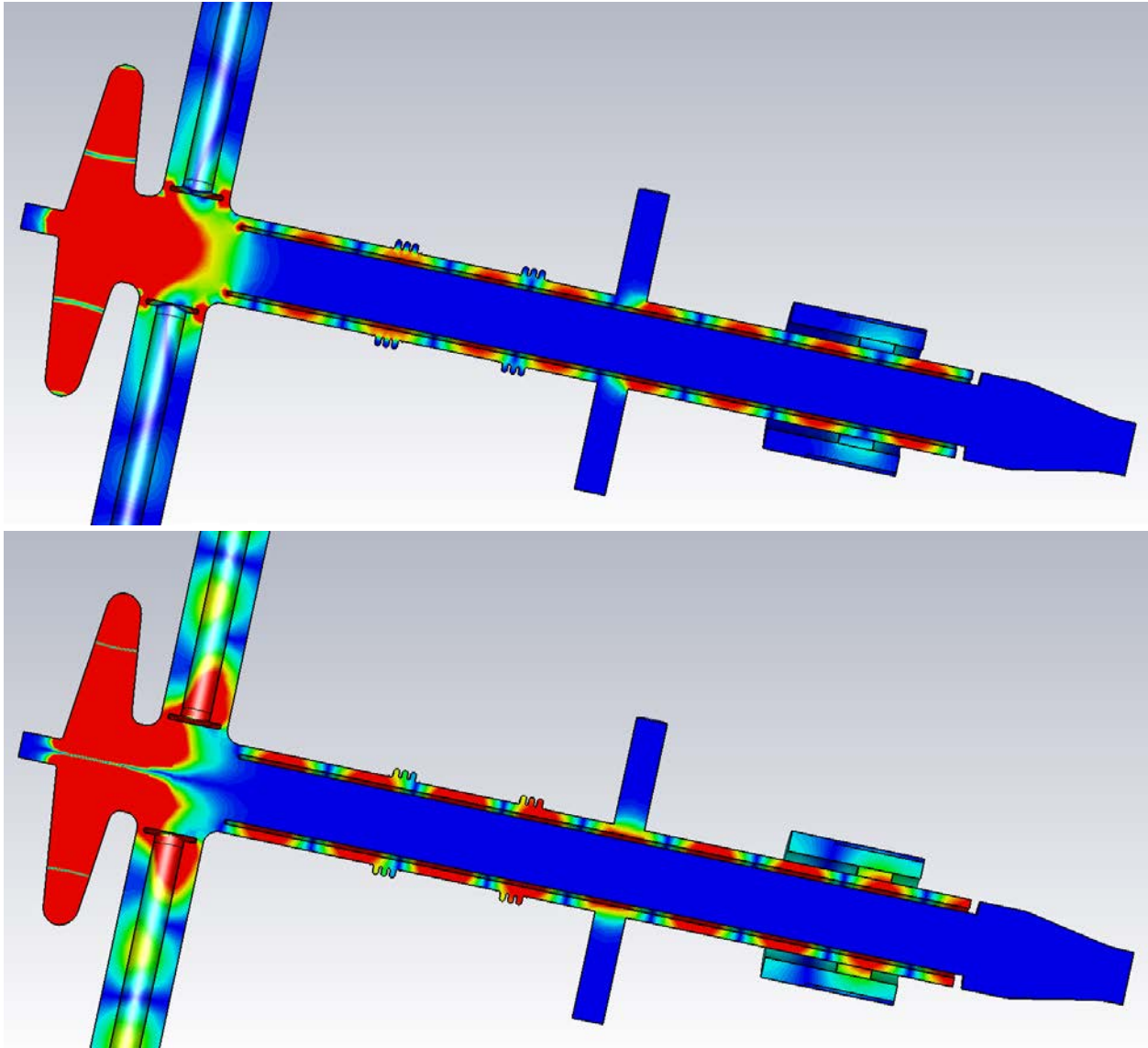


Fig. 8 Field pattern of the 1.478 GHz HOM in the 704 MHz SRF booster cavity: a) E field on the cross section; b) H field on the cross section.

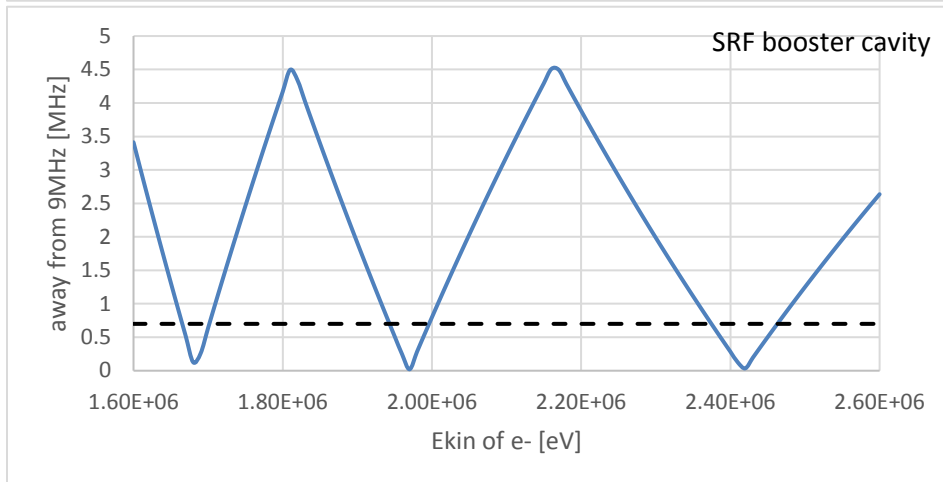
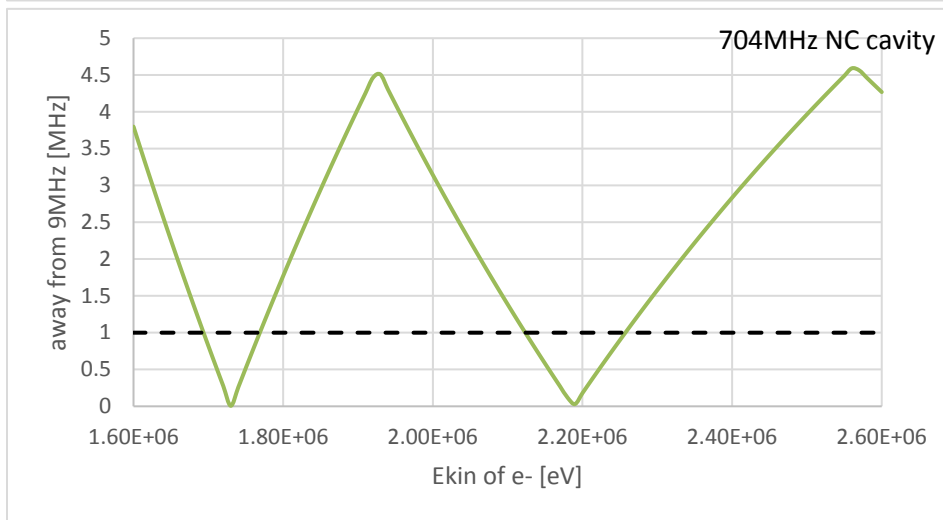
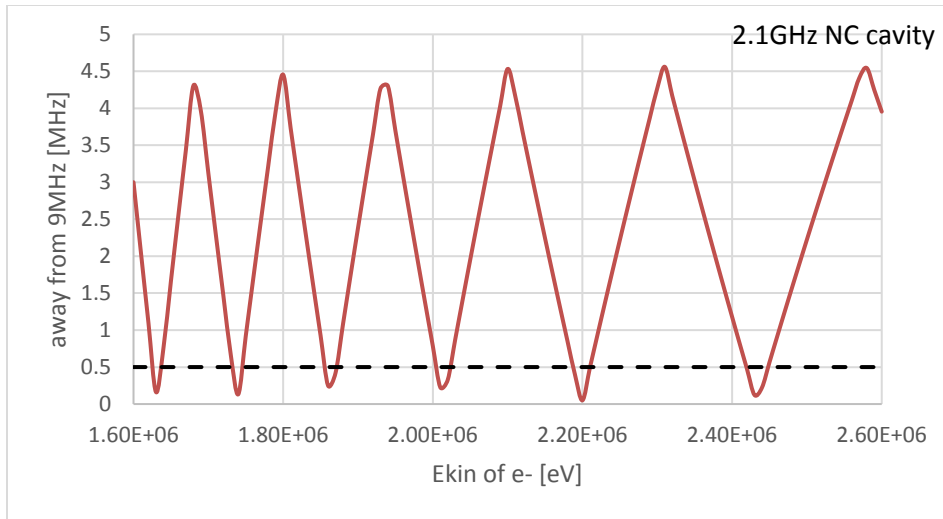


Fig. 9 Dangerous modes that should be away from the harmonic of 9 MHz. The x axis is the kinetic energy of the electron beam, and y axis is the frequency distance from the closest harmonic of 9 MHz: a) 2.1 GHz warm cavity; b) 704 MHz warm cavity; c) 704 MHz SRF booster cavity.

6, Choice of the beam energy

Since the RHIC revolution frequency changes with beam energy, the 9MHz frequency will also change with beam energy. In the LEReC system, the kinetic energy of the electron beam ranges from 1.6 MV to 2.6 MV. The RHIC revolution frequency ranges from 75.87 kHz to 77.14 kHz, and the 9 MHz frequency ranges from 9.10 MHz to 9.26 MHz.

The dangerous modes should be put away from the harmonic of the 9 MHz. For the 2.1 GHz warm cavity, the 3.2808 GHz (simulation result) mode should be 0.5 MHz away, for the 704 MHz warm cavity, the 1.087 GHz (simulation result) mode should be 1 MHz away, and for the 704 MHz SRF booster cavity, the 1.478 GHz (measurement result) mode should be 0.7 MHz away. Fig. 9 shows the possible choice of the beam energy. The x axis is the kinetic energy of the electron beam, and y axis is the frequency distance from the closest harmonic of 9 MHz. The region with solid line below dashed line should be avoided. More accurate analysis will be done based on the measurement results of the dangerous modes in the 2.1 GHz and 704 MHz warm cavities once received.

Emittance growth from the transverse modes

To calculate the energy spread from the longitudinal modes, Eigen mode simulation is first done using CST Microwave Studio™, with the simulation frequency ranges from the fundamental mode to the first transverse cut-off of the beam pipe. Then the single bunch wake potential is constructed using the Eigen mode simulation results. The multi bunch wake potential is calculated by shifting – adding the single bunch wake potential.

1, Eigen mode simulation

As we discussed earlier, for the 704 MHz SRF booster cavity, we calculate the HOMs using CST Microwave Studio™ Eigen mode simulation with frequency up to 3.81 GHz, the cut-off frequency of the TM₀₁ mode. For all transverse modes, we use the **accelerator definition** $(R/Q)_T$ at $\beta=1$:

$$\left(\frac{R}{Q}\right)_T = \frac{|V_z(r_0) - V_z(0)|^2}{\omega U \left(\frac{\omega}{c} r_0\right)^2}$$

2, Transverse wake potential

For each transverse HOM, the impedance is given by:

$$Z_{\perp}(\omega) = \frac{\omega_r}{j\omega} \frac{R_{\perp}}{1 + jQ_r(\omega/\omega_r - \omega_r/\omega)} \frac{\omega_r r_0}{c}$$

where R_{\perp} is the transverse shunt impedance in accelerator definition, Q_r the quality factor, ω_r the resonant frequency, r_0 the displacement and c the speed of light. Please note in (3) the impedance is in circuit definition with Ω/m in unit and here we use Ω . The wake function can be found from its Inverse Fourier transform, and its real part with $Q_r \gg 1$ (narrow band HOMs) is (3):

$$G_{\perp}(\tau) = \frac{\omega_r r_0}{c} \frac{\omega_r R_{\perp}}{2Q_r} e^{-\frac{\omega_r \tau}{2Q_r}} \sin(\omega_r \tau)$$

The single bunch is noted as $Q\lambda(t)$, the $\lambda(t)$ will be defined later for bunch with different shape, and Q the charge of the bunch.

The wake potential is the convolution of the wake function and the normalized density (3):

$$W_{\perp}(\tau) = \int_0^{\infty} dt G_{\perp}(t) \lambda(\tau - t)$$

For a simplified model with a single bunch $Q\lambda(t)$ in (half of the) Gaussian form with Q charge, with normalized density:

$$\lambda(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t)^2}{2\sigma^2}}$$

$$\text{and } \int_{-\infty}^{+\infty} \lambda(t) dt = 1.$$

After integration the wake potential can be written (use $Q_r \gg 1$ again) (3):

$$W_{\perp}(\tau) = \frac{\omega_r r_0}{c} \frac{\omega_r R_{\perp}}{4Q_r} e^{-\frac{\tau^2}{2\sigma^2}} \text{Im} \left\{ w \left(\frac{z_1}{\sqrt{2}} \right) \right\}$$

with $z_1 = \omega_r \sigma + j \left[\frac{\omega_r \sigma}{2Q_r} - \frac{\tau}{\sigma} \right]$ and w -function related to the error function of complex argument:

$$\int_0^{+\infty} dt e^{-a^2 t^2 + jzt} = \frac{\sqrt{\pi}}{2a} w \left(\frac{z}{2a} \right)$$

The multi bunch (31 bunches for example) single train wake potential is:

$$\text{MultiBunch}_W W_{\perp}(\tau) = \sum_{N=1}^{31} \sum_{HOMs} W_{\perp}(\tau - (N-1)T_1)$$

and the multi train wake potential is:

$$MultiTrain_W_{\perp}(\tau) = \sum_k MultiBunch_W_{\perp}(\tau - (k - 1)T_2)$$

Here k is a number large enough to guarantee the wake potential to saturate. Considering that it is not easy to accurately simulate the frequency of the HOMs, we use the same assumption as before that **the modes that are close (± 20 MHz) to the multiple of the frequency f_1 will beat the multiple, and all modes will beat the frequency f_2 .**

Since the modes that are close to the multiple of the frequency f_1 have low $(R/Q)_T$, none of these HOMs can accumulate voltage quickly within a train. In this case it is easy to understand that the highest $(R/Q)_T$ will give the most perturbation since we assumed all modes will beat the frequency f_2 . The most critical mode for vertical kick (aligned with FPC) is at 1.0057 GHz and the most critical mode for horizontal kick (perpendicular to FPC) is at 1.0049 GHz. The vertical mode is measured at around 1.0065 GHz at 2 K liquid helium bath temperature. Two modes are not well separated in this measurement. They are both TM11 modes, and their polarizations are perpendicular to each other. The estimated maximum vertical kick is 5.2 kV and for horizontal it is 1.43 kV for 0.5 mm displacement on each direction. Please note these two effect will not get stacked together, these two resonances are 0.8 MHz away and they will not beat ~ 9 MHz simultaneously. In this case $\Delta x'$ is estimated to be 1.8 mrad ($5.2\text{kV}/\sqrt{2}/2\text{MV}$) for vertical case and 0.51 mrad for horizontal case, and $\Delta \epsilon$ to be 7.7 mm*mrad ($4.3\text{mm} \cdot 1.8\text{mrad}$) for vertical and 2.2 mm*mrad for horizontal. This is a rough upper limit estimation. With the specification ϵ at 2.5 mm*mrad, the SRF booster cavity will contribute at most 308% of the emittance for vertical case and 88% for horizontal case. One can always limit the displacement at a smaller number, i.e, 0.02 mm, so that the contribution will decrease to 12% for vertical case and 4% for horizontal case. This can be achieved by steering the beam right after the DC gun using dipoles. With the resonance frequencies of these two modes away from the harmonic of f_2 , the transverse emittance will reduce. In case the actual result turns out to be much better, the displacement limitation can be relaxed accordingly.

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